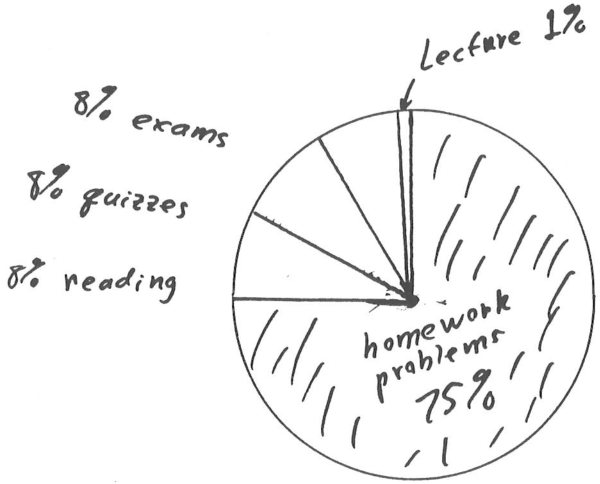
Where learning occurs



We have 3 hours of lecture time per week. You should spend <u>at least</u> that much time on this course outside of class.

Chapter 23

Electric Charge

Some observations:

- Electric charge appears to come in two "flavors." Call them
  - ~~vanilla and chocolate~~
  - ~~up and down~~
  - positive and negative

It doesn't matter which is which!

- Like charges repel each other while unlike charges attract each other.

demo

One kind of charge is applied to rubber by stroking it with hair (wool, fur, human hair)

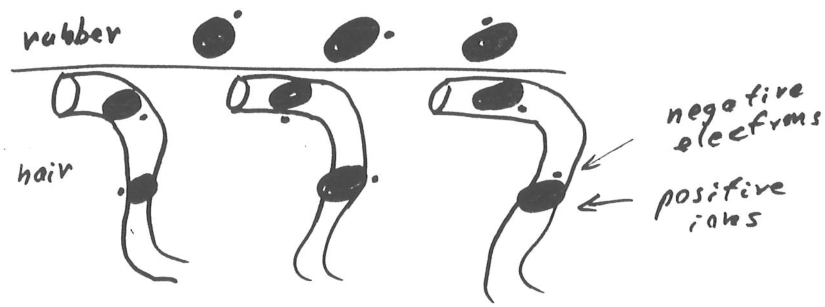
negative on rubber
positive charge on fur

The other kind of charge is applied to glass by stroking it with silk.

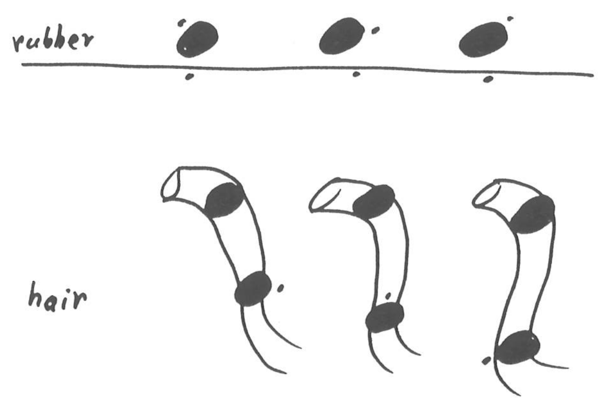
Positive on glass
negative on silk

Charge is <u>conserved</u>.

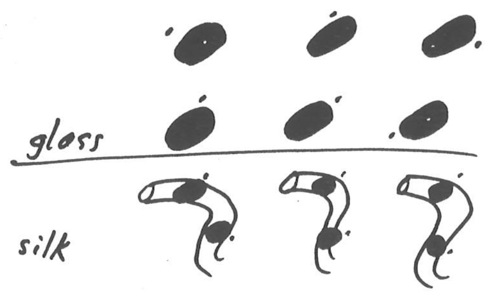
What's really going on here? Microscopically:



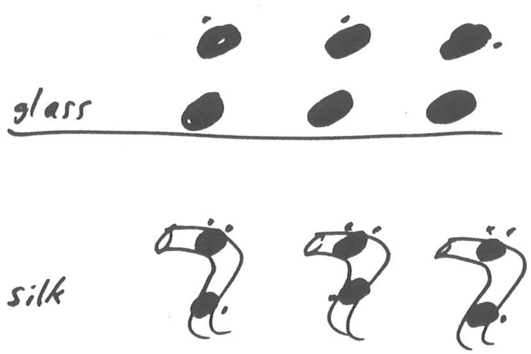
Electrons from the hair remain on the surface of the rubber. By convention, electrons are said to have negative charge.



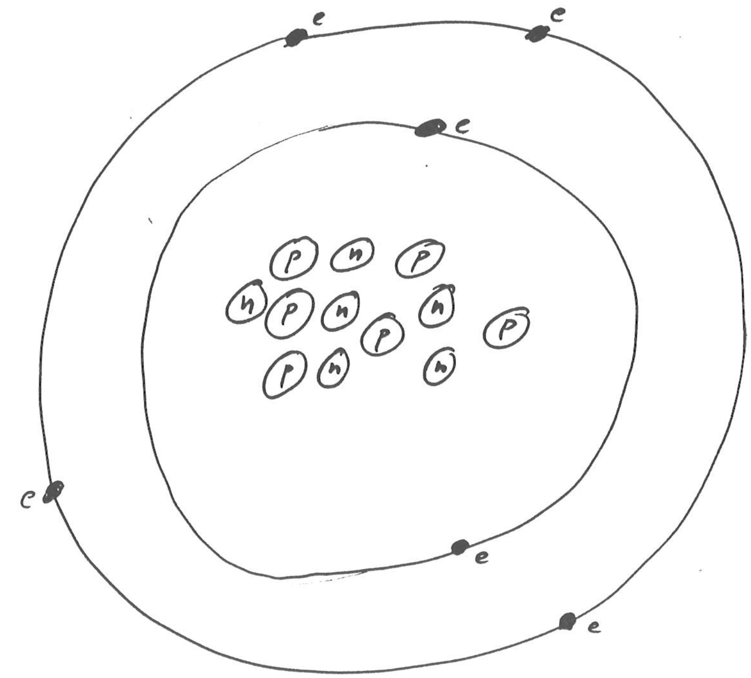
Again, microscopically:



Silk removes electrons from the surface of the glass, leaving the glass positively charged.



The Atom (e.g. Carbon)



Not to scale!

(p) proton - positively charged
(n) neutron - zero charge (neutral)
e• electron - negatively charged

The atom is electrically neutral.

All of the forces that you studied in Mechanics (1303) except gravity were electrical in origin.

- contact forces
  - normal force
  - friction
  - viscous drag
- string tension
- Hooke's Law spring force

More Observations

Gravity

Newton's Law of Universal Gravitation

magnitude: $F_G = \dfrac{G\, m_1 m_2}{(r_{12})^2}$

direction: along the line joining the masses

always attractive

Electric Force

Coulomb's Law

magnitude: $F_E = \dfrac{k\, |q_1|\, |q_2|}{(r_{12})^2}$

direction: along the line joining the charges

attractive if $\begin{cases} q_1 \text{ positive},\ q_2 \text{ negative} \\ q_1 \text{ negative},\ q_2 \text{ positive} \end{cases}$

repulsive if $\begin{cases} q_1 \text{ positive},\ q_2 \text{ positive} \\ q_1 \text{ negative},\ q_2 \text{ negative} \end{cases}$

G is the gravitational constant.

$$G = 6.67 \times 10^{-11} \ \frac{N \cdot m^2}{kg^2} \quad \text{(in MKS units)}$$

---

k is the electrostatic constant.

$$k = 8.99 \times 10^{9} \ \frac{N \cdot m^2}{C^2}$$

for later convenience

$$k \equiv \frac{1}{4\pi\epsilon_0} \qquad \epsilon_0 = \frac{1}{4\pi k}$$

ϵ_0 (epsilon nought) is the permittivity of free space

$$\epsilon_0 = 8.85 \times 10^{-12} \ \frac{C^2}{N \cdot m^2}$$

$$F_E = \frac{1}{4\pi\epsilon_0} \frac{|q_1| \ |q_2|}{(r_{12})^2}$$

Vector nature of $\vec{F}_E$

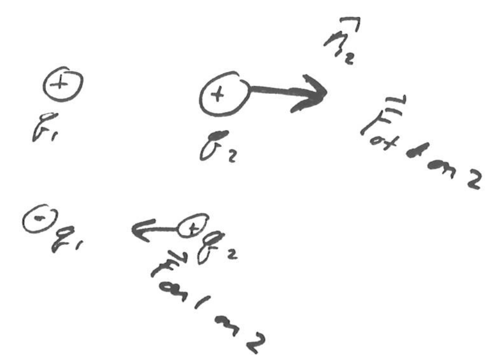
$$\vec{F}_{\text{of 1 on 2}} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{(r_{12})^2} \hat{r}_{12}$$

q_1, q_2 can be positive or negative

r_{12} is the distance between q_1 and q_2

$\hat{r}_{12}$ is a unit vector pointing from q_1 to q_2

Ex.



The MKS unit of charge

is the coulomb (C)

1 C is an <u>enormous</u> amount of charge.

Ex Find the force between two 1 C charges separated by 1 m.

$$F_E = k \frac{|q_1|\,|q_2|}{(r_{12})^2} = 8.99\times10^9 \frac{N\cdot m^2}{C^2} \frac{(1C)(1C)}{(1m)^2}$$

$$= \approx 9\times10^9 \cancel{N} \left(\frac{0.225\ lb}{1\cancel{N}}\right)$$

$$= 2\times10^9\ lbs \left(\frac{1\ ton}{2000\ lb}\right)$$

$$= 10^6\ tons.$$

The coulomb (C) is not one of the fundamental units (base units):

meter	m
kilogram	kg
second	s
ampere	A (amp) unit of current

The coulomb is a derived unit

$$1\ C = 1\ A\cdot s$$

like the newton (N)

$$1\ N = 1\ \frac{kg\cdot m}{s^2}$$

Electric charge is a scalar quantity, like mass. The only difference is that mass is always positive while charge can be positive <u>or</u> negative.

Charge adds like a scalar:

<u>Ex.</u> $3\,C + (-4\,C) = -1\,C$

<u>Ex.</u> $3\text{ kg} + 5\text{ kg} = 8\text{ kg}$ (mass)

<u>Ex.</u> $1.2\text{ kg}\cdot\text{m}^2 + 2.7\text{ kg}\cdot\text{m}^2 = 3.9\text{ kg}\cdot\text{m}^2$ (moment of inertia or rotational inertia)

The forces on one charge due to all the rest add like vectors.

$$\vec{F}_{\substack{\text{total}\\\text{on }1}} = \vec{F}_{\substack{\text{of }2\\\text{on }1}} + \vec{F}_{\substack{\text{of }3\\\text{on }1}} + \dots + \vec{F}_{\substack{\text{of }87\\\text{on }1}} + \dots$$

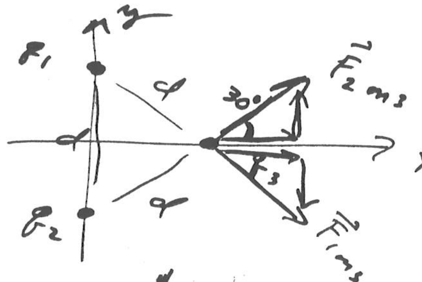
$$\vec{F}_{\substack{\text{of }87\\\text{on }1}} = \frac{1}{4\pi\epsilon_0}\frac{q_{87}\,q_1}{(r_{87,1})^2}\,\hat{r}_{87,1}$$

$\hat{r}_{87,1}$ points from q_{87} to q_1.

Charge q_1 cannot exert a force on itself!

This additive property of the vector force is called "super position."

Ex Consider 3 identical positive charges at the corners of an equilateral triangle of side d. What is the force on one charge due to the other two?



Find $\vec{F}_{\substack{total \\ on\ 3}}$

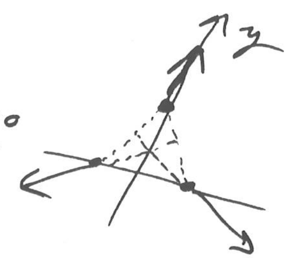
$q_1 = q_2 = q_3$

$$\left|F_{\substack{of\ 1\\ on\ 3}}\right| = \left|F_{\substack{of\ 2\\ on\ 3}}\right| = \frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}$$

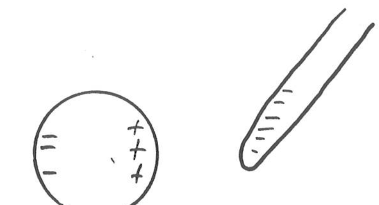
y-components cancel
x-components add up

$$\vec{F}_{\substack{total\\ on\ 3}} = (\hat{\imath})\,2\,\underbrace{\left|F_{1\,on\,3}\right|\cos 30^\circ}_{F_{1\,on\,3,\,x}}$$

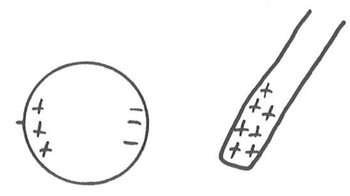
$$= 2\,\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\frac{\sqrt{3}}{2}\,\hat{\imath} = \sqrt{3}\,\frac{1}{4\pi\epsilon_0}\frac{q^2}{d^2}\,\hat{\imath}$$



An uncharged metal can is attracted by each kind of charge, never repelled. What's going on?



charge is induced (moved around) in the metal can.

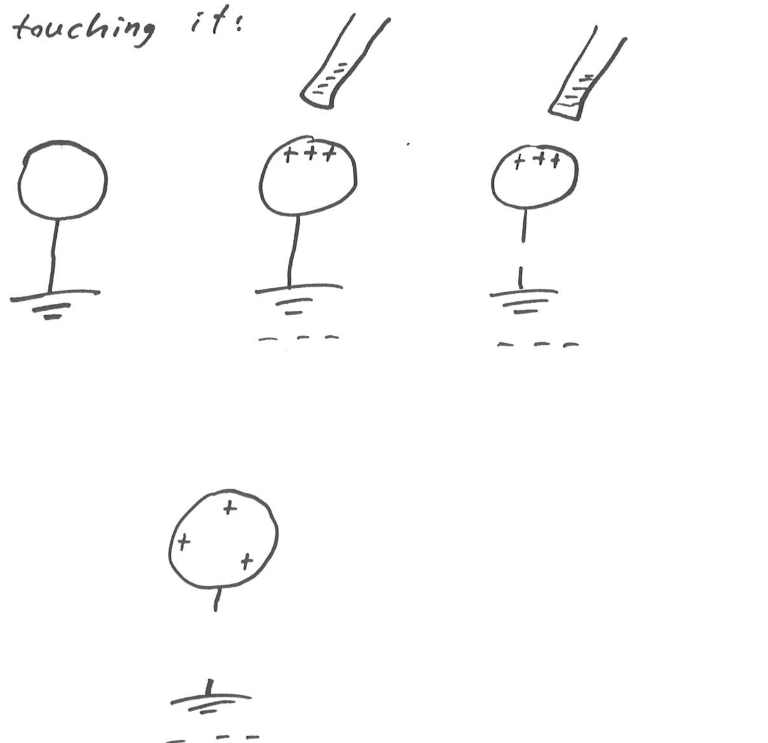


In a conductor, charge can flow freely.

In an insulator, charge is stuck in place.

A conductor that can take as many electrons as it needs from a large supply (like the earth), or give up as many electrons as it needs is said to be grounded.

How to charge a conductor without touching it:



Ex How much charge is in you?

$m = 75\ kg = 75{,}000\ g$ all water

molecular mass of water $18\ g$

I contain $\frac{75{,}000\ g}{18\ g} = 4100$ moles of water

each mole 6.02×10^{23} molecules

I have 2.5×10^{27} molecules

H_2O $\left.\begin{matrix} 1e \text{ in } H \\ 8e \text{ in } O \end{matrix}\right\}$ 10 electrons in H_2O molec.

How many C of negative charge?

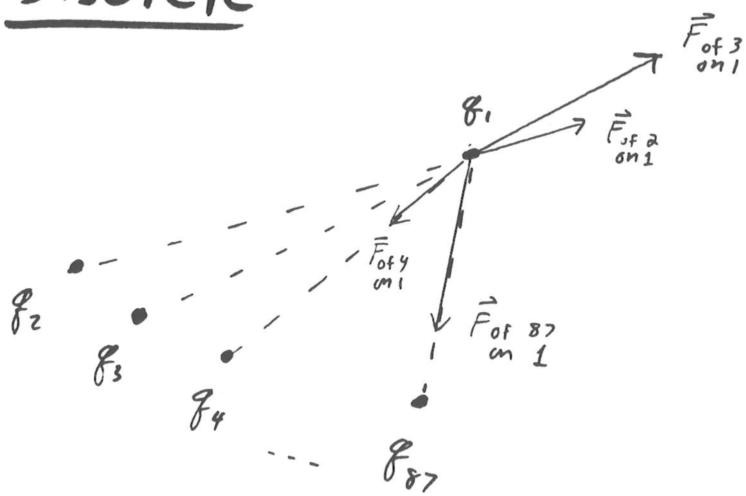
$2.5 \times 10^{27} \cdot 10 \cdot \underbrace{(-1.6 \times 10^{-19}\ C)}_{\text{charge on electron}}$

$= -4 \times 10^{9}\ C$

And just as much positive charge.

Superposition

Discrete

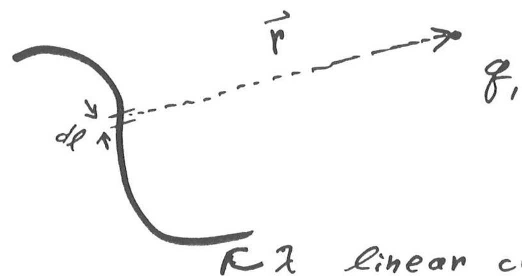


$$\vec{F}_{\substack{\text{total}\\ \text{on } 1}} = \frac{q_1}{4\pi\epsilon_0} \sum_{i=2}^{87} \frac{q_i}{(r_{i,1})^2}\, \hat{r}_{i,1}$$

points from charge i to charge 1.

Superposition

Continuous



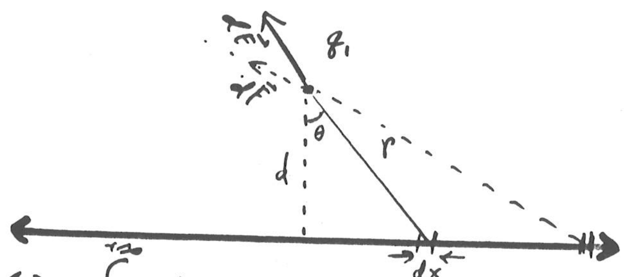
λ linear charge density "charge per meter"

$$dq = \lambda\, d\ell$$

$$\vec{F}_{\substack{\text{total}\\ \text{on } 1}} = \frac{q_1}{4\pi\epsilon_0} \int \frac{\lambda}{r^2}\, \hat{r}\, d\ell$$

$$= \frac{q_1}{4\pi\epsilon_0} \int dq\, \frac{1}{r^2}\, \hat{r}$$

Ex Find the force on a point charge $\underline{q_1}$ a distance $\underline{d}$ from an infinite uniform line of charge.



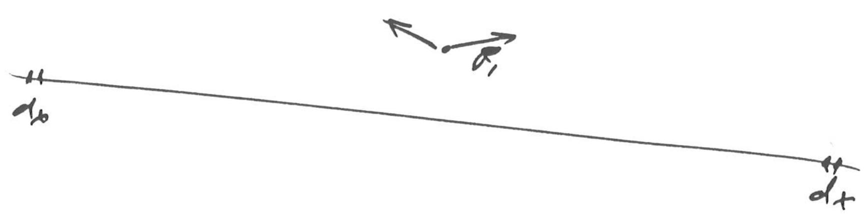
λ = constant

$$Q_{total} = \int_{-\infty}^{+\infty} \lambda \, dx = \lambda \int_{-\infty}^{\infty} dx = \infty$$

$$dq = \lambda \, dx$$

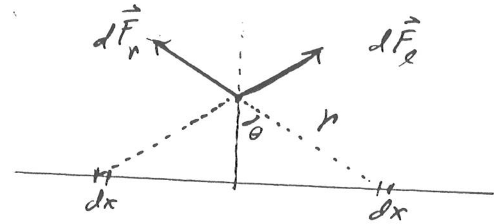
$$dF = \frac{1}{4\pi\epsilon_0} \frac{q_1 \, dq}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{q_1 \lambda \, dx}{r^2}$$

Direction?



$d\vec{F}$ has both x and y components.

For each infinitesimal charge dq on the right of q_1, there is a corresponding dq on the left of q_1.

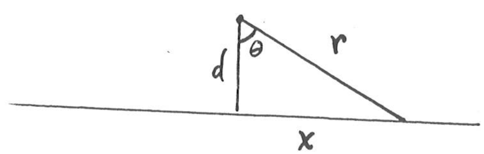


The x-components will cancel, leaving a force in the y-direction only.

$$dF_y = dF \cos\theta$$

$$F_y = \int dF_y = \int_{x=-\infty}^{\infty} \frac{1}{4\pi\epsilon_0} \frac{q_1 \lambda \cos\theta}{r^2} dx$$

r, θ, and x are related:



$$\tan\theta = \frac{opp}{adj} = \frac{x}{d} \Rightarrow x = d\tan\theta$$

$$\cos\theta = \frac{adj}{hyp} = \frac{d}{r} \Rightarrow r = \frac{d}{\cos\theta}$$

We need dx, so differentiate both sides of

$$x = \underline{d}\tan\theta$$

$$dx = \underline{d}\,\frac{1}{\cos^2\theta}\,d\theta$$

Finally, $x = -\infty$ corresponds to $\theta = -\frac{\pi}{2}$

$x = +\infty$ corresponds to $\theta = +\frac{\pi}{2}$

$\left[x = d\tan\theta \, , \quad \tan\frac{\pi}{2} = +\infty \right]$

$$F_y = \int_{x=-\infty}^{+\infty} \frac{1}{4\pi\epsilon_0}\frac{q_1\lambda\cos\theta}{r^2}\,dx$$

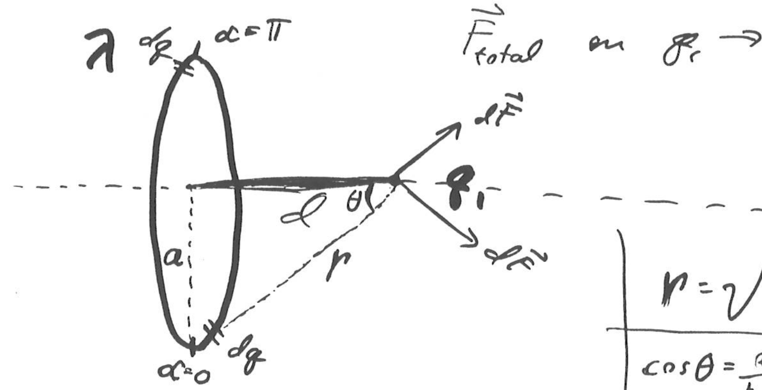
$$= \frac{q_1\lambda}{4\pi\epsilon_0}\int_{\theta=-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{\cos\theta}{\left(\frac{d}{\cos\theta}\right)^2}\,d\,\frac{1}{\cos^2\theta}\,d\theta$$

$$= \frac{q_1\lambda}{4\pi\epsilon_0 d}\int_{-\frac{\pi}{2}}^{+\pi/2}\cos\theta\,d\theta$$

$$= \frac{q_1\lambda}{4\pi\epsilon_0 d}\sin\theta\Big|_{-\pi/2}^{+\pi/2} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{2q_1\lambda}{d}}$$

$$\vec{F}_{total} = \frac{1}{4\pi\epsilon_0}\frac{2q_1\lambda}{d}\,\hat{j}$$

Find the electric field on the axis of a ring of charge, a distance $\underline{d}$ from the center.



$\vec{F}_{total}$ on $q_1 \rightarrow$ in x direction

$r = \sqrt{a^2 + d^2}$

$\cos\theta = \frac{adj}{hyp} = \frac{d}{r} = \frac{d}{\sqrt{a^2+d^2}}$

$dq = \lambda\, a\, d\alpha$

$$dF = \frac{1}{4\pi\epsilon_0} \frac{q_1\, dq}{r^2}$$

$$dF_x = \frac{1}{4\pi\epsilon_0} \frac{q_1\, dq}{r^2} \cos\theta$$

$$F_x = \int dF_x = \int_{\alpha=0}^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{q_1\, dq}{r^2}\cos\theta$$

$$= \frac{1}{4\pi\epsilon_0} q_1 \int_{\alpha=0}^{2\pi} \frac{1}{(a^2+d^2)} \frac{d}{\sqrt{a^2+d^2}} \lambda a \; d\alpha$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q_1 d \lambda a}{(a^2+d^2)^{3/2}} 2\pi$$

dimensions?

$[F] = \left[\frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}\right]$

$q_1 (\lambda a) \frac{L}{(L^2)^{3/2}}$

Charge is Quantized $\nearrow$ discrete

proton charge: $+e = +1.6 \times 10^{-19}\,C$

electron charge: $-e = -1.6 \times 10^{-19}\,C$

When dealing with lines of charge, we treat the charge as continuous. This approximation is valid as long as "e" is small compared to the other charges in the system.

The approximation fails in Atomic Physics.

Quarks

$$\left.\begin{array}{l} u = \text{up quark, charge } +\frac{2}{3}e \\ d = \text{down quark, charge } -\frac{1}{3}e \end{array}\right\} \text{never free}$$

proton = (uud) neutron = (udd)

electron = electron

What is the force on a charge $2q_1$?

$$\left[\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\right](2q_1)$$

$15.7\,q_1$?

$$\left[\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{d}\right](15.7\,q_1)$$

The Force on the charge at position 1 is

(Electric Field at position 1) q_1

The Electric Field is the "Force per unit charge" at position 1.

Since the Force is a <u>vector</u>, the Electric Field is also a <u>vector</u>.

The Electric Field exists at position 1 <u>even if</u> there is no charge at position 1 to feel a force.

(The Electric Field at position 1 is created by all of the <u>other</u> charges in the problem.)

Mechanical Analog:

The gravitational field (acceleration, or "force per unit mass") of the Earth exists even if there is no small mass m, to feel the attractive force.

If you jiggle the other charges (2→87) you will create waves in the electric field. These waves travel at the speed of light, and are called radiation.

Radiation is extremely complicated, so we will first study statics, in which the charges will <u>not</u> move.

Units

MKS units of the electric field are

$$\frac{N}{C} = \frac{\text{newtons}}{\text{coulomb}}$$

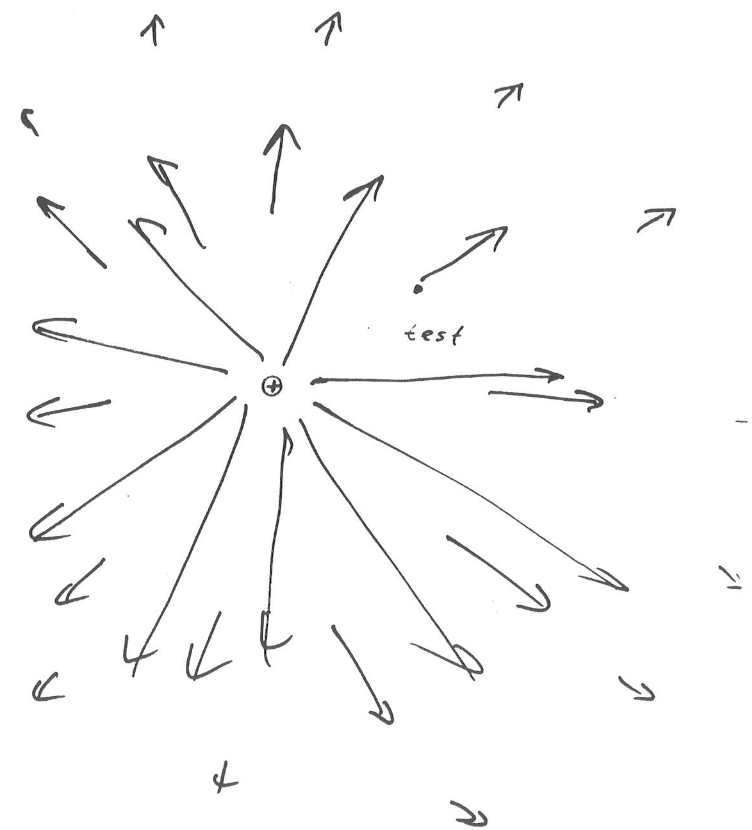
The $\vec{E}$ field is a <u>vector</u>, so we can represent it graphically with directed lines.

<u>Convention:</u> $\vec{E}$ field lines point away from positive charges and toward negative charges.

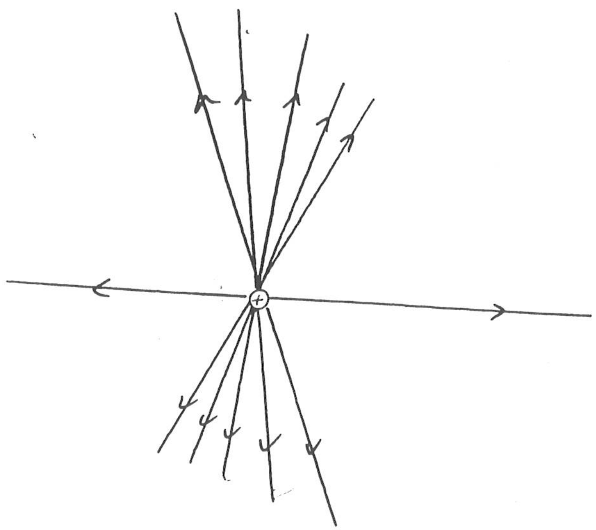
We will use a infinitely small positive "test charge" to map the electric field.

$\vec{F}$orce on a <u>positive</u> test charge is in the direction of $\vec{E}$ field.

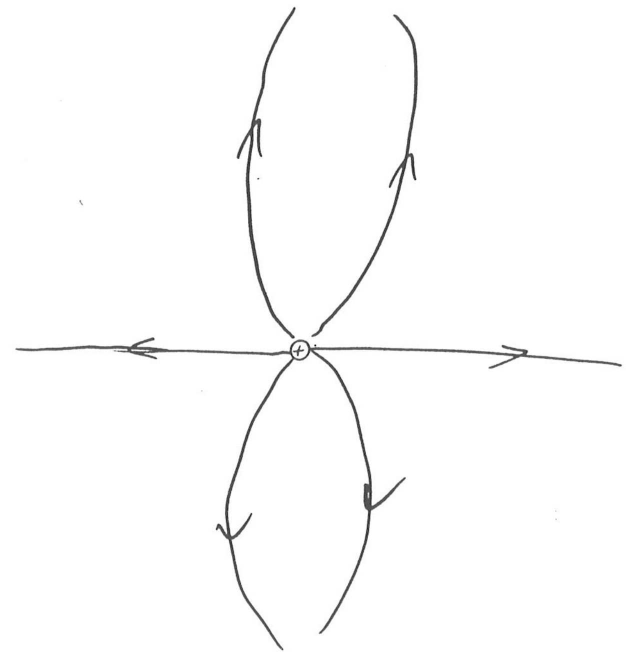
$\vec{E}$ field due to a positive <u>point</u> charge:



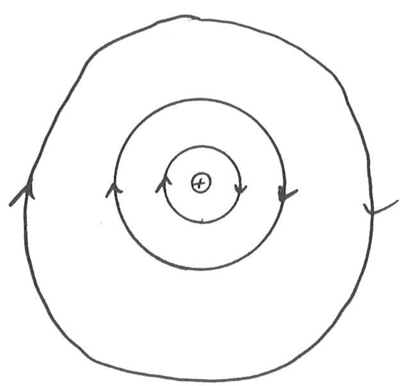
A symmetry argument:



A symmetry argument:



A symmetry argument:

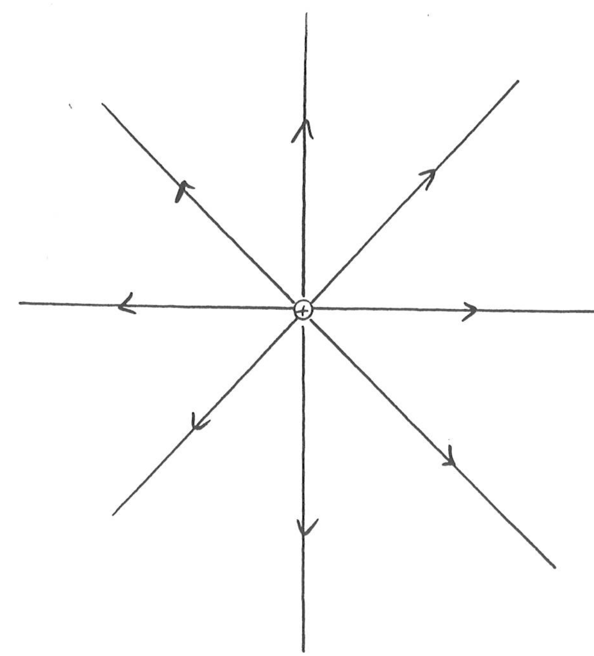


Use a test charge to eliminate this possibility.

The $\vec{E}$ field lines are radially outward.

They are infinitely long.

They are radially symmetric (not bunched up on one side).



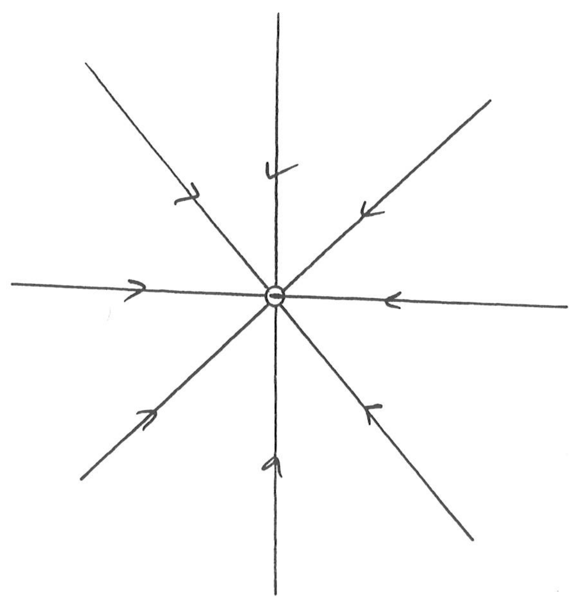
density of lines = strength of $\vec{E}$ field.

How many $\vec{E}$ field lines are there?

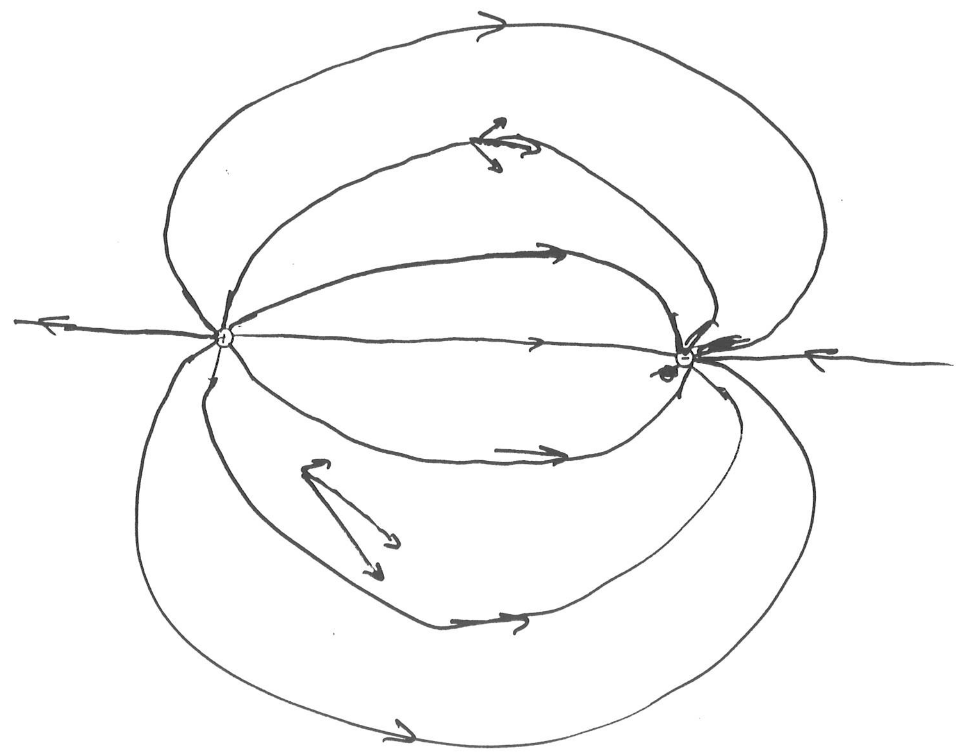
Infinitely many.

Draw as many as you like.

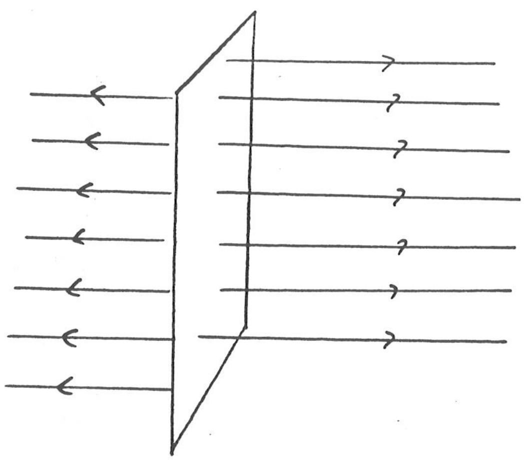
$\vec{E}$ field due to a negative point charge



$\vec{E}$ field for a pair of positive and negative point charges: (Dipole) of the same magnitude



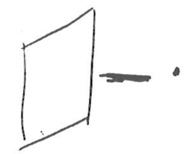
$\vec{E}$ field due to an infinite uniformly charged plate:



uniformly spaced

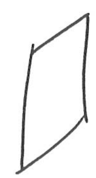
The electric field <u>any</u> distance from an infinite plate is constant.

Symmetry argument:



$|\vec{E}| = 10\,\frac{N}{C}$

Binoculars (the right way)



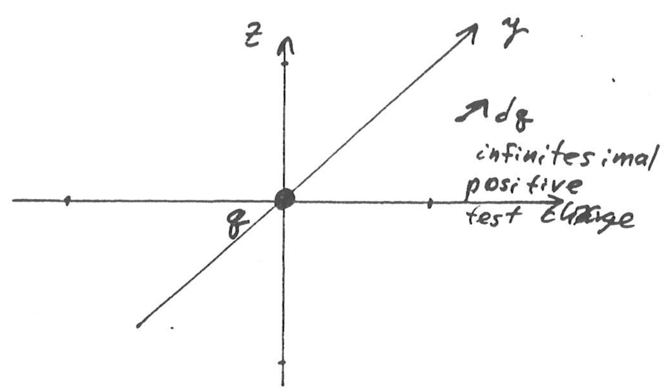
$|\vec{E}| = 10\,\frac{N}{C}$

Binoculars (the wrong way)

$|\vec{E}| = 10\,\frac{N}{C}$

$\vec{E}$ field due to a positive point charge

q at the origin:



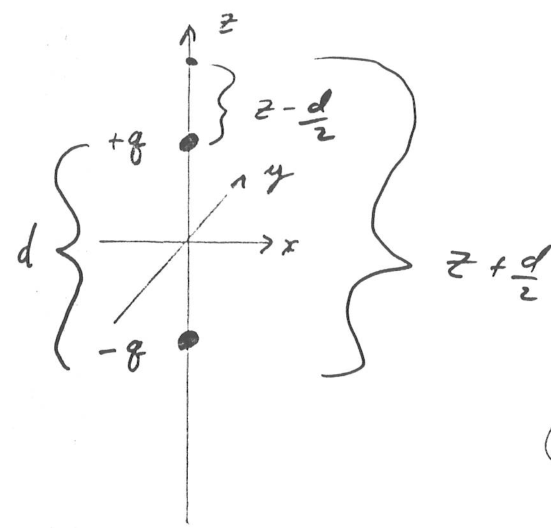
Force:

$$\underset{\substack{\text{on}\\\text{test}\\\text{charge}}}{\vec{F}(\vec{r})} = \frac{1}{4\pi\epsilon_0} \frac{q\,dq}{r^2} \hat{r}$$

Electric Field:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$\vec{E}$ field of an electric dipole (on axis):



$$\left(z - \frac{d}{2}\right)^2 = z^2\left(1 - \frac{d}{2z}\right)$$

For points on the z-axis:

$$E(z) = E_{(+)} + E_{(-)}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{\left(z - \frac{d}{2}\right)^2} + \frac{1}{4\pi\epsilon_0} \frac{-q}{\left(z + \frac{d}{2}\right)^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{z^2}\left[\left(1 - \frac{d}{2z}\right)^{-2} - \left(1 + \frac{d}{2z}\right)^{-2}\right]$$

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$|x| < 1$$

$$\left(1-\frac{d}{2z}\right)^{-2} = 1 + (-2)\left(\frac{-d}{2z}\right) + \cdots$$
$$= 1 + \frac{d}{z} + \cdots$$

$$\left(1+\frac{d}{2z}\right)^{-2} = 1 + (-2)\left(\frac{d}{2z}\right) + \cdots$$
$$= 1 - \frac{d}{z} + \cdots$$

$\vec{E}$ field of an electric dipole on the dipole axis:

$$E(z) = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\left(1-\frac{d}{2z}\right)^{-2} - \left(1+\frac{d}{2z}\right)^{-2}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\left(1+\frac{d}{z}+\cdots\right) - \left(1-\frac{d}{z}+\cdots\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\frac{2d}{z}+\cdots\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{2(qd)}{z^3} + \cdots = \frac{1}{4\pi\epsilon_0}\frac{2p}{z^3} + \cdots$$

First term in an infinite series. This is a good approximation as long as

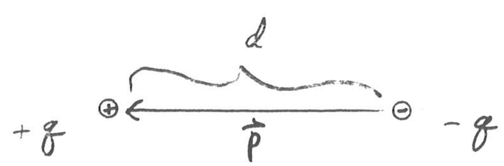
$$z \gg d$$

Electric Dipole Moment

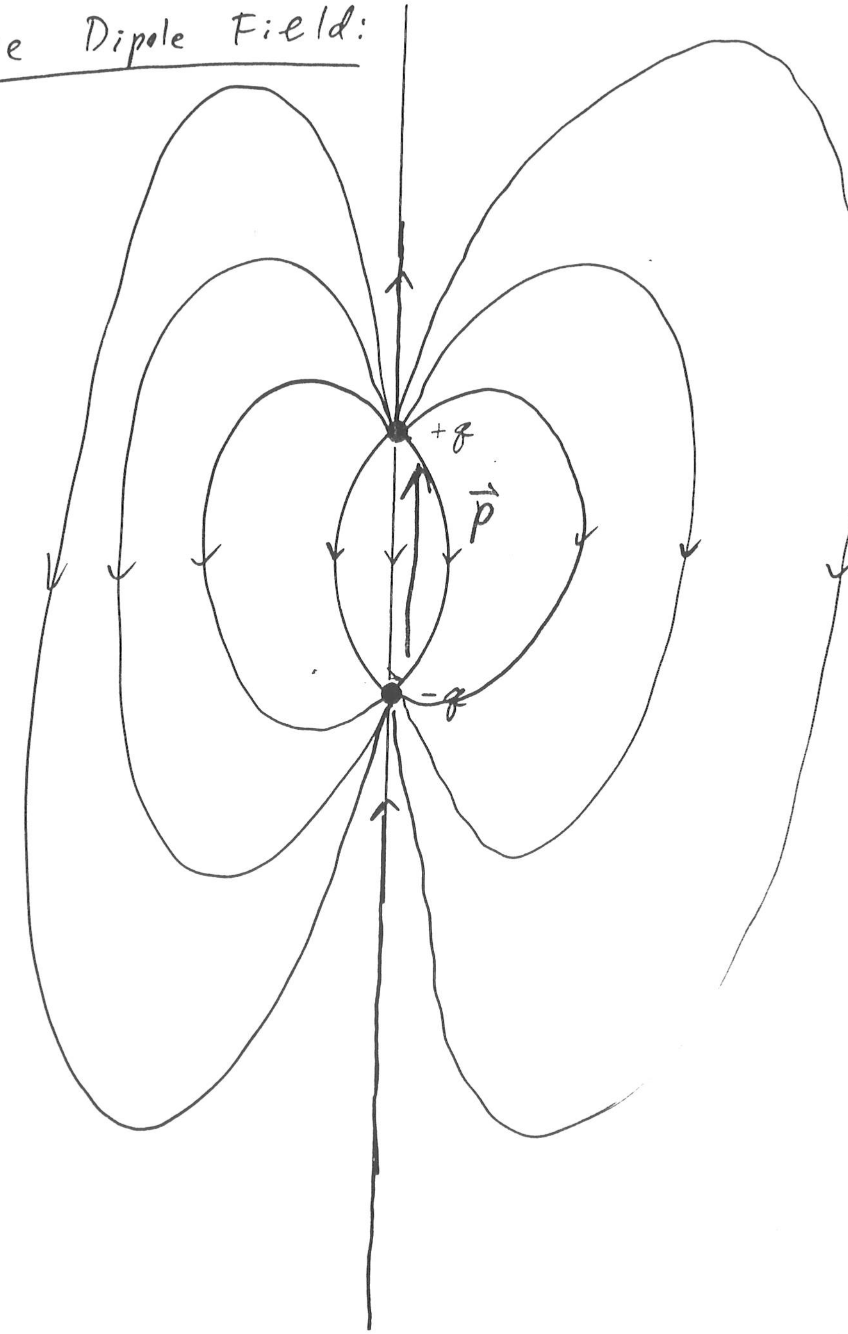
If two charges, each of magnitude q, but of opposite sign are separated by a distance d, that configuration has an electric dipole moment

$$|\vec{p}| = q\,d$$

The direction of $\vec{p}$ is from the negative charge to the positive one, Opposite to the $\vec{E}$ field.



The Dipole Field:



Mechanics

$\vec{F} = m\vec{a}$ Newton's 2nd Law

Electricity + Magnetism

4 Maxwell's Equations

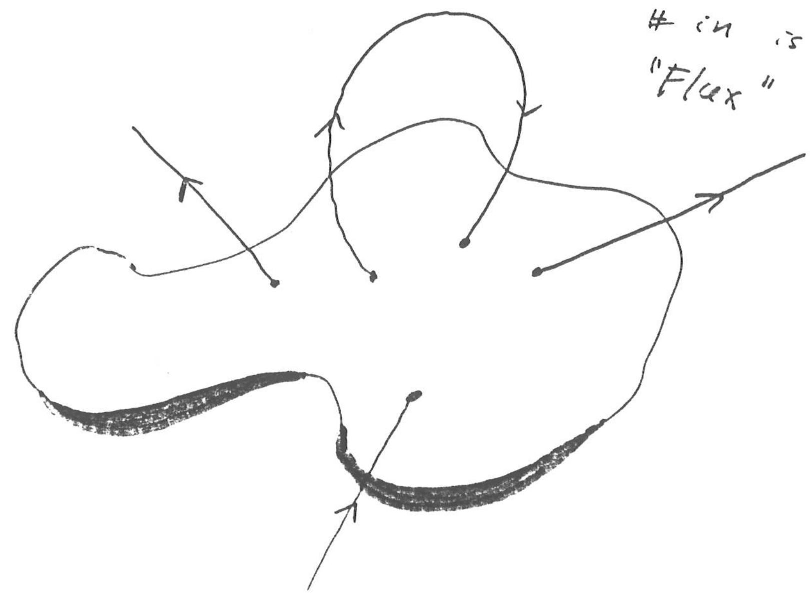
the first one that we will study is

⋮

Gauss' Law

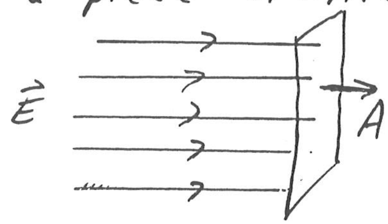
The number of electric field lines poking out through a closed surface minus the number of field lines poking into the surface tells you something about the net charge enclosed within that surface.

\# out − # in is called "Flux"



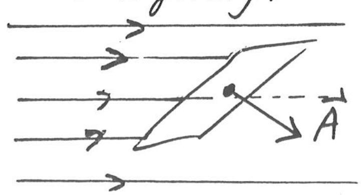
Special Cases

1) Constant $\vec{E}$ field perpendicular to a piece of flat Area:



number of field lines $\propto |\vec{E}|\, A$

2) Constant $\vec{E}$ field striking a piece of flat Area obliquely:



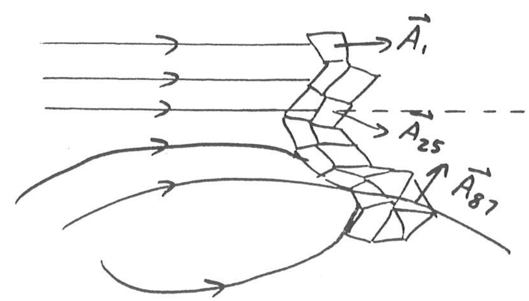
number of field lines <u>out</u> $\propto \vec{E} \cdot \vec{A}$

$= |\vec{E}|\,|\vec{A}| \cos\theta$

$\vec{A}$ points <u>out</u> of the closed surface

What if the electric field is <u>not</u> constant, and the piece of Area is <u>not</u> flat?

Break the surface into tiny regions which are <u>almost</u> flat and over which the $\vec{E}$ field does not vary considerably.



number of field lines out

$$\propto \sum_{i=1}^{N} \vec{E}_i \cdot \vec{A}_i$$

To obtain a more accurate answer, let the size of each piece of Area shrink to zero while increasing the number N of pieces:

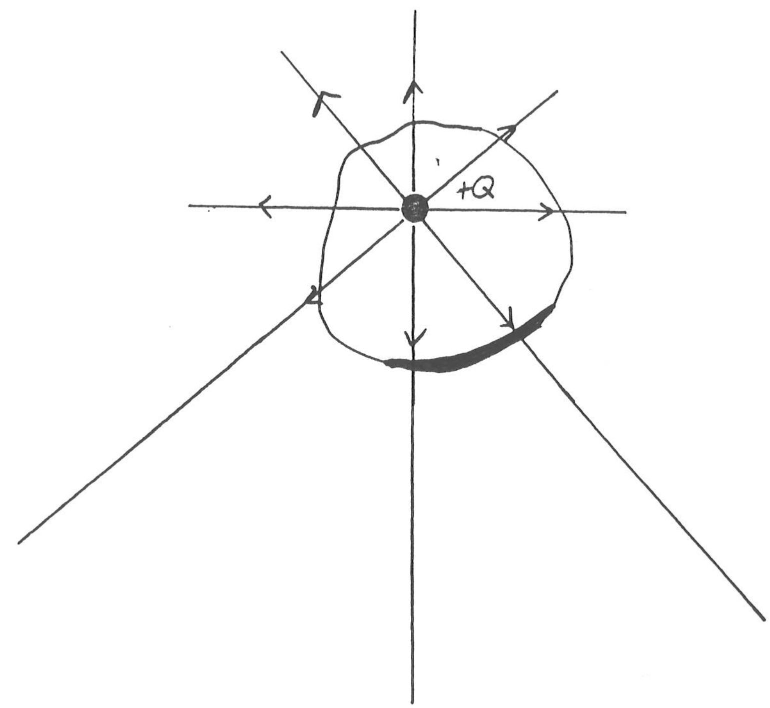
$$\lim_{N \to \infty} \sum_{i=1}^{N} \vec{E}_i \cdot \vec{A}_i = \iint \vec{E}(\vec{r}) \cdot d\vec{A}$$

Gauss' Law

$$\oiint \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

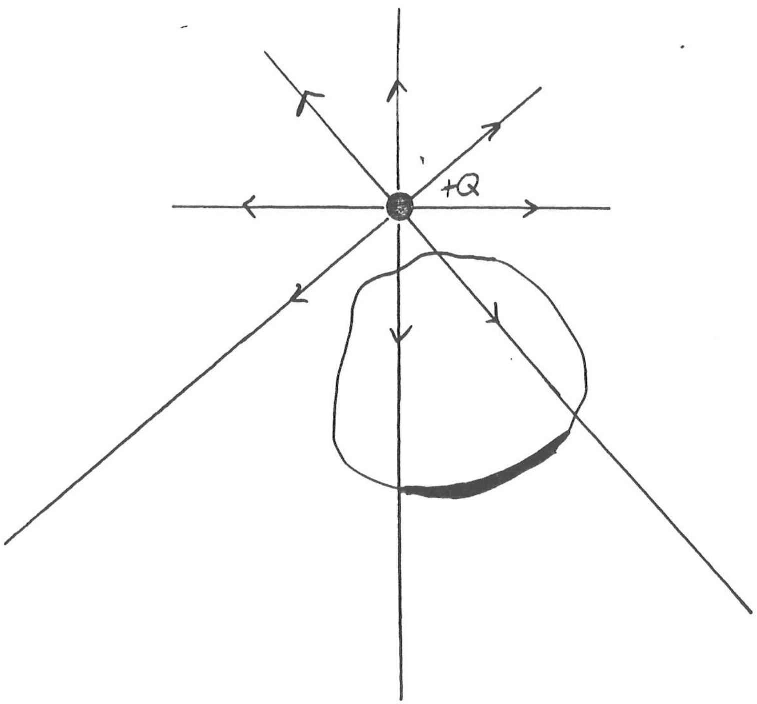
surface is closed

only the charge enclosed by the surface



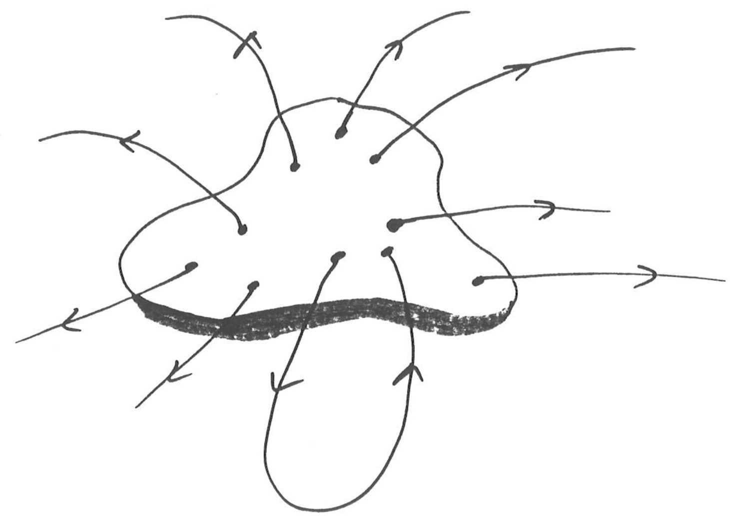
8 lines pierce the surface on their way out.

Suppose that you couldn't see inside the <u>mathematical</u> surface.



2 lines enter,
2 lines leave the surface.

The net flux through the surface is <u>zero</u>.



What is inside?

<u>net</u> charge $+Q$ inside.

could be ($3Q$ and $-2Q$)
or ($47Q$ and $-46Q$)

Gauss' Law: $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$

and Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_{12}$

contain exactly the <u>same</u> physics.

Recall Newton's 2nd Law: $\vec{F} = m\vec{a}$

and Energy Conservation:

$$U_i + K_i = U_f + K_f$$

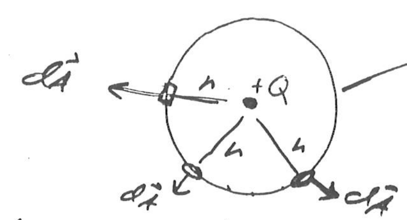
also contain the same physics.

They are different methods for obtaining the same answer. Usually, one is easier than the other for a given problem.

Gauss' Law is easy when the problem has a high degree of <u>symmetry</u>.

Use Gauss' Law to derive the electric field due to a point charge.

radial or spherical symmetry



magnitude of $\vec{E}$ field depends only on r

$d\vec{A}$ points radially outward

$\vec{E}$ is parallel to $d\vec{A}$

$$\oiint \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} = \frac{+Q}{\epsilon_0}$$

$$= \oiint |\vec{E}(r)|\, \hat{r} \cdot d\vec{A}$$

$$= |\vec{E}(r)| \oiint \hat{r} \cdot d\vec{A}$$

$$= |\vec{E}(r)| \oiint dA$$

$$= |\vec{E}(r)|\,(\text{Area})$$

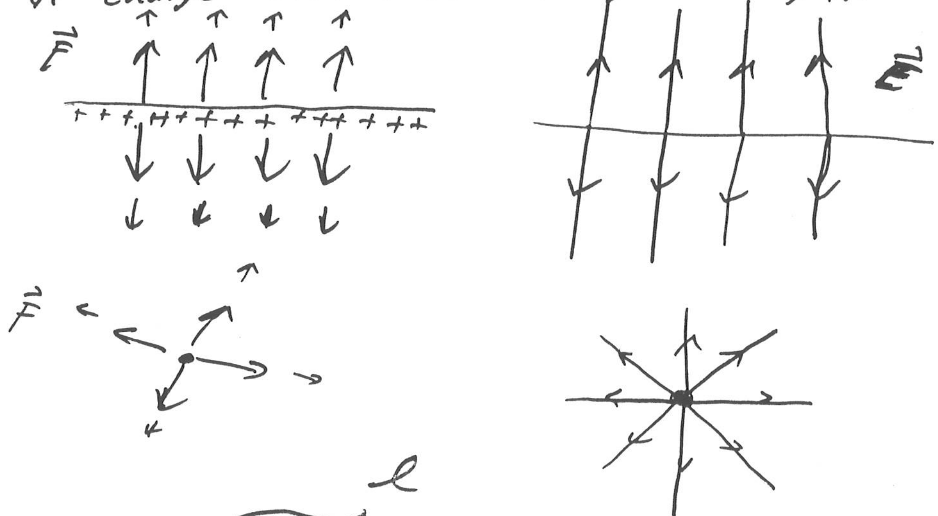
$$= |\vec{E}(r)|\, 4\pi r^2 = \frac{+Q}{\epsilon_0}$$

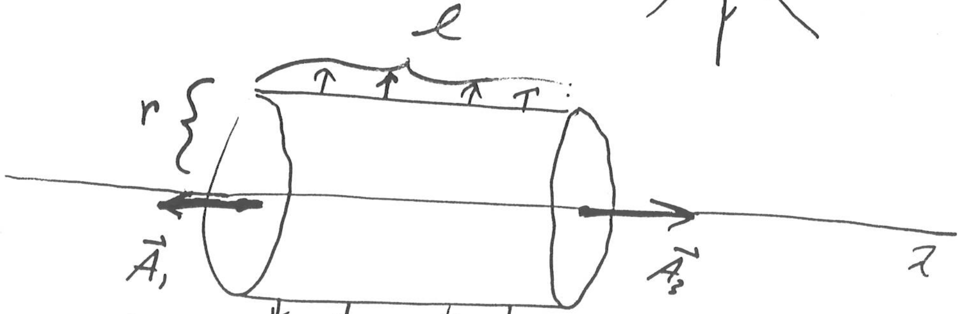
direction by symmetry. $\hat{r}$

$$\therefore \quad |\vec{E}(r)| = \frac{+Q}{4\pi\epsilon_0 r^2}$$

magnitude same as Coulomb's Law

Use Gauss' Law to derive the electric field due to an infinite line of charge with linear charge density λ.





$|\vec{A}_1| = \pi r^2$ $\qquad$ $|\vec{A}_3| = \pi r^2$

$|\vec{A}_2| = 2\pi r \ell$

$\vec{E}(\vec{r}) \cdot \vec{A}_1 = 0 \qquad \vec{E}(\vec{r}) \cdot \vec{A}_3 = 0$

$\vec{E}(\vec{r})$ is parallel to $\vec{A}_2$ at all points

$$\oint\!\!\oint \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$= \iint_{A_1} \cancel{\vec{E} \cdot d\vec{A}_1}^{\,0} + \iint_{A_2} \vec{E} \cdot d\vec{A}_2 + \iint_{A_3} \cancel{\vec{E} \cdot d\vec{A}_3}^{\,0}$$

$$= |\vec{E}_{(r)}| \iint_{A_2} dA_2 \qquad \vec{E} \cdot d\vec{A} = |\vec{E}|\,|dA|$$

$$= |\vec{E}_{(r)}| (2\pi r \ell) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}$$

elec. field a distance r away from line

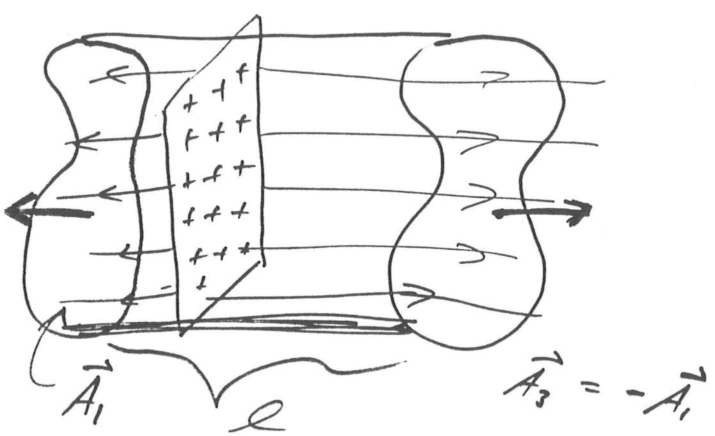
$$|\vec{E}(r)|\, 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\epsilon_0}$$

$$|\vec{E}(r)| = \frac{\lambda}{2\pi\epsilon_0 r} = \boxed{\frac{2\lambda}{4\pi\epsilon_0 r}}$$

direction by symmetry away from wire

same as Coulomb's law from 2 lectures ago.

Use Gauss' Law to derive the electric field due to an infinite sheet of charge with surface charge density $\underline{\sigma}$.



insulator
infinitely thin

$|A_1| = |A_3|$

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$= \iint_{A_1} \vec{E}\cdot d\vec{A}_1 + \underbrace{\iint_{A_2} \vec{E}\cdot d\vec{A}_2}_{0} + \iint_{A_3} \vec{E}\cdot d\vec{A}_3$$

$$\iint_{A_1} |E|\,dA_1 + 0 + \iint_{A_3} |E|\,dA_3$$

$$|E| \iint_{A_1} dA_1 + 0 + |E| \iint_{A_3} dA_3$$

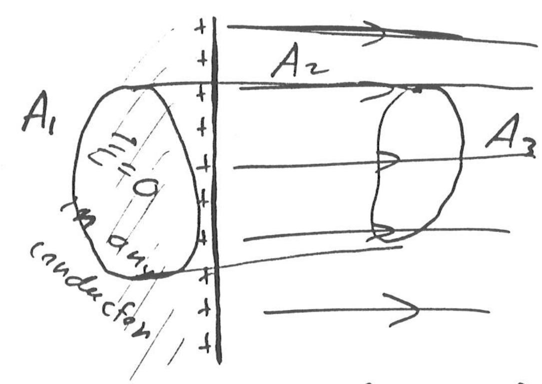
$$|E|\,A_1 + |E|\,A_3$$

$$2|E|\cancel{A_1} = \frac{\sigma \cancel{A_1}}{\epsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\sigma}{2\epsilon_0}}$$

at $\underline{\text{any}}$ distance away

Find the electric field due to a surface charge density σ on one side of an infinite <u>conducting</u> sheet.



$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$\underbrace{\iint_{A_1} \vec{E}\cdot dA_1}_{0} + \underbrace{\iint_{A_2} \vec{E}\cdot d\vec{A}_2}_{0} + \iint_{A_3} \vec{E}\cdot d\vec{A}_3$$

since $\vec{E} = 0$ $\qquad$ $\vec{E} \perp d\vec{A}_2$

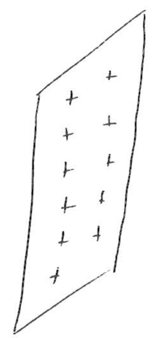
$$|E| \iint_{A_3} dA_3$$

$$|\vec{E}|\cancel{A_3} = \frac{\sigma \cancel{A_3}}{\epsilon_0}$$

$$\boxed{|\vec{E}| = \frac{\sigma}{\epsilon_0}}$$

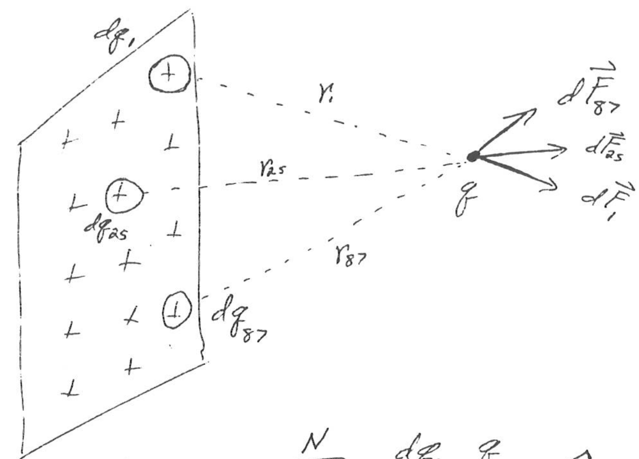
at $\underline{\text{any}}$ distance away

To see that Gauss' Law is easier than Coulomb's law for the case of an infinite sheet of charge, we will set up the Coulomb's law calculation (but not solve it!)



1) Find the total vector force $\vec{F}$ acting on the test charge q.

2) Divide by q to get the electric field $\vec{E}$ (force per unit charge).



$$\vec{F}_{\substack{total\\ on\ q}} = \sum_{i=1}^{N} \frac{dq_i\, q}{(r_i)^2}\,\hat{r}_i$$

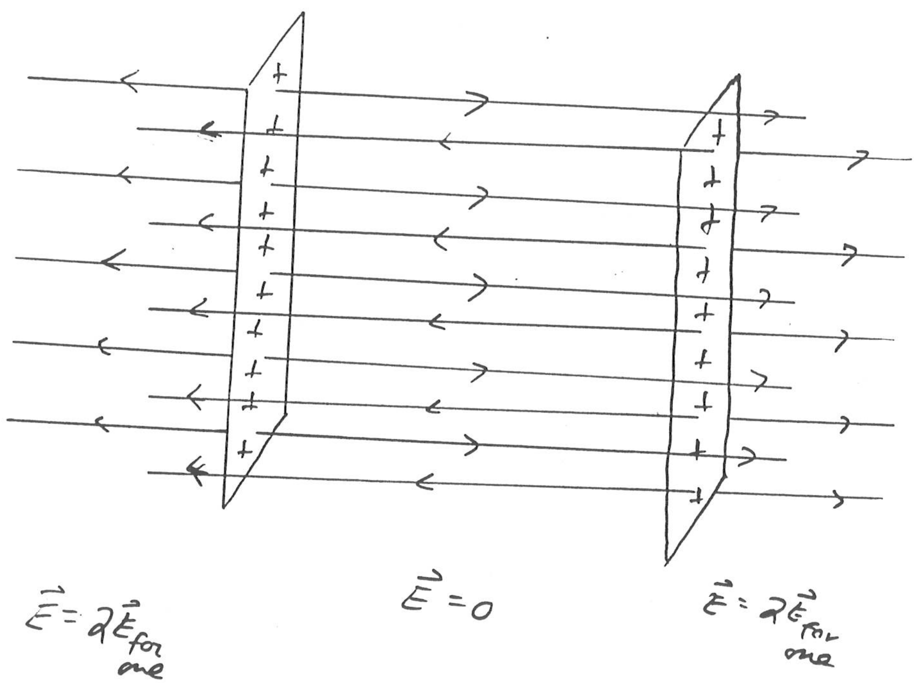
$$= q\sum_{i=1}^{N} \frac{dq_i}{(r_i)^2}\,\hat{r}_i$$

$$\xrightarrow[N\to\infty]{} q\iint \frac{dq}{r^2}\,\hat{r} \qquad \boxed{dq = \sigma\, dx\, dy}$$

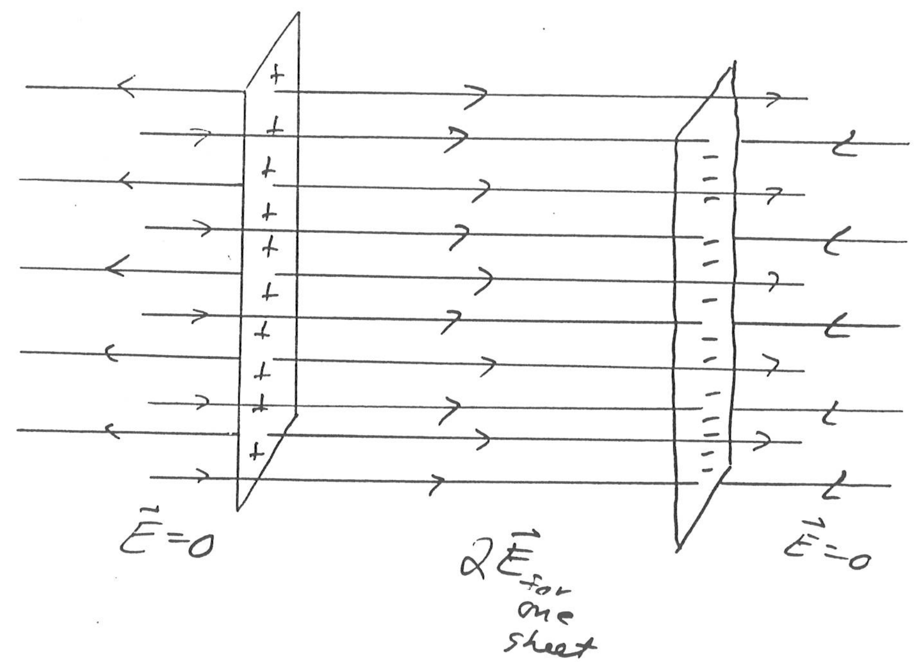
$$= q\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \frac{\sigma\, dx\, dy}{r^2}\,\hat{r}$$

Obviously Gauss' Law is much easier!

The electric field due to many sheets of charge. Superposition

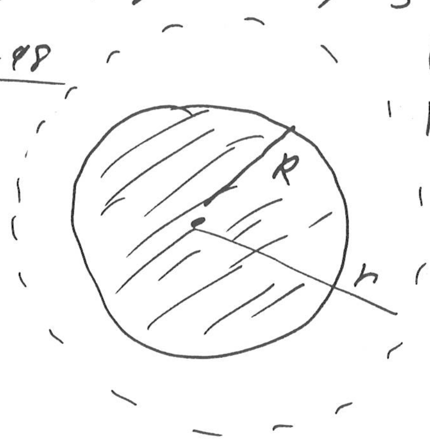


The electric field due to many sheets of charge. Superposition



Find the electric field due to a uniform ball of charge with radius R and volume charge density ρ.

P25-98



$r > R$

$r < R$

$r > R$

Gauss' Law $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$

by symmetry $\vec{E}$ field is radial

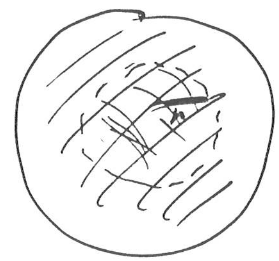
$$|\vec{E}(r)| \oiint dA = \frac{\rho \text{ Volume } \text{vol(sphere)}}{\epsilon_0}$$

$$E(r)\, 4\pi r^2 = \frac{\rho \frac{4}{3}\pi R^3}{\epsilon_0}$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \frac{\rho \frac{4}{3}\pi R^3}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{Q_{total}}{r^2}$$

$r < R$



Gauss

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

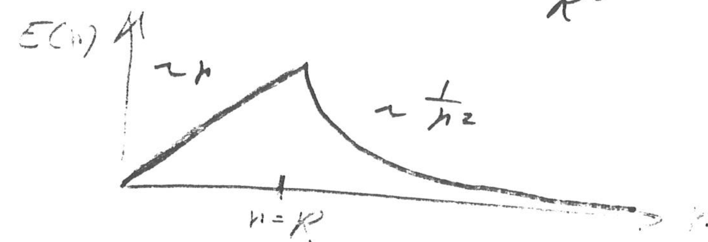
$$E \oiint dA = \frac{\rho \frac{4}{3}\pi r^3}{\epsilon_0}$$

$$E\, 4\pi r^2$$

$$E(r) = \frac{1}{4\pi\epsilon_0} \quad \rho \tfrac{4}{3}\pi r$$

$$\rho = \frac{Q_{total}}{\text{Volume of } \text{vol}} = \frac{Q_{total}}{\frac{4}{3}\pi R^3}$$

$$E(r) = \frac{1}{4\pi\epsilon_0} Q_{total} \frac{r}{R^3}$$



Electric Potential Energy

Recall from Mechanics:

Gravitational Potential Energy

$$\Delta U_{grav} \equiv -W_{grav\,i\to f} = -\int_i^f \vec{F}_{grav}\cdot d\vec{s}$$

change in P.E. (unique)

Work done <u>by</u> the force of gravity <u>on</u> some object with mass.

$$U_{grav}(\vec{r}) = -\int \vec{F}_{grav}\cdot d\vec{s}$$

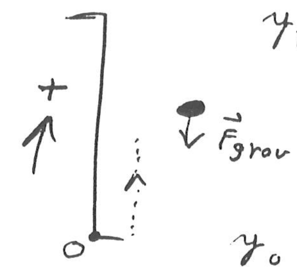
indefinite integral

Potential Energy Function

(defined up to an arbitrary constant of integration)

Consequence: The zero of $U_{grav}(\vec{r})$ can be chosen arbitrarily.

<u>Ex.</u> Find ΔU_{grav} for a stone of mass <u>m</u> lifted up to the top of a tower of height <u>h</u>.



$y_f = h$ $\qquad \vec{F}_{grav} = mg(-\hat{j})$

$d\vec{s} = dy\,(\hat{j})$

$y_0 = 0$ $\qquad \hat{j}\cdot\hat{j} = 1$

$$W_{grav} = \int_i^f \vec{F}_{grav}\cdot d\vec{s} = \int_{y=0}^{h} mg(-\hat{j})\,dy\,\hat{j}$$

$$= \int_{y=0}^{h} -mg\,dy = -mgy\Big|_0^h = -mgh$$

$$\Delta U_{grav} = -W_{grav} = +mgh$$

$$U_{grav}(y) = -\int \vec{F}_{grav}\cdot d\vec{s}$$

$$= \int +mg\,dy = +mgy + \text{constant of int.}$$

Electric Potential Energy

$$\Delta U_{elec} \equiv - W_{elec\, i \to f} = - \int_i^f \vec{F}_{elec} \cdot d\vec{s}$$

unique

Work done by the Coulomb force on some object with charge.

$$U_{elec}(\vec{r}) = - \int \vec{F}_{elec} \cdot d\vec{s}$$

The zero of $U_{elec}(\vec{r})$ can be chosen arbitrarily.

The usual choice is:

$$\boxed{U_{elec}(\vec{r}) = 0 \quad \text{at} \quad |\vec{r}| = \infty}$$

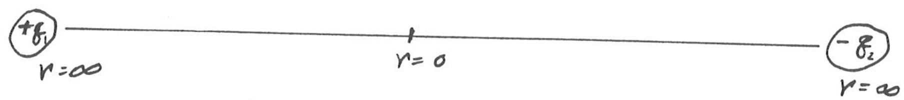
Note: Both the force of gravity and the Coulomb force are CONSERVATIVE.
The work W and the potential energy are independent of the path from i to f.

The electric potential energy of a configuration of charges with the choice $U(\infty) = 0$ is the negative of the work done on the charges by the electric field as the charges are brought together from infinitely far apart.

NOT the work that you do to assemble them!

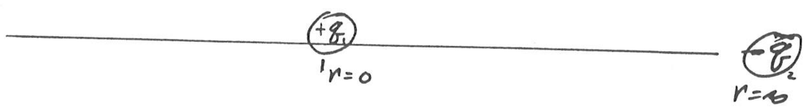
This is the electric field due to the charges already assembled, not including the one you are bringing in from infinity.

Ex. What is the potential energy of two charges $(+q_1)$ and $(-q_2)$ separated by distance $\underline{d}$?

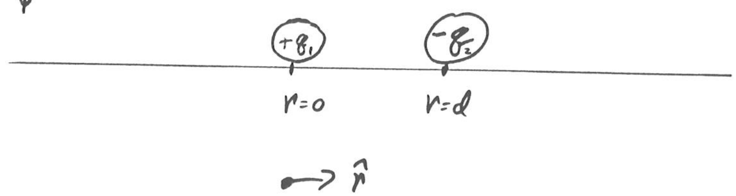


work done by field?

$W_{elec} = 0$ no charges assembled yet



work done by field?



$$W_{elec} = \int_{r=\infty}^{d} \underbrace{\vec{F}_{elec}}_{\text{on } -q_2 \text{ due to } +q_1} \cdot d\vec{s}$$

choose a simple path — straight line

$$= \int_{r=\infty}^{d} \left[\frac{1}{4\pi\epsilon_0} \frac{(+q_1)(-q_2)}{r^2} (\hat{r}) \right] \cdot \left[dr\, \hat{r} \right]$$

$\hat{r} \cdot \hat{r} = 1$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \int_{r=\infty}^{d} \left(-\frac{1}{r^2} \right) dr$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \left[+\frac{1}{r} \right]_{\infty}^{d}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0} \left[\frac{1}{d} - \cancel{\frac{1}{\infty}}^{\,0} \right] = \frac{q_1 q_2}{4\pi\epsilon_0 d}$$

This work is positive.

$$U_{elec} = -W_{elec} = \frac{-q_1 q_2}{4\pi\epsilon_0 d} = \frac{(+q_1)(-q_2)}{4\pi\epsilon_0 d}$$

Electric Potential

Voltage

$$V_{(\vec{r})} \equiv \frac{U_{elec}(\vec{r})}{q}$$

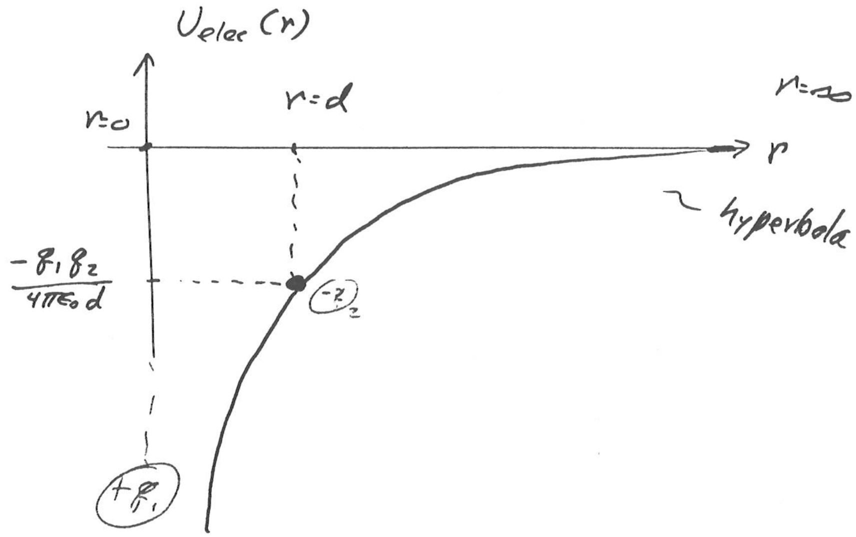
same arbitrariness in choosing the zero.

Electric Potential is Electric Potential Energy per unit charge.

$$\Delta U_{elec} = -\int_i^f \vec{F}_{elec} \cdot d\vec{s}$$

$$\Delta V = \frac{\Delta U_{elec}}{q} = -\int_i^f \frac{\vec{F}_{elec}}{q} \cdot d\vec{s} = -\int_i^f \vec{E} \cdot d\vec{s}$$

Electric field is Electric Force per unit charge



Think of this as a hill on a golf course. All systems seek to lower their potential energy as much as possible. How can charges (+q) and (−q) decrease their potential energy?

Ex What is the electric potential $\underline{V}$ a distance $\underline{d}$ away from a point charge $(+q_1)$ with the choice $V=0$ at infinity?

$$U_{\substack{elec \\ +q_1 \leftrightarrow -q_2}} = \frac{(+q_1)(-q_2)}{4\pi\epsilon_0 d}$$

$$V_{\substack{due \\ to\ +q_1}} = \frac{U_{elec}}{(-q_2)} = \frac{+q_1}{4\pi\epsilon_0 d}$$

$$V_{\substack{due\ to \\ -q_2}} = \frac{U_{elec}}{+q_1} = \frac{-q_2}{4\pi\epsilon_0 d}$$

$$(\text{Voltage})(\text{Charge}) = \text{Energy}$$

$$1\,V \quad 1\,C = 1\,J$$

$$1\,V = 1\,\frac{J}{C}$$

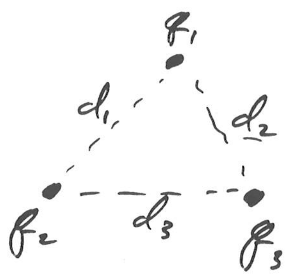
$$(1\,V)(1\,e) = 1\,eV$$

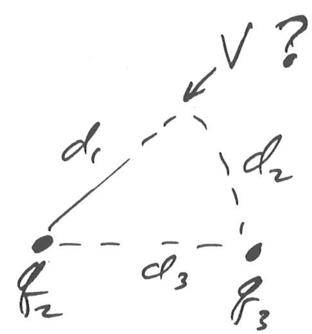
$$= 1 \text{ electron-volt}$$

$$= (1\,V)(1.6\times10^{-19}\,C)$$

$$= 1.6\times10^{-19}\,J = 1\,eV$$

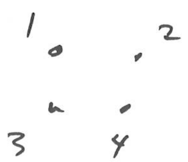
Find U_{elec} of





$$U_{elec} = \frac{q_1 q_2}{4\pi\varepsilon_0 d_1} + \frac{q_2 q_3}{4\pi\varepsilon_0 d_3} + \frac{q_1 q_3}{4\pi\varepsilon_0 d_2}$$

with the choice $U(\infty) \to 0$



\# terms $= 6$

1-2 2-3 3-4
1-3 2-4
1-4

\# terms $= \frac{n(n-1)}{2}$

$$\Delta V_{1\to 2} = -\int_1^2 \vec{E}\cdot d\vec{s}$$

$$E_x = -\frac{\partial}{\partial x} V$$

$$E_y = -\frac{\partial}{\partial y} V$$

$$E_z = -\frac{\partial}{\partial z} V$$

$$V_{(x,y,z)} = 2x^2y + yz$$

$$E_x = -\frac{\partial}{\partial x}(2x^2y + yz) = -(4xy + 0)$$

$$E_y = -\frac{\partial}{\partial y}(2x^2y + yz) = -(2x^2 + z)$$

$$E_z = -\frac{\partial}{\partial z}(2x^2y + yz) = -(0 + y)$$

Capacitance

What is a capacitor?

A device that stores energy and discharges it as we need it.

How do we make one?

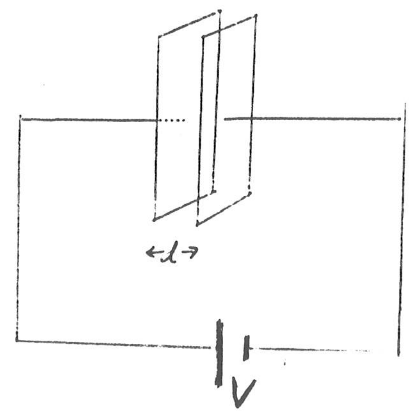
Two isolated <u>conductors</u> of any size and shape will work

The conductors are called "plates" even if they are not plate-like.

A special case:

The Parallel-Plate Capacitor

This is constructed from two conducting sheets, each of area A. The plates are parallel to each other and separated by distance d.



Now connect the capacitor to a battery.

Voltage difference between plates $= V$
Net charge on capacitor $= 0$
Charge on each plate $= +Q, -Q$

The charge on each plate is proportional to the applied voltage:

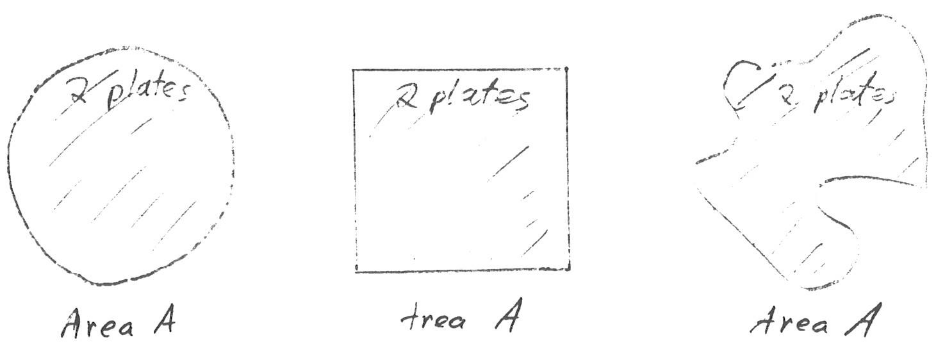
$$\boxed{q = CV}$$

The constant of proportionality is called the capacitance.

The capacitance, C, depends only on the <u>geometry</u> of the plates.

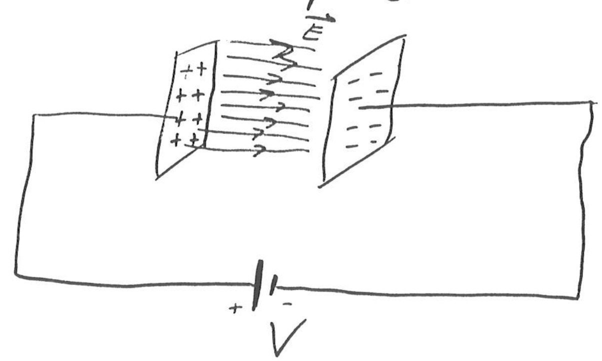
$\left.\begin{array}{l}\text{- their area} \\ \text{- their separation}\end{array}\right\}$ For parallel-plate, just <u>A</u> and <u>d</u>!

- their shape (if not parallel-plate)



What happens while a capacitor is charging?

Electrons are being <u>removed</u> from one plate and are being <u>deposited</u> on the other plate.



Eventually, the electric field that is created between the charged plates prevents any more charge from accumulating on the plates.

When this occurs, the potential difference between the plates is equal to V, the battery voltage.

Units

The MKS unit of capacitance is the Farad (F).

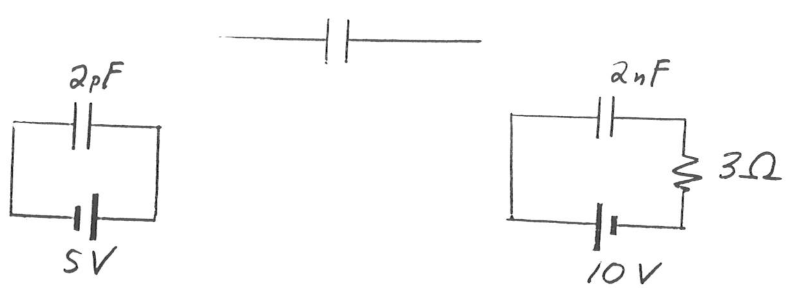
$$1 \text{ Farad} = 1 \frac{\text{Coulomb}}{\text{Volt}} \quad , \quad 1\,F = 1\,\frac{C}{V}$$

I Farad is a <u>huge</u> capacitance!

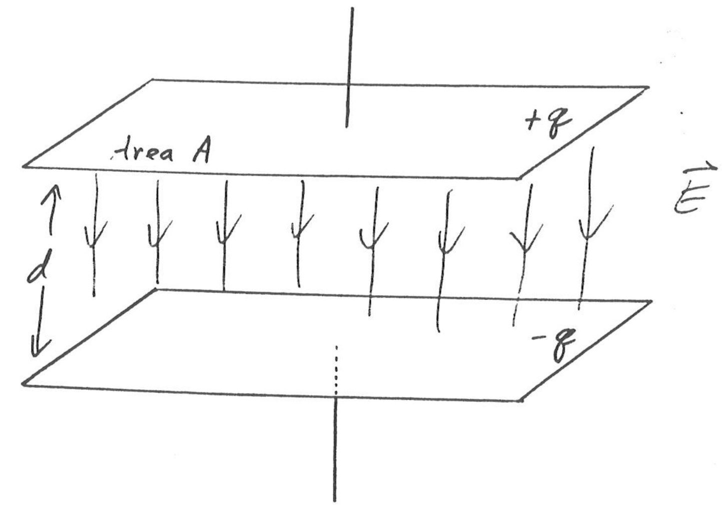
$mF = 10^{-3}\,F$
$\mu F = 10^{-6}\,F$
$nF = 10^{-9}\,F$
$pF = 10^{-12}\,F$

Most devices use pF capacitors.

Graphical Symbol



What is the electric field between the parallel plates of a capacitor?

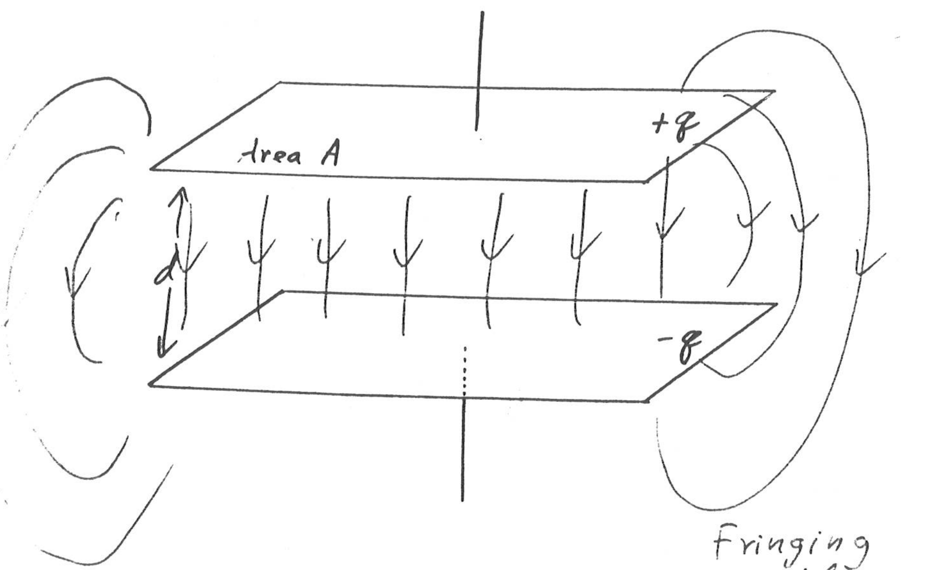


What does the electric field look like?

$|\vec{E}|$ = constant between the plates,

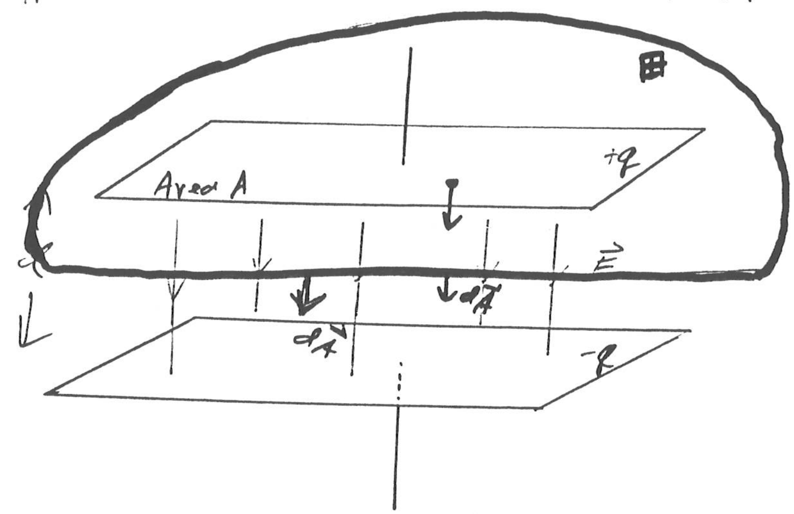
$\vec{E} = 0$ outside the plates.

What is the electric field between the parallel plates of a capacitor?



What does the electric field look like?

How do we calculate the $\vec{E}$ field?



Choose a Gaussian surface around one of the plates. The surface shown here is a plane between the plates, and some arbitrary shape outside.

Gauss' Law: $\epsilon_0 \oint \vec{E} \cdot d\vec{A} = Q_{inside} = +q$

$$\left.\begin{array}{l} \vec{E} \parallel d\vec{A} \\ |\vec{E}| = const \end{array}\right\} \quad \begin{array}{l} \epsilon_0 E A = +q \\ E = \dfrac{q}{\epsilon_0 A} \end{array}$$

What is the <u>voltage</u> in terms of the electric field?

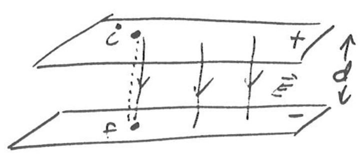
$$\Delta V = V_f - V_i = -\int_i^f \vec{E} \cdot d\vec{s}$$

i = some point on the + plate

f = some point on the − plate

We will choose the path of the line integral to follow an $\vec{E}$ field line.

Then $d\vec{s} \parallel \vec{E}$



The "V" in $q = CV$ is the absolute value of the potential difference ΔV.

$$V = |\Delta V| = \int_+^- \vec{E} \cdot d\vec{s} = \int_+^- E\,ds = Ed$$

$$E = \frac{q}{\epsilon_0 A} \qquad V = Ed$$

$$V = \frac{qd}{\epsilon_0 A} \quad \text{or} \quad q = \left(\frac{\epsilon_0 A}{d}\right) V$$

But $q = CV$

therefore $C_{\text{parallel plate}} = \dfrac{\epsilon_0 A}{d}$

geometry A, d

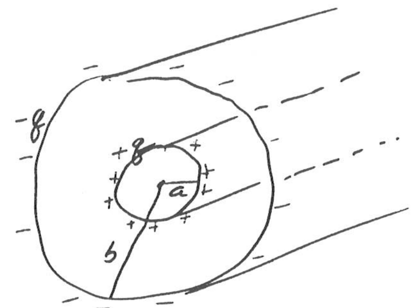
Ex. What is the area of a 1 Farad capacitor made from two parallel plates separated by 1 mm?

$$C_{\text{parallel plate}} = \frac{\epsilon_0 A}{d} \qquad A = \frac{Cd}{\epsilon_0}$$

$$A = \frac{(1\ \text{Farad})(1\ \text{mm})\left(\frac{1\ \text{m}}{1000\ \text{mm}}\right)}{8.85\times 10^{-12}\ \text{F/m}} = 1.13\times 10^{8}\ \text{m}^2$$

$$A = 10\ \text{km} \times 10\ \text{km}$$

Cylindrical Capacitor

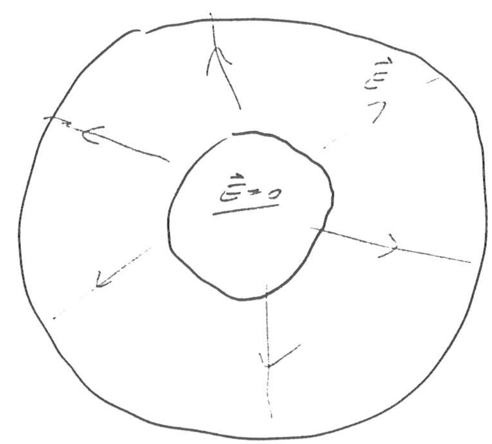


Two coaxial cylinders of radii a and b.

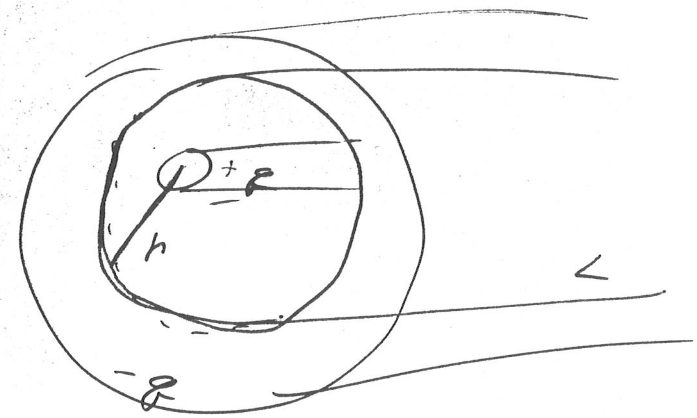
We assume the length L is large compared to a or b.

What is the electric field everywhere?

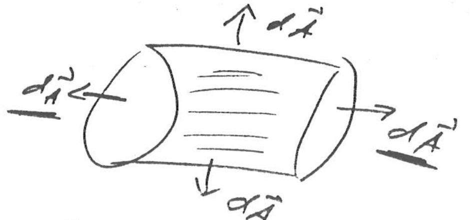
1. Find $\vec{E}$
2. Find ΔV
3. Find C



$\vec{E}=0$



Gauss' Law $\quad \epsilon_0 \oiint \vec{E}\cdot d\vec{A} = Q_{inside} = +q$



$\epsilon_0 E(r)\, 2\pi r L = +q$

$\iint_{lateral} d\vec{A} = 2\pi r L$

$$\boxed{E(r) = \frac{q}{2\pi r L \epsilon_0}}$$

$\Delta V = -\int_+^- \vec{E}\cdot d\vec{s}$ $\quad$ choose path to be a radius from $\underline{a}$ to $\underline{b}$.

$$V = \int_a^b \frac{q}{2\pi r L\epsilon_0}\, dr = \frac{q}{2\pi L\epsilon_0}\int_a^b \frac{1}{r}\, dr$$

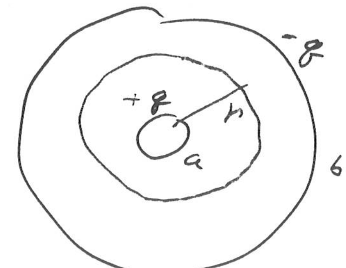
$$\boxed{V = \frac{q}{2\pi L\epsilon_0} \ln\left(\frac{b}{a}\right)}$$

$\ln(b) - \ln(a) = \ln\left(\frac{b}{a}\right)$

$q = CV \qquad C = \frac{q}{V}$

$$\underline{C_{cylind}} = \frac{q}{\frac{q}{2\pi L\epsilon_0}\ln\left(\frac{b}{a}\right)} = \frac{2\pi L\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

Spherical Capacitor.



1) find $E(r)$

$\epsilon_0 \oiint \vec{E}\cdot d\vec{A} = +q$

$\epsilon_0 E(r)\, 4\pi r^2 = +q$

$\oiint d\vec{A} = 4\pi r^2$ $\qquad E(r) = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}$

$\underline{a < r < b}$

2) find ΔV

$$\Delta V = -\int_+^- \vec{E}\cdot d\vec{s} = -\int_a^b \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\, dr$$

If $\vec{E} \parallel d\vec{s}$, then $\vec{E}\cdot d\vec{s} = |E|\,|ds|$

$$V = |\Delta V| = \int_a^b \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\,dr$$

$$\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r}\right)\Bigg|_a^b = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$$q = CV$$

$$C = \frac{q}{V} = \frac{q}{\frac{q}{4\pi\epsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]} =$$

$$C_{spherical} = 4\pi\epsilon_0 \frac{1}{\frac{1}{a}-\frac{1}{b}}$$

$$= 4\pi\epsilon_0\left(\frac{ab}{b-a}\right)$$

Capacitance for a <u>single</u> sphere

$b \to \infty$

$$C_{\substack{single\\sphere\\radius = a}} = \underline{\underline{4\pi\epsilon_0 a}}$$

Capacitance of a head:

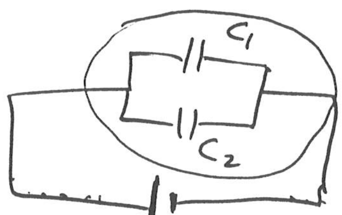
$$C_{\substack{single\\sphere}} = 4\pi\epsilon_0 r$$

$$= 4\pi\left(8.85\times10^{-12}\,\frac{F}{m}\right)(0.15\,m)$$

$$= 16.7\times10^{-12}\,F = 16.7\,pF$$

Equivalent Capacitance

Parallel



Potential across $C_1 = V$
Potential across $C_2 = V$

$q_1 = C_1 V$ $\quad$ $q_2 = C_2 V$

$q_{total} = q_1 + q_2 = C_{equiv} V$

$C_1 V + C_2 V = C_{equiv} V$

$C_{equiv\ parallel} = C_1 + C_2$

In Series

$q_1 = q_2 = q$ $\quad$ $q_1 = C_1 V_1$ $\quad$ $q_2 = C_2 V_2$

$V_1 = \frac{q}{C_1}$ $\quad$ $V_2 = \frac{q}{C_2}$

Total Potential drop across circuit

$V = V_1 + V_2 = \frac{q}{C_1} + \frac{q}{C_2} = q\left(\frac{1}{C_1} + \frac{1}{C_2}\right)$

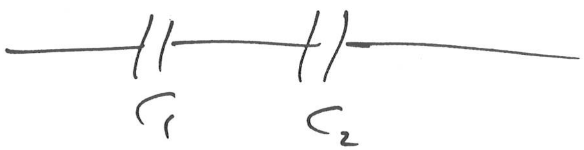
$q = C_{equiv} V$

$q = \left[\left(\frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}\right)\right] V$

$C_{equiv\ series} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} = \frac{C_1 C_2}{C_1 + C_2}$

$C_1 = 1\,nF$

$C_2 = 1\,nF$



$$C_{\substack{equiv\\series}} = \frac{C_1 C_2}{C_1 + C_2} = \frac{(10^{-9}F)(10^{-9}F)}{10^{-9}F + 10^{-9}F}$$

$$= \frac{10^{-18}F^2}{2 \times 10^{-9}F} = \frac{1}{2} \times 10^{-9}F$$

$$= 0.5\,nF$$

Energy Stored in a Capacitor

It takes <u>energy</u> to separate the charges on the plates of a capacitor.

Suppose that I have already transferred charge q from one plate to the other

And suppose that the potential difference (the voltage) is V.

$$V = \frac{q}{C}$$

more charge $\Rightarrow$ higher voltage

How much work must be done on an infinitesimal piece of charge dq to move it from one plate to the other?

$$dW = V\,dq = \left(\frac{q}{C}\,dq\right)$$

The total work required to charge the capacitor from $q = 0$ to $q = Q$ is:

$$W = \int dW = \int_{q=0}^{Q} \left(\frac{q}{C}\,dq\right) = \boxed{\frac{q^2}{2C}}$$

$$q = CV$$

$$\boxed{W = \frac{(CV)^2}{2C} = \frac{1}{2} C V^2}$$

energy stored on a capacitor

Faraday Cage

Conductors are equipotentials

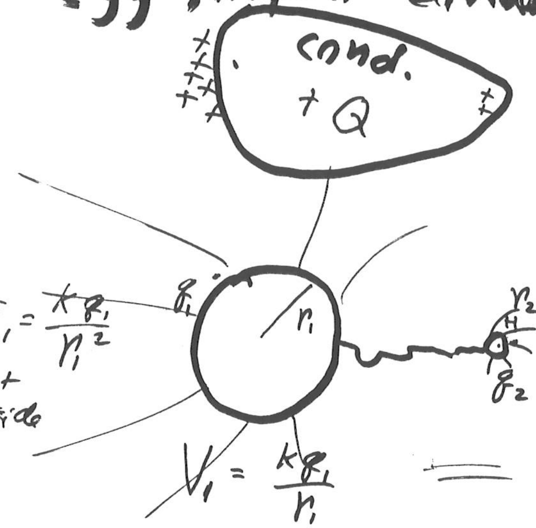
$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{s}$$

$$0 =$$

if path is arbitrary, then $\vec{E} = 0$ every where inside

Egg shaped conductor

Point charge $V = \frac{kQ}{r}$



$E_1 = \frac{kq_1}{r_1^2}$ right outside

$E_2 = \frac{kq_2}{r_2^2}$

$V_1 = \frac{kq_1}{r_1}$ $=$ $V_2 = \frac{kq_2}{r_2}$

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} \Longrightarrow \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

since $r_1 > r_2$ then $q_1 > q_2$

$$\frac{E_1}{E_2} = \frac{\frac{kq_1}{r_1^2}}{\frac{kq_2}{r_2^2}} = \frac{q_1 r_2^2}{q_2 r_1^2} = \frac{r_2}{r_1}$$