Plane Wave solutions to Maxwell's Equations:

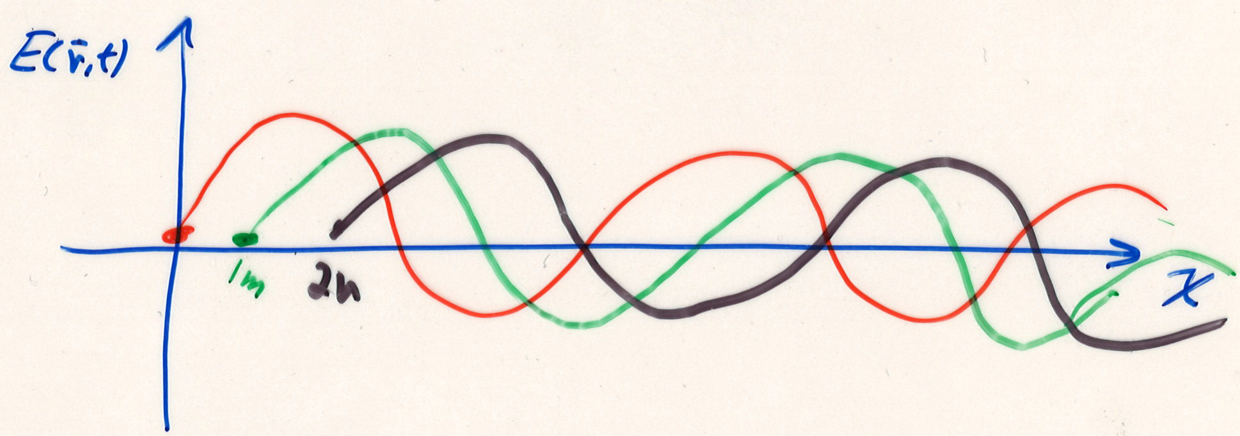
Suppose the wave is travelling along the x-axis from $-\infty$ to $+\infty$.

The function

$$E(\vec{r},t) = E_{max} \sin\left[\omega\left(\frac{x}{c} - t\right)\right]$$

satisfies Maxwell's Eq's in free space and describes a wave moving along the x-axis with speed c.

$t=0$ $\quad \vec{E} = E_{max} \sin\left(\frac{\omega x}{c}\right)$



$t = \frac{1m}{c} = \frac{1m}{3\times10^8\ m/s} = \frac{1}{3}\times10^{-8}\,s$

$\left(\frac{x}{c} - t\right) = 0$ when $x = 1m$

Notice that there is no y or z dependence in $E(\vec{r},t)$.

- What does this look like?

- What about the magnetic field?

- $\vec{E}(\vec{r},t)$ and $\vec{B}(\vec{r},t)$ are <u>vector</u> fields.

  They are perpendicular to each other <u>and</u> to the direction of propagation.

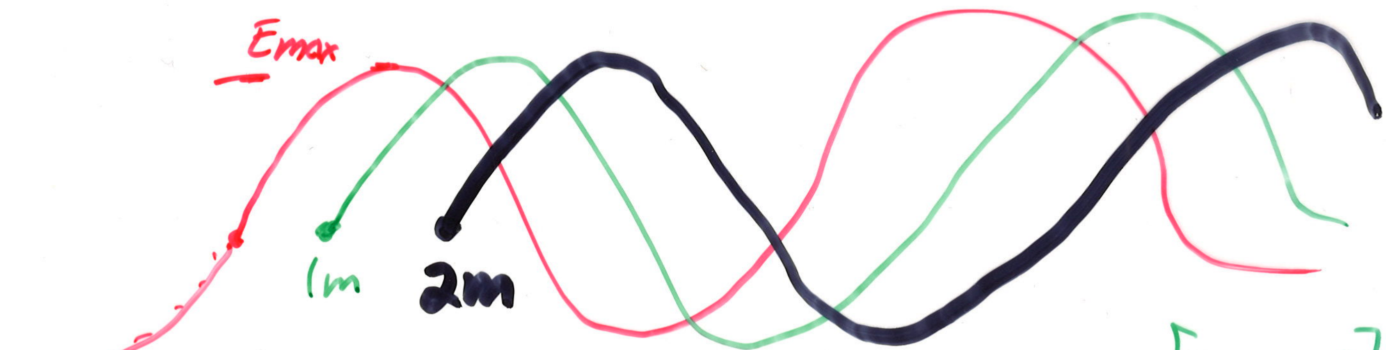
<u>For example:</u> $\vec{E}$ along y

$\vec{B}$ along z

motion along x

$E(x) = E_{max} \sin\left(\frac{\omega x}{c}\right)$

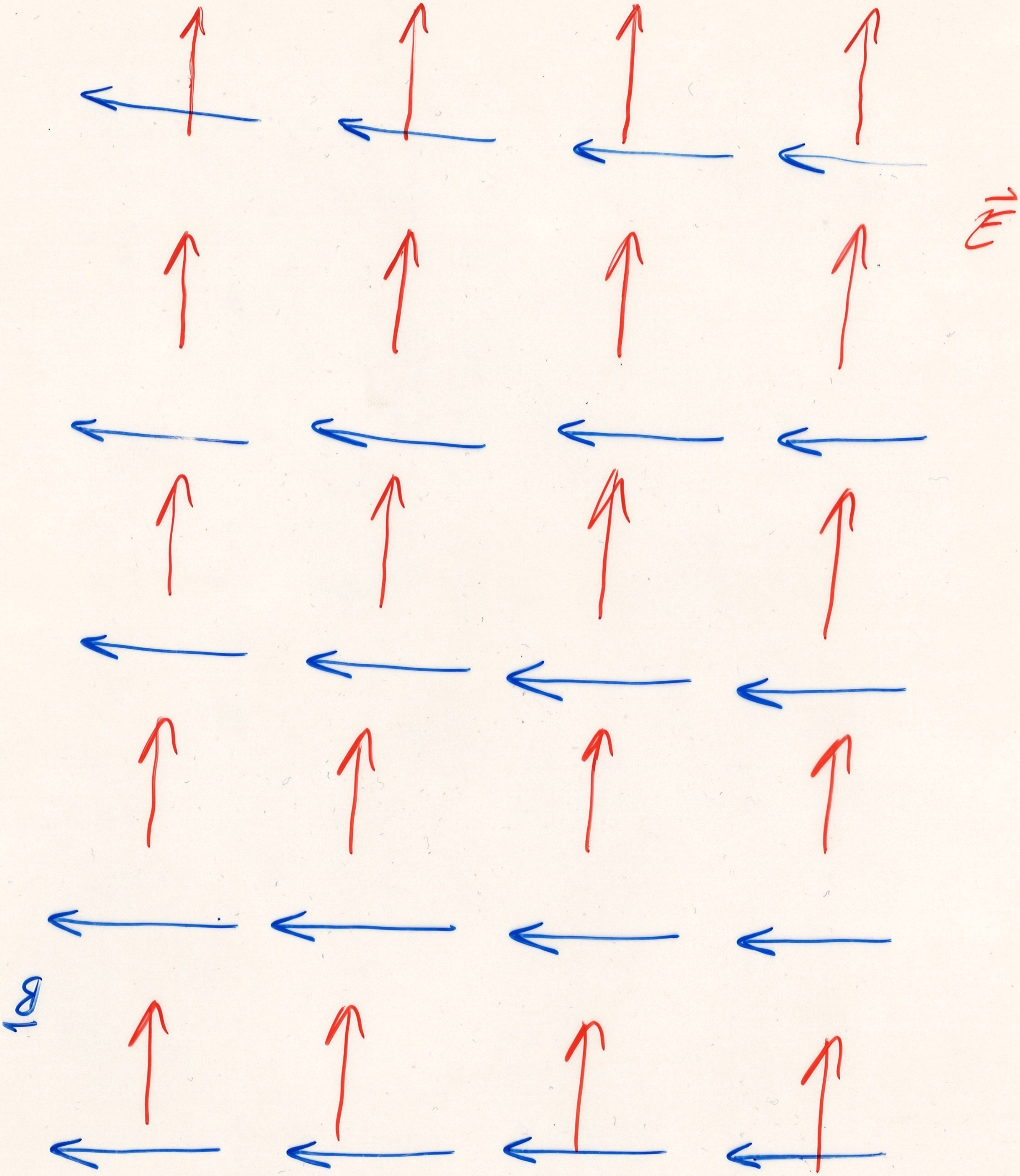
$t = 0$

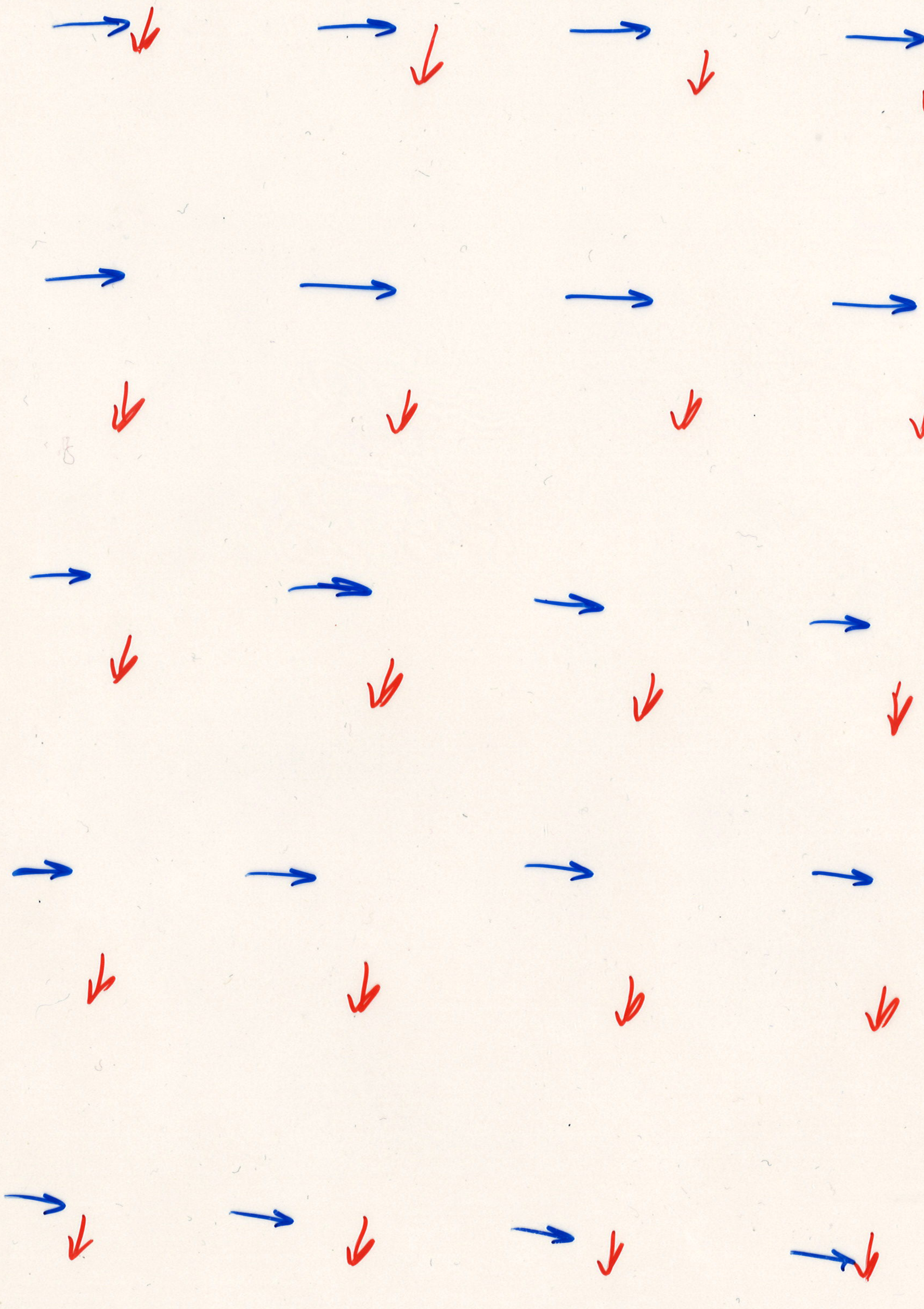


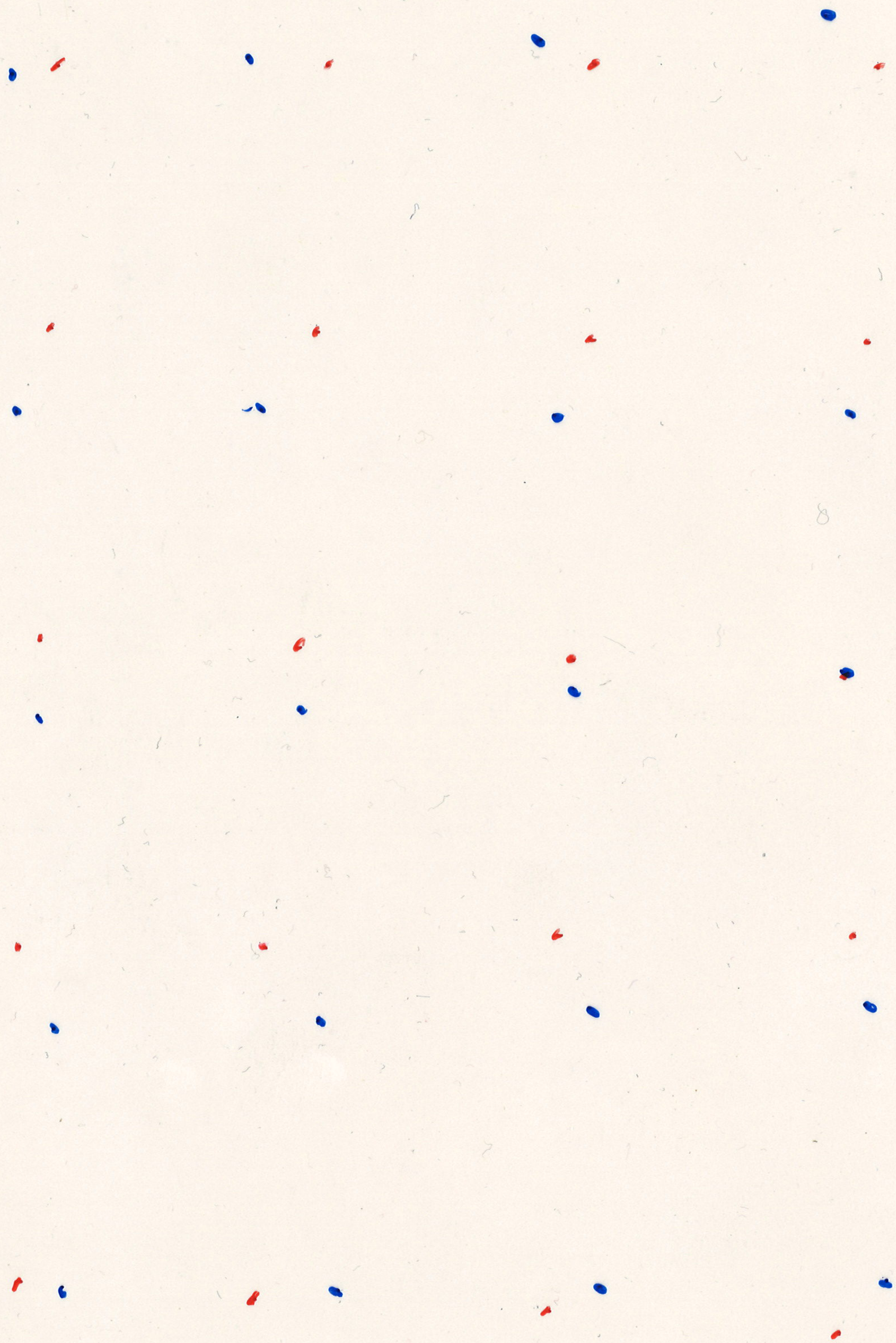
$E(x) = E_{max} \sin\left[\frac{\omega x}{c} - \frac{\omega 1}{c}\right]$

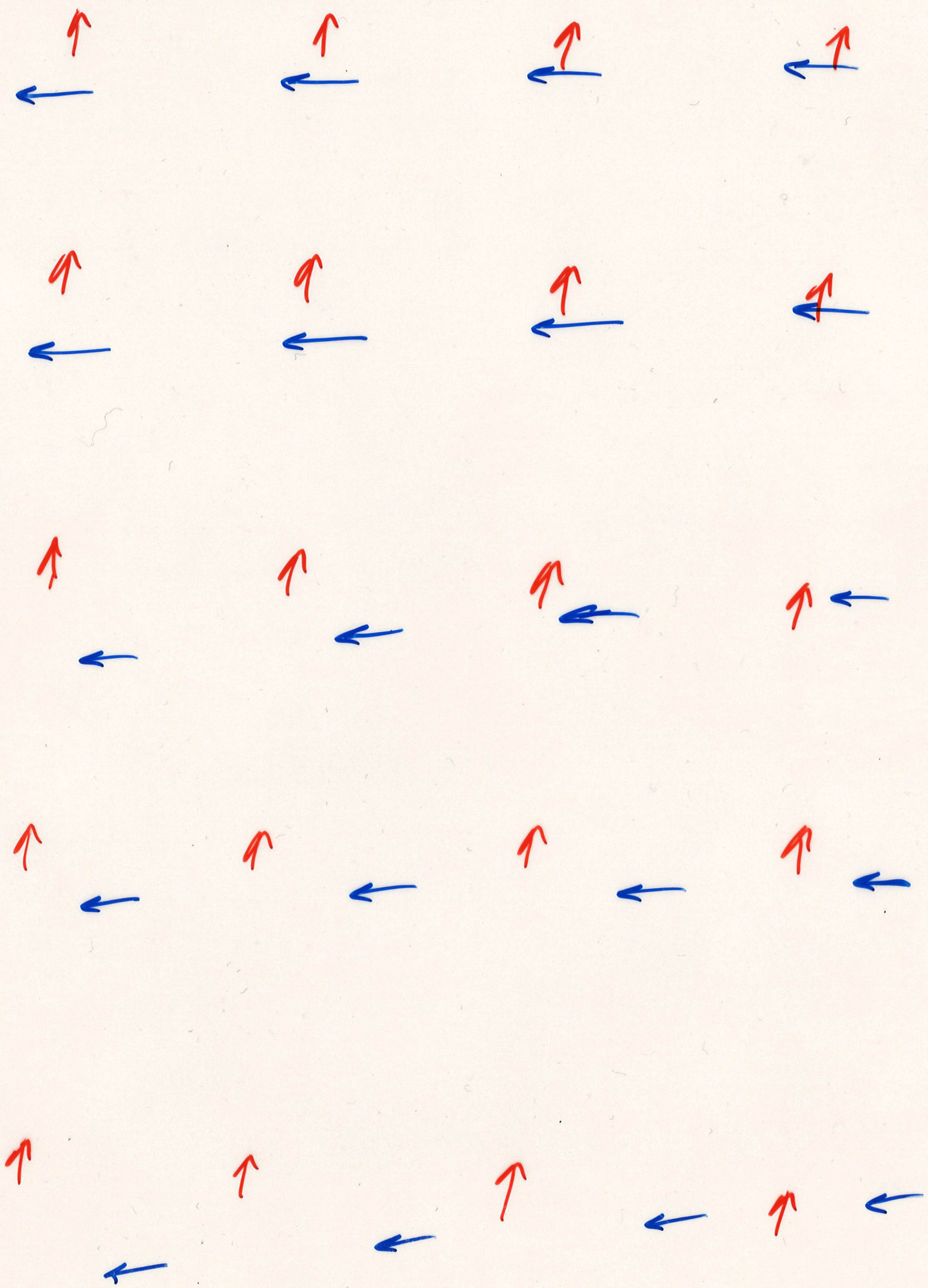
$t = \frac{1m}{c} \sim 3ns$

$t = \frac{2m}{c} \sim 6ns$


$\vec{E}$
$\vec{B}$







Maxwell's Equations in vacuum put a constraint on E_{max} and B_{max}.

They are not both arbitrary. You can select one, then the other must satisfy

$$\frac{E_{max}}{B_{max}} = c$$ (speed of light)

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

In fact, the electromagnetic wave moves along the vector

$$\vec{E} \times \vec{B}$$

The direction of $\vec{E}$ is called the polarization of the wave.

$$\vec{E}(\vec{r},t) = E_{max} \sin\left[\omega\left(\frac{x}{c} - t\right)\right] \hat{j}$$

$$\vec{B}(\vec{r},t) = B_{max} \sin\left[\omega\left(\frac{x}{c} - t\right)\right] \hat{k}$$

$\frac{\omega}{c}$ is often denoted as k, the wave number.

$$\vec{E}(\vec{r},t) = E_{max} \sin(kx - \omega t) \hat{j}$$

Energy in

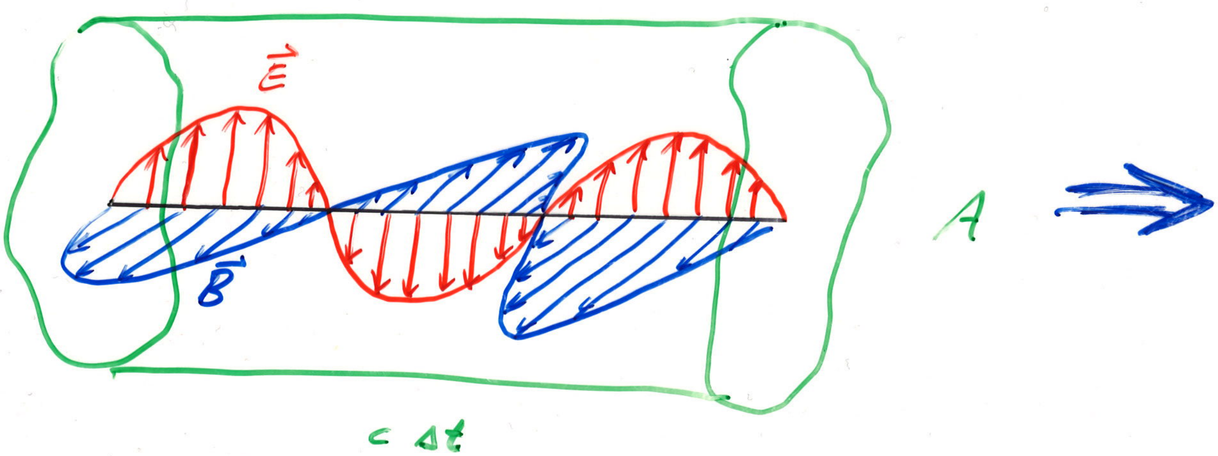
Electromagnetic Radiation

$$E(\vec{r},t) = E_{max} \sin(kx - \omega t) = E_{max} \sin\left[\omega\left(\frac{x}{c} - t\right)\right]$$

$$B(\vec{r},t) = B_{max} \sin(kx - \omega t) = \frac{E_{max}}{c} \sin\left[\omega\left(\frac{x}{c} - t\right)\right]$$

$$B_{max} = \frac{E_{max}}{c} \qquad c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Consider a box of cross-sectional Area A and length $c\Delta t$. In a time Δt, all the energy in this box will fall on a screen (detector).



electric energy density $\quad u_E(\vec{r},t) = \frac{1}{2}\epsilon_0 E^2(\vec{r},t)$

magnetic energy density $\quad u_B(\vec{r},t) = \frac{1}{2\mu_0} B^2(\vec{r},t) = \frac{1}{2\mu_0}\frac{E^2(\vec{r},t)}{c^2}$

$$= \frac{1}{2}\epsilon_0 E^2(\vec{r},t) = u_E(\vec{r},t)$$

energy in box: $\quad \Delta U = (u_E + u_B)\, A\, c\Delta t$

$$= \epsilon_0 E^2(\vec{r},t)\, A\, c\, \Delta t$$

rate of energy hitting the detector per unit area:

$$\frac{\Delta U}{\Delta t}\frac{1}{A} = \epsilon_0 c E^2(\vec{r},t) = \frac{1}{c\mu_0} E^2(\vec{r},t) \equiv S(\vec{r},t)$$

$$= \frac{1}{\mu_0} E(\vec{r},t) B(\vec{r},t)$$

$S(\vec{r},t)$ is the magnitude of the

Poynting vector.

$$\vec{S}(\vec{r},t) = \frac{1}{\mu_0}\vec{E}(\vec{r},t) \times \vec{B}(\vec{r},t)$$

direction of travel, direction of energy flow

Intensity

Intensity I is the time-averaged Poynting vector.

$$I = \frac{1}{T}\int_0^T S(\vec{r},t)\,dt = \frac{1}{c\mu_0}\,\frac{1}{T}\int_0^T E^2(\vec{r},t)\,dt$$

$$= \frac{1}{c\mu_0} E_{max}^2 \underbrace{\frac{1}{T}\int_0^T \sin^2\left[\omega\left(\frac{x}{c}-t\right)\right] dt}_{=\frac{1}{2}}$$

$$= \frac{1}{c\mu_0}\frac{E_{max}^2}{2} = \frac{1}{c\mu_0} E_{rms}^2$$

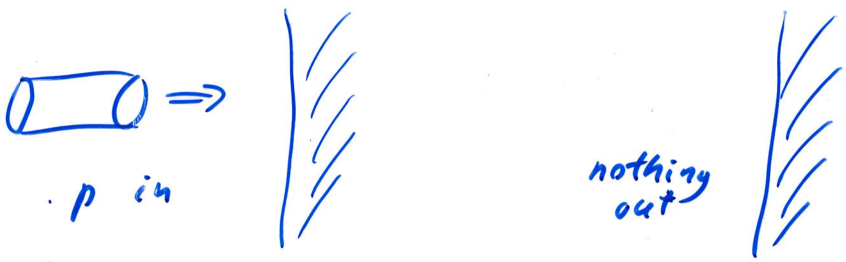
$$E_{max} = \sqrt{2}\, E_{rms}$$

Light carries momentum

If the energy in the box is ΔU,

then the momentum in the box is $p = \frac{\Delta U}{c}$.

Case 1: Total absorption



momentum transferred to screen : $\frac{\Delta U}{c} = \Delta p$

$= p_f - p_i$

Case 2: total reflection

momentum transferred to screen : $\frac{2\Delta U}{c} = \Delta p$

Radiation Pressure

Average force : $F = \frac{\Delta p}{\Delta t}$

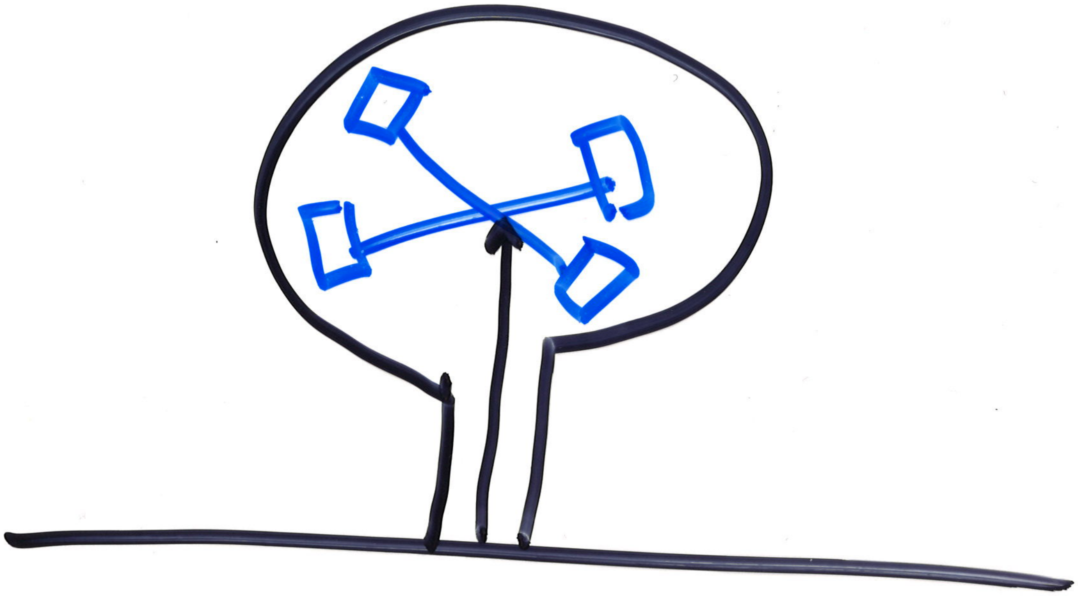
Average pressure : $\mathcal{P} = \frac{F}{A}$ force per unit area

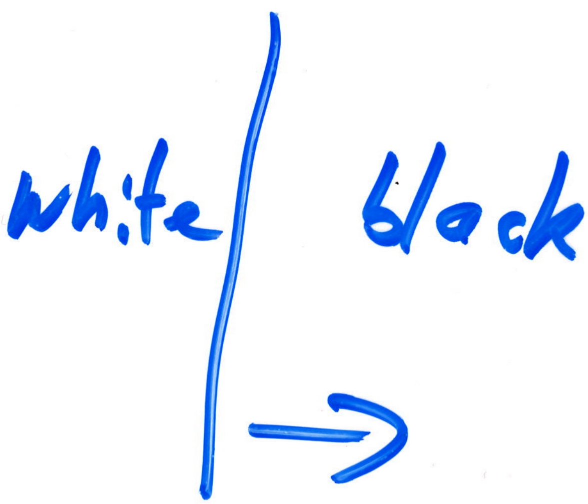
pressure

momentum

$$\mathcal{P} = \frac{\Delta p}{A \Delta t} = \begin{cases} \frac{\Delta U}{c A \Delta t} & \text{total absorption} \\ \frac{2\Delta U}{c A \Delta t} & \text{total reflection} \end{cases}$$

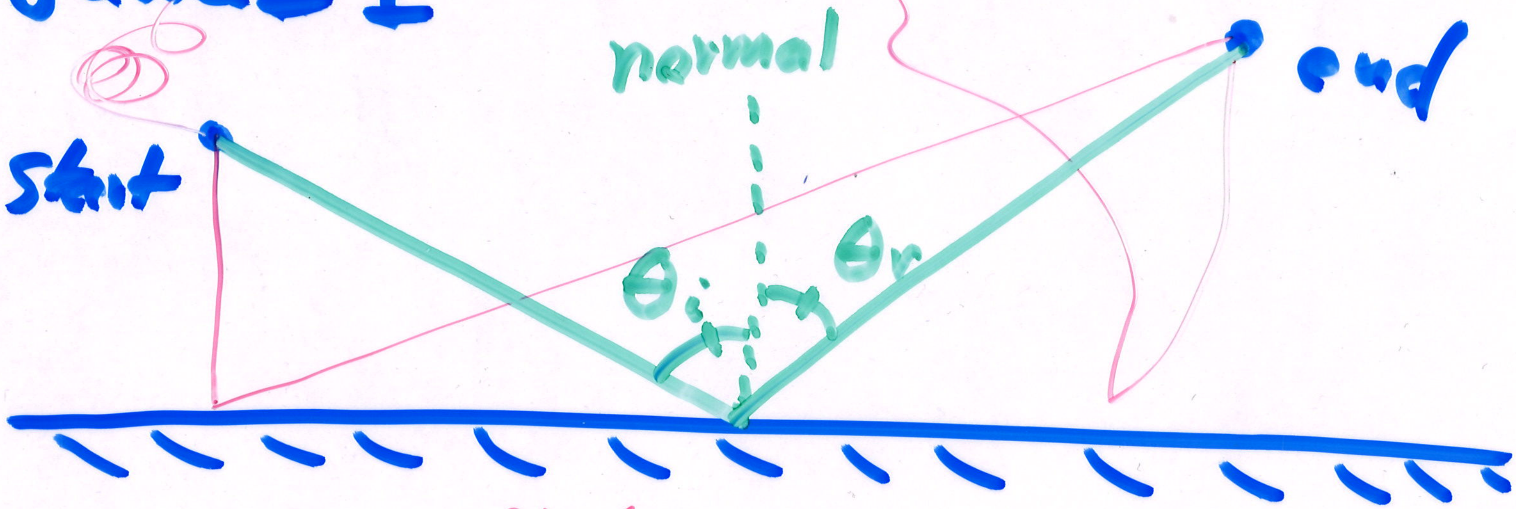
$$\mathcal{P} = \begin{cases} \frac{I}{c} & \text{total absorption} \\ \frac{2I}{c} & \text{total reflection} \end{cases}$$



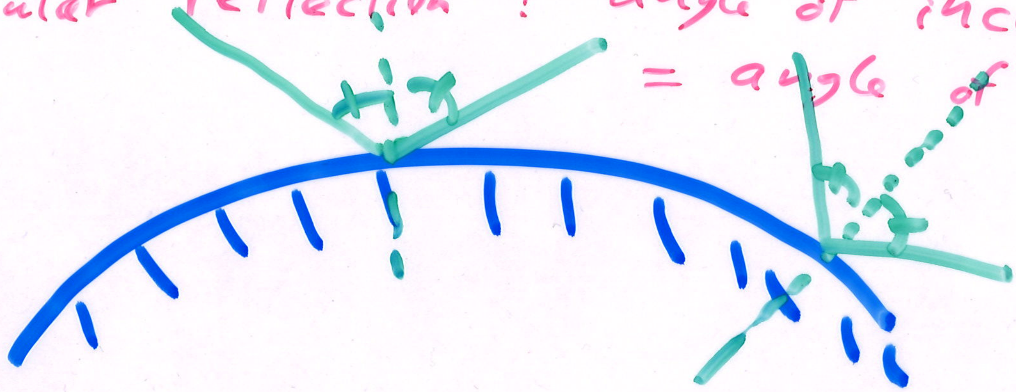

white
black

Reflection + Refraction
(Bending)

Game #1

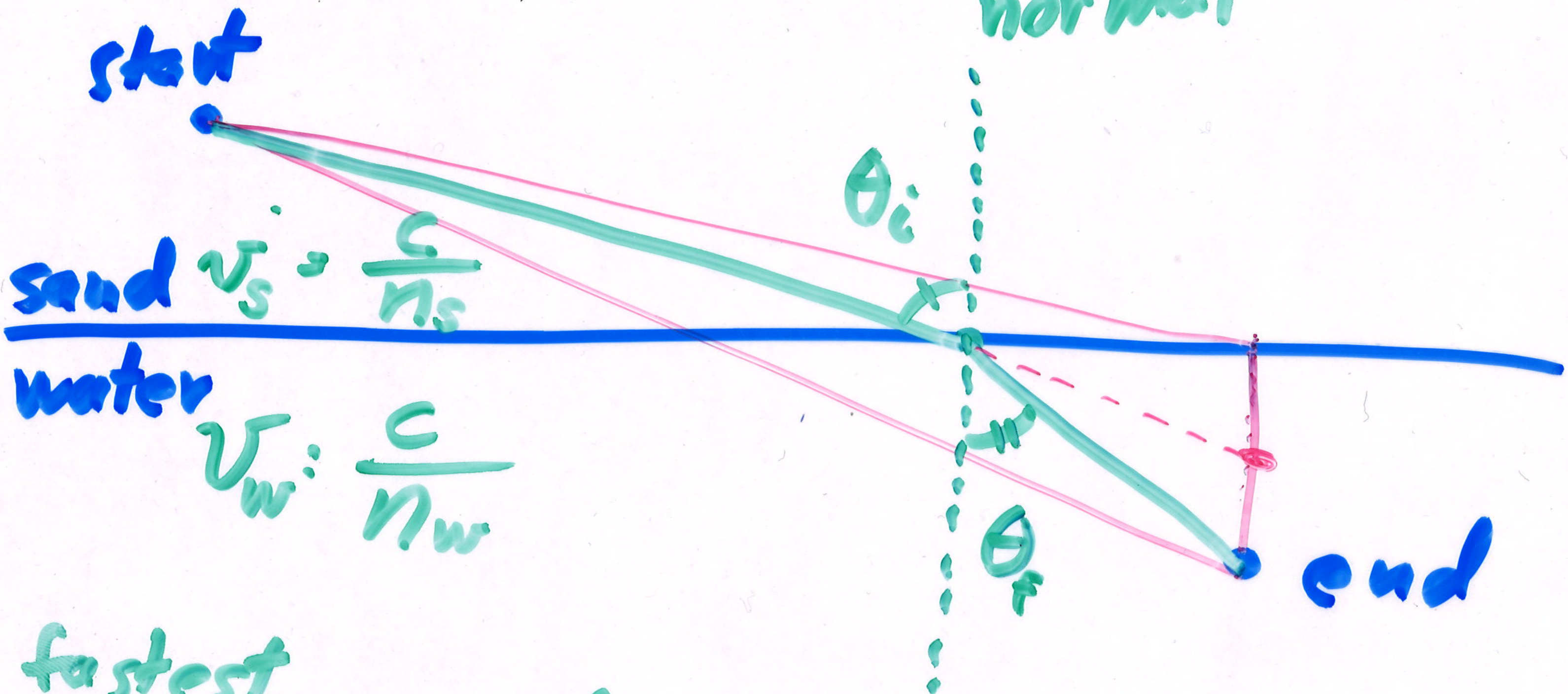


specular reflection : angle of incidence
= angle of reflection



Fermat law of least time

Game #2



fastest running speed : c

$1 \leq n_i \leq \infty$

Snell's Law

Snell ≡

$$n_i \sin\theta_i = n_f \sin\theta_f$$

incident

refracted

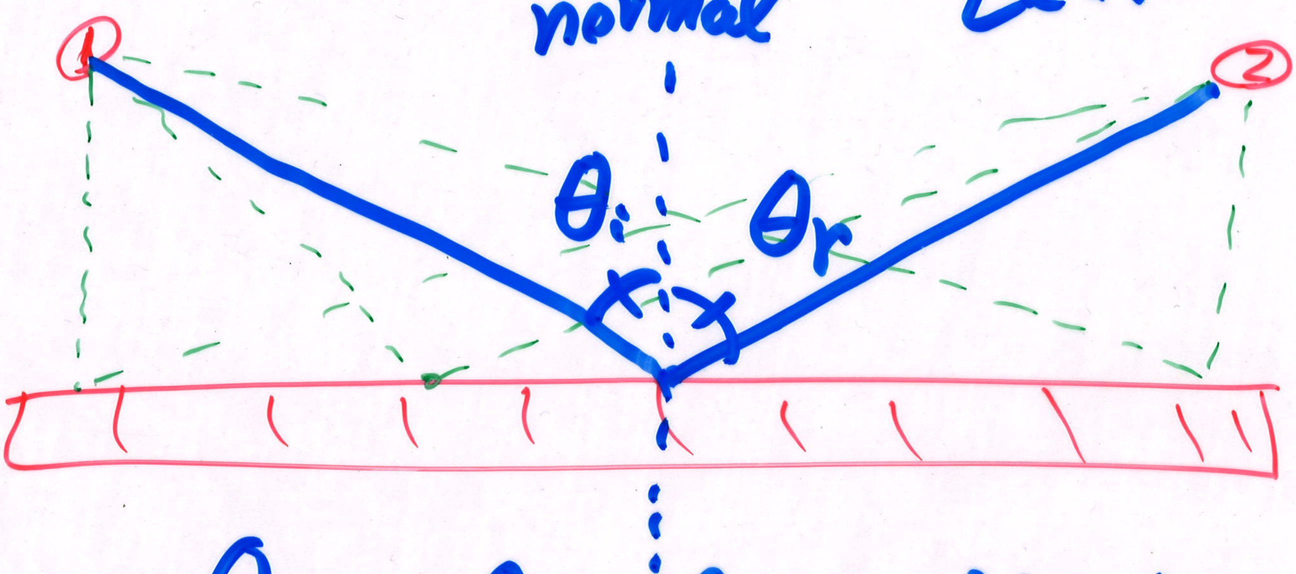
$n_{air} \approx 1.00$?

$n_{water} \approx \frac{4}{3} \sim 1.33$

Fermat's Principle of Least time

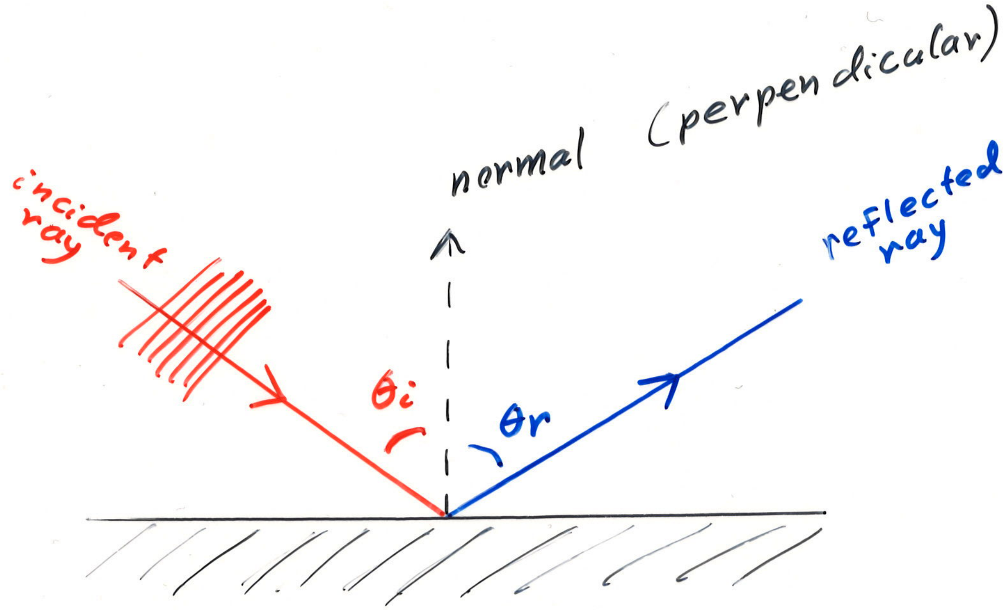
$$c^2 = a^2 + b^2$$

$$5^2 = 3^2 + 4^2$$



$\theta_i = \theta_r$ for reflection

Specular Reflection



$$\theta_i = \theta_r$$

angle of incidence = angle of reflection

The incident and reflected rays are in the same plane as the normal vector.

Refraction (Bending)

- The speed of light in a medium is less than the speed of light in vacuum.

$$v \leq c$$

- In fact, $v = \frac{c}{n}$ where $n \geq 1$
  n is called the index of refraction.

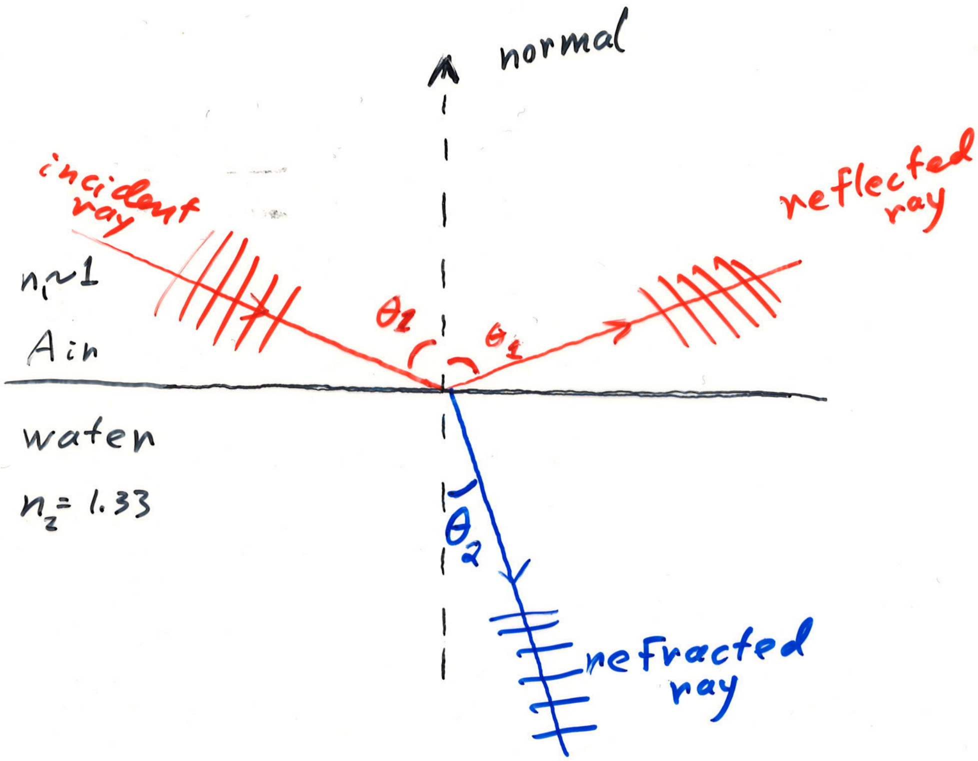
$n_{air} \approx 1$
$n_{vac} = 1$
$n_{H_2O} = 1.33$
$n_{Glass} = 1.33$

- The frequency of light does not change in a medium.

- Because $\lambda f = v$, the wavelength changes. λ is shorter in a medium than in vacuum.

big density $\Rightarrow$ n large $\Rightarrow$ v small

All of this has the effect of bending (refracting) a light ray.



Snell's Law

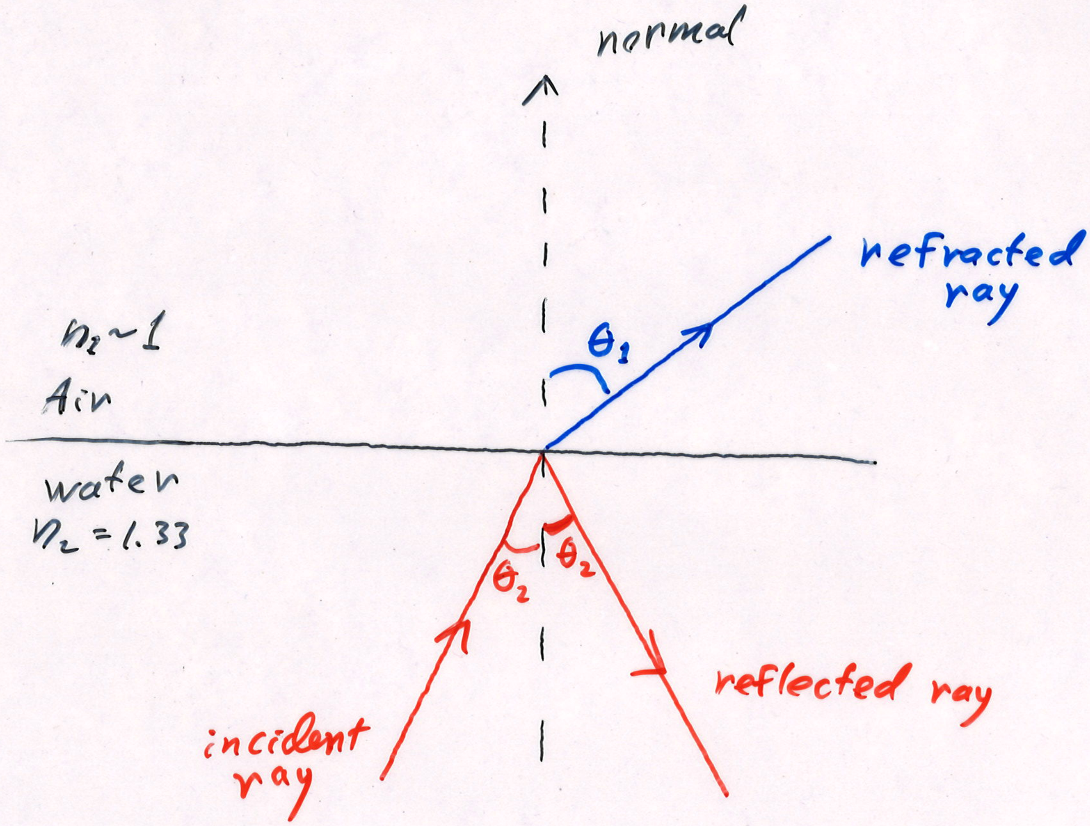
$$\boxed{n_1 \sin \theta_1 = n_2 \sin \theta_2}$$

$n_1 < n_2$

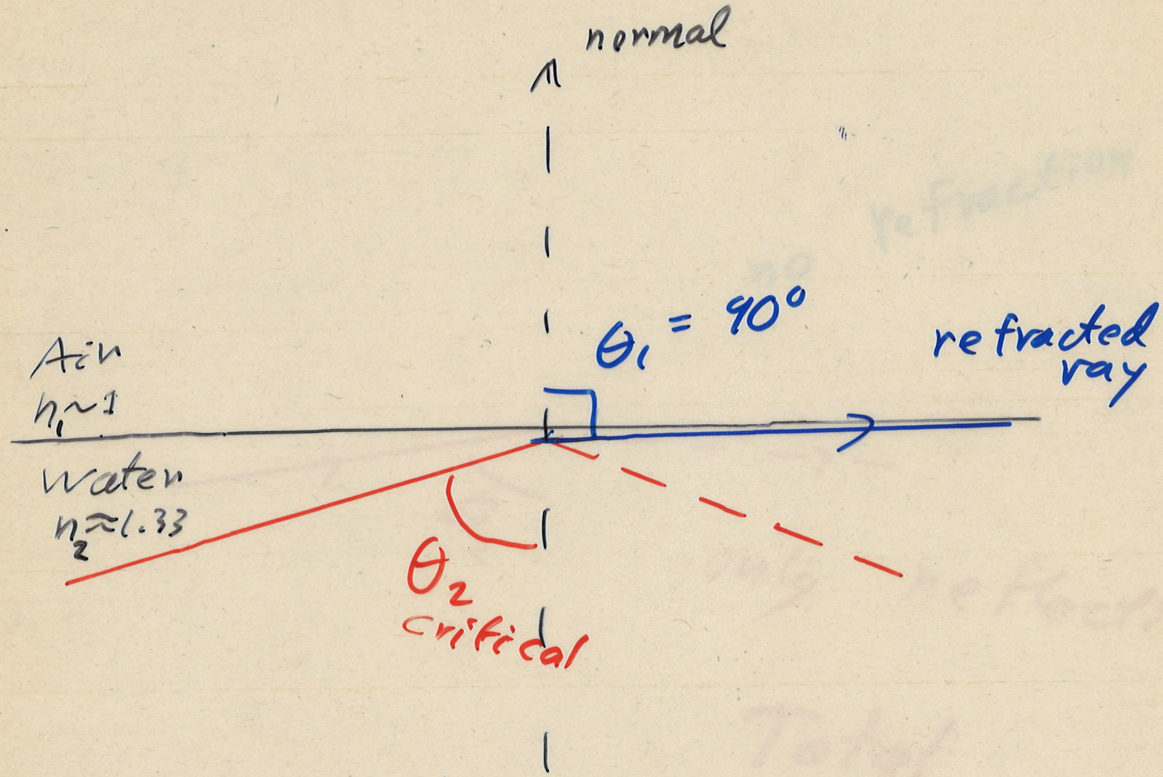
$\sin \theta_2 < \sin \theta_1$

$\theta_2 < \theta_1$

The incident ray can also originate in the denser medium (the one with larger n).



$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$



Total internal reflection

— no refracted ray

Snell's Law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

Ex: What is the critical angle for total internal reflection at a water → air interface?

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

$n_1 \sim 1$ (Air) $\qquad n_2 \sim 1.33$ (water)

when $\theta_2 = \theta_{2\,critical}$ then $\theta_1 = 90°$

$$\sin\theta_1 = \sin 90° = 1$$

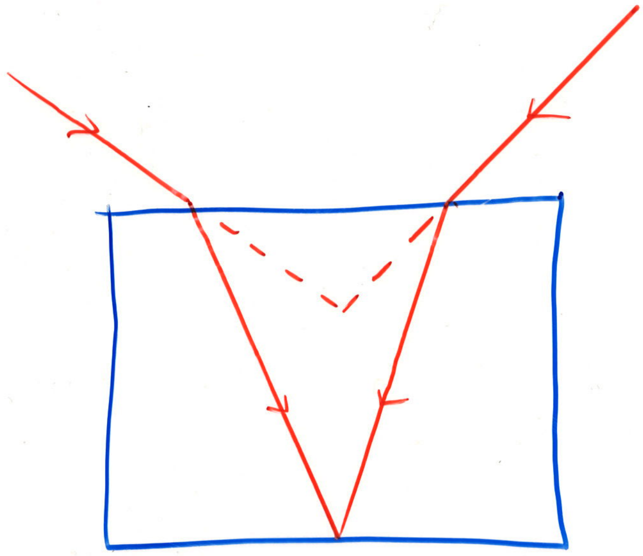
$$n_1 (1) = n_2 \sin(\theta_{2\,critical})$$

$$\frac{n_1}{n_2} = \sin(\theta_{2\,crit.})$$

$$\sin^{-1}\left(\frac{n_1}{n_2}\right) = \theta_{2\,critical}$$

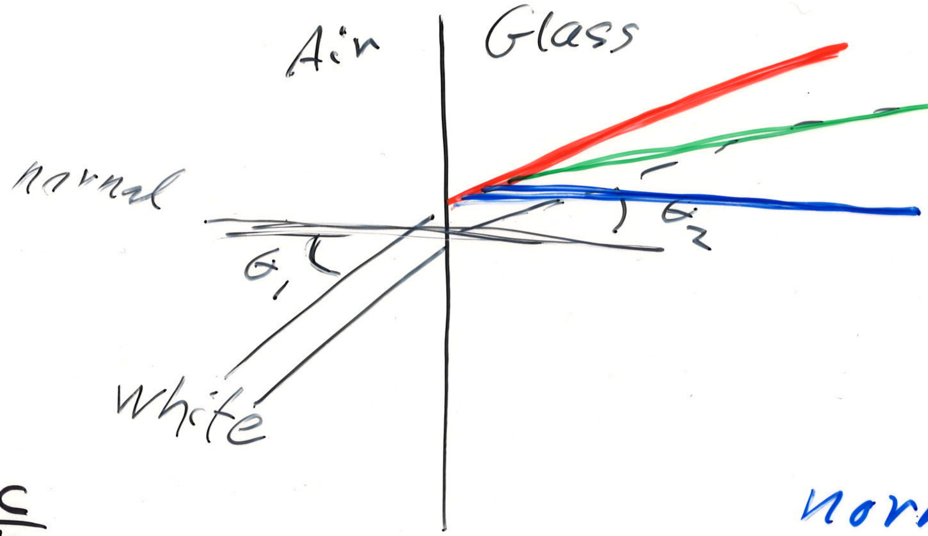
$$\sin^{-1}\left(\frac{1}{1.33}\right) \cong 49°$$

The Glass Block



Objects appear <u>higher</u> than the bottom of the block.

Dispersion (spreading)



$v = \frac{c}{n}$

normal dispersion

$v_{red} > v_{blue}$

$n_{red} < n_{blue}$

$\lambda f = v$

Anomalous dispersion

$v_{red} < v_{blue}$