Chapter 9: Linear Momentum + Collisions

Momentum: $\vec{p} = m\vec{v}$

$$\begin{cases} p_x = m v_x \\ p_y = m v_y \\ p_z = m v_z \end{cases}$$

Newton's Second Law

$$\boxed{\Sigma \vec{F} = \frac{d\vec{p}}{dt}}$$

$$\Sigma \vec{F} = \frac{d}{dt}(m\vec{v}) = \left(\frac{dm}{dt}\right)\vec{v} + m\left(\frac{d\vec{v}}{dt}\right)$$

$$= \left(\frac{dm}{dt}\right)\vec{v} + \underline{m\vec{a}}$$

when is $\vec{F} = m\vec{a}$ true?

when the mass is <u>constant</u>.

$$m = \text{const} \Rightarrow \frac{dm}{dt} = 0$$

Impulse

$\vec{F}_{NET} = \frac{d\vec{p}}{dt}$ can be inverted

to give $\Delta\vec{p} = \vec{p}_f - \vec{p}_i = \int_{t_i}^{t_f} \vec{F}\,dt \equiv \vec{I}$

$\vec{I}$ is the impulse.

Ex. A 1 kg rubber ball initially moving at 10 m/s to the left strikes a wall and recoils at 8 m/s to the right. What is the impulse? of the ball?

$\longrightarrow x$

$\vec{p}_i = -(10\ \text{m/s})(1\ \text{kg}) = -10\ \frac{\text{kg}\cdot\text{m}}{\text{s}}$

$\vec{p}_f = +8\ \frac{\text{kg}\cdot\text{m}}{\text{s}}$

$\vec{I} = \vec{p}_f - \vec{p}_i = 8 - (-10) = \boxed{+18\ \frac{\text{kg}\cdot\text{m}}{\text{s}}}$ to the right

Ex. The same ball was in contact with the wall for 0.001 seconds. What average force is exerted on the ball in this time?

$\vec{F}_{AVG} = \frac{\vec{I}}{\Delta t} = \frac{\Delta\vec{p}}{\Delta t}$

$= \frac{+18\ \frac{\text{kg}\cdot\text{m}}{\text{s}}}{0.001\ \text{s}} = 18{,}000\ \text{N}$ to the right

---

Ex. Given the same impulse $\vec{I}$, which block at rest will acquire the larger velocity, a small mass block or a large mass block?

$\vec{p}_i = 0 \qquad \vec{p}_f = m\vec{v}$

$\vec{I} = \Delta\vec{p} = \vec{p}_f - \cancel{\vec{p}_i}_{0} = \vec{p}_f = m\vec{v}$

$\downarrow\ \uparrow$

Demo:

Conservation of Momentum

Consider two particles that interact with each other (exert forces on each other) but are isolated from the environment.

$$\underset{\substack{\text{on } 1\\ \text{by } 2}}{\vec{F}_{12}} = \frac{d\vec{p}_1}{dt} \qquad \underset{\substack{\text{on } 2\\ \text{by } 1}}{\vec{F}_{21}} = \frac{d\vec{p}_2}{dt}$$

Newton's third law: $\underbrace{\vec{F}_{12} = -\vec{F}_{21}}_{\text{action-reaction pair}}$

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \quad \underline{\underline{\text{or}}} \quad \frac{d}{dt}(\vec{p}_1+\vec{p}_2) = 0$$

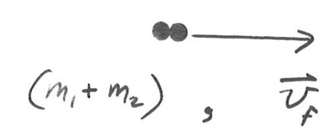
$$\underline{\underline{\text{or}}} \quad \vec{p}_1+\vec{p}_2 = \overrightarrow{\text{constant}} \qquad \underset{\substack{\parallel \\ (\vec{p}_1+\vec{p}_2)_i}}{p_{\text{total } i}} = \underset{\substack{\parallel \\ (\vec{p}_1+\vec{p}_2)_f}}{p_{\text{total } f}}$$

Collisions (1-D)

First, a special case: one particle at rest.

Before:

After:



Suppose the two particles stick together and emerge with a common final velocity. Such a collision is called perfectly inelastic.

How can we find $\vec{v}_f$?

Conservation of Momentum!

$$\vec{P}_i = \vec{P}_f$$

$$m_1 \vec{v}_{1i} + m_2\, 0 = (m_1+m_2)\vec{v}_f$$

$$\vec{v}_f = \frac{m_1 \vec{v}_{1i}}{m_1+m_2}$$

Perfectly Inelastic Collisions:
The general case

$$\vec{P}_i = \vec{P}_f$$

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1+m_2)\vec{v}_f$$

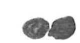
$$\boxed{\vec{v}_f = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1+m_2}}$$

Ex. A particle of unknown mass m is moving to the right at speed $10\ m/s$. Another particle of the same mass is moving to the left with speed $10\ m/s$. They stick together. What is the common final velocity.

Before:

$m, \vec{v}_{1i}$ $\quad\quad$ $m, \vec{v}_{2i}$

After:



$2m, \vec{v}_f$

$\vdash \rightarrow x$ $\quad$ $\vec{v}_{1i} = +10\ m/s$ $\quad$ $\vec{v}_{2i} = -10\ m/s$

$$\vec{v}_f = \frac{m\vec{v}_{1i} + m\vec{v}_{2i}}{2m} = \frac{m(10) + m(-10)}{2m} = 0$$

Is energy conserved in an inelastic collision?

Ex. $K_i = \frac{1}{2}m(v_{1i})^2 + \frac{1}{2}m(v_{2i})^2$

$= \frac{1}{2}m(10)^2 + \frac{1}{2}m(-10)^2$

$= 100(m)$

$K_f = \frac{1}{2}(2m)(v_f)^2 = \frac{1}{2}(2m)(0)^2$

$= 0$

In general $K_f < K_i$ for an inelastic collision. Where did the mechanical energy go?

Is it possible to arrange a collision in which mechanical energy is conserved? Yes.

Such a collision is called ELASTIC.

Ex. A special case:

Suppose the two particles have equal mass, particle one is moving with speed v toward particle two which is at rest.

Demo

Before:

After:

To solve, we must satisfy both momentum and energy conservation.

If $m_1 = m_2$, $v_{1i} = v$, $v_{2i} = 0$

Momentum: $p_i = p_f$

$$\cancel{m_1} v_{1i} = \cancel{m_1} v_{1f} + \cancel{m_2} v_{2f}$$

$$\boxed{v = v_{1f} + v_{2f}}$$

Energy: $KE_i = KE_f$ ⇒ elastic

$$\frac{1}{2}\cancel{m_1} v_{1i}^2 = \frac{1}{2}\cancel{m_1} v_{1f}^2 + \frac{1}{2}\cancel{m_2} v_{2f}^2$$

$$\boxed{v^2 = v_{1f}^2 + v_{2f}^2}$$

The energy equation is quadratic so once again, there are two solutions. Which one is physically reasonable?

C of E: $v^2 = v_{1f}^2 + v_{2f}^2$

$$(v_{1f} + v_{2f})^2 = v_{1f}^2 + v_{2f}^2$$

$$\cancel{v_{1f}^2} + 2v_{1f}v_{2f} + \cancel{v_{2f}^2} = \cancel{v_{1f}^2} + \cancel{v_{2f}^2}$$

$$2v_{1f}v_{2f} = 0$$

⇒ either $v_{1f} = 0$ or $v_{2f} = 0$ but not both!

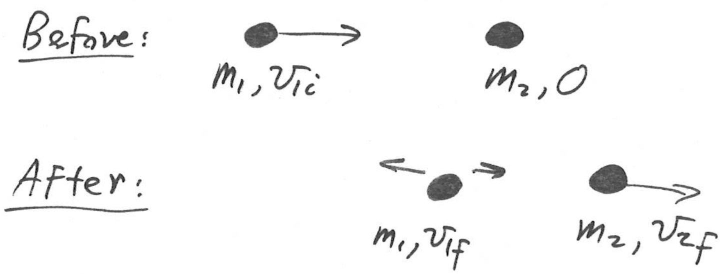
Collision	Pass Through
$v_{1f} = 0$	$v_{1f} = v$
$v_{2f} = v$	$v_{2f} = 0$
physically reasonable	

Unequal Masses

Elastic Collisions -

Use Conservation of Momentum

and Conservation of Energy



$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \qquad \text{Eq 9-22}$$

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \qquad \text{Eq 9-23}$$

Derive these yourself!

One Last Complication

Unequal masses: $m_1 \neq m_2$

and m_2 is moving initially.

This is the general case:

Eq 9-2

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} + \left(\frac{2m_2}{m_1 + m_2}\right) v_{2i}$$

Eq 9-2

$$v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} + \left(\frac{m_2 - m_1}{m_1 + m_2}\right) v_{2i}$$

Elastic

Ex. Two particles of equal mass collide. The first is moving at 10 m/s to the right, the second at 10 m/s to the left. After the collision, they both move away at ± 8 m/s.

Is momentum conserved? Yes

$$\vec{P}_i = m(+10) + m(-10) = 0$$
$$\vec{P}_f = m(+8) + m(-8\ m/s) = 0 \quad \checkmark$$

Is mechanical energy conserved? No!

$$K_i = \frac{1}{2}m(10\ m/s)^2 + \frac{1}{2}m(-10\ m/s)^2 = 100\cdot m$$
$$K_f = \frac{1}{2}m(8\ m/s)^2 + \frac{1}{2}m(-8\ m/s)^2 = 64\cdot m$$

Is the collision ~~perfectly~~ inelastic? No

Summary

Elastic: { Conservation of Momentum; Conservation of Energy }

Inelastic: Conservation of Momentum

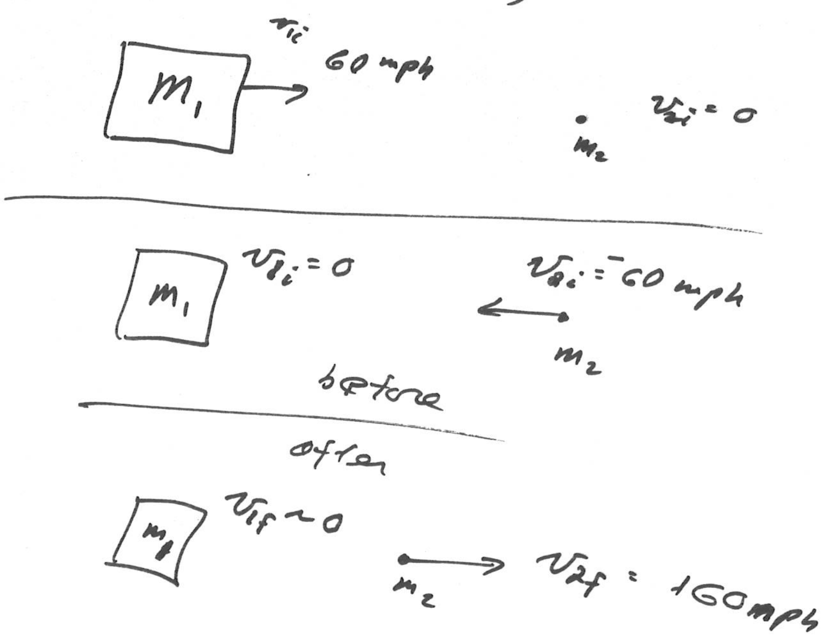
special case → Perfectly Inelastic: { Conservation of Momentum; Final velocities equal; Particles stick togethe }

★ Momentum is always conserved in any type of collision.

Ex A 20 ton truck hits a stationary ping-pong ball. The initial speed of the truck is 60 mph. What is the final speed of the ping-pong bal?

Elastic!

① Galilean Relativity



Collisions in Multiple Dimensions

In 1-dimension, we learned that in <u>any</u> kind of collision, momentum is always conserved.

In 2-d and 3-d, momentum is conserved as a <u>vector</u>.

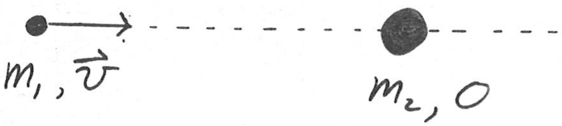
$$\vec{P}_i = \vec{P}_F$$

shorthand notation for

$$\begin{cases} P_{i,x} = P_{F,x} \\ P_{i,y} = P_{F,y} \end{cases}$$

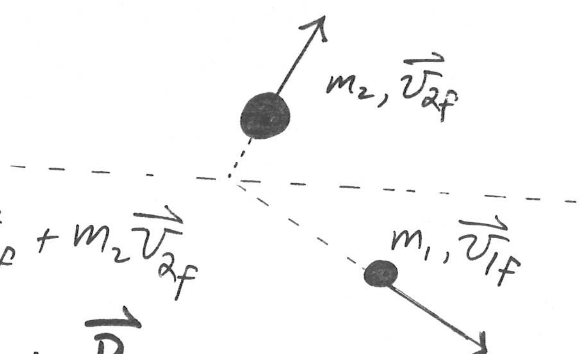
each component is conserved separately!

before



$$\vec{P}_i = m_1 \vec{v} + m_2 \cdot 0$$

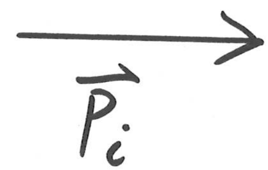
after

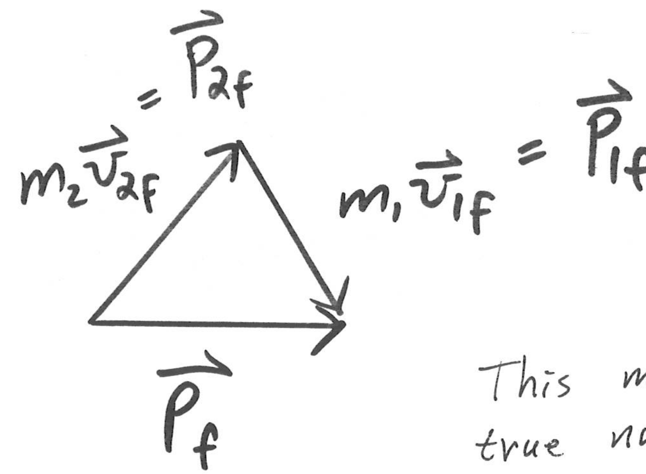


$$\vec{P}_f = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f}$$
$$= \vec{P}_{1f} + \vec{P}_{2f}$$

Demo

Graphically



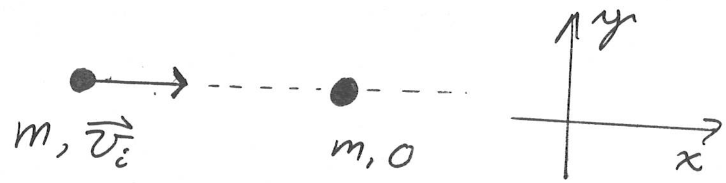


This must be true numerically as well.

[redacted]

Numerically

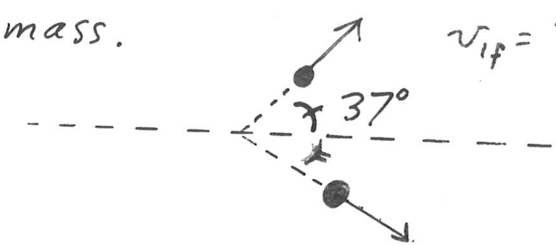
Let's consider a 2-dimensional collision between two packs of equal mass, one at rest.



$\vec{v}_i = 5\,\hat{i}$ m/s

After the collision, we are told the direction and speed of the red mass.

$v_{1f} = 3$ m/s



What is the final velocity <u>vector</u> of the green mass $\vec{v}_{2f}$?

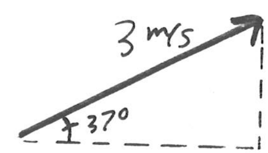
Approach?

$$\vec{P_i} = \vec{P_f}$$

$$\vec{P_i} = 5 \cdot m\,\hat{i} + 0\,\hat{j}$$

$$\vec{P}_{1f} = ? = m v_{1f,x}\,\hat{i} + m v_{1f,y}\,\hat{j}$$

$$= m 3 \cos 37°\,\hat{i} + m 3 \sin 37°\,\hat{j}$$



$3 \sin 37°$ m/s $= v_{1f,x}$

$v_{1f,x} = 3 \cos 37°$ m/s

$$\vec{P}_{2f} = ? = m v_{2f,x}\hat{\imath} + m v_{2f,y}\hat{\jmath}$$

$$\vec{P}_i = \vec{P}_f$$

$$\vec{P}_i = \vec{P}_{1f} + \vec{P}_{2f}$$

$$5m\hat{\imath} + 0\hat{\jmath} = 3m\cos 37^\circ\hat{\imath} + 3m\sin 37^\circ\hat{\jmath} + m v_{2fx}\hat{\imath} + m v_{2fy}\hat{\jmath}$$

How do we solve this?

by components!

x - components $\hat{\imath}$ direction

$$5\cancel{m} = 3\cancel{m}\cos 37^\circ + \cancel{m}\, v_{2fx}$$

$$v_{2fx} = 5 - 3\cos 37^\circ = \boxed{2.6 \text{ m/s}}$$

y - components $\hat{\jmath}$ direction

$$0 = 3\cancel{m}\sin 37^\circ + \cancel{m}\, v_{2fy}$$

$$v_{2fy} = -3\sin 37^\circ = \boxed{-1.8 \text{ m/s}}$$

$$\vec{v}_{2f} = (2.6\hat{\imath} - 1.8\hat{\jmath}) \text{ m/s}$$

Polar Coordinates

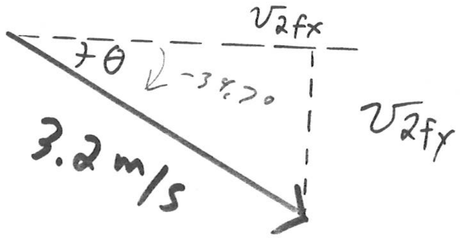
What is the angle and magnitude of $\vec{v}_{2f}$

$$|\vec{v}_{2f}| = \sqrt{(v_{2fx})^2 + (v_{2fy})^2}$$

$$= \boxed{3.2 \text{ m/s}}$$

SCALAR speed

$$\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \boxed{-34.7^\circ}$$



Is this collision elastic?

Is mechanical energy conserved?

$$K_i = \tfrac{1}{2} m (v_i)^2 = \tfrac{1}{2} m (5)^2 = \tfrac{25}{2} \cdot m$$

$$K_f = \tfrac{1}{2} m (v_{1f})^2 + \tfrac{1}{2} m (v_{2f})^2$$

$$= \tfrac{1}{2} m (3)^2 + \tfrac{1}{2} m (3.2)^2 = \tfrac{19.2}{2} \cdot m$$

This collision is <u>not</u> elastic,

and it is not <u>completely</u> <u>inelastic</u>.

Now, consider a completely inelastic collision. The two masses stick together.

Demo

Approach?

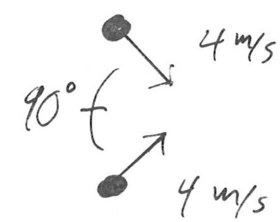
How do we solve for the common final velocity vector?

$$\vec{P}_i = \vec{P}_f$$

[redacted]

Example: <u>Perfectly Inelastic</u> Collision

Consider a collision between two particles $m_1 = 2\,kg$ and $m_2 = 1\,kg$. After the collision, the particles stick together. Initially, they both have velocities of magnitude $4\ m/s$, but at right angles to each other.



How much energy is lost in the collision?

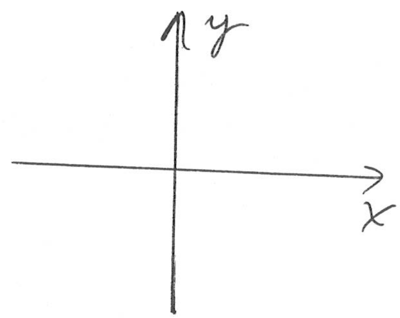
Approach?

Momentum Conservation

$$\vec{P_i} = \vec{P_f}$$

Choose a coordinate system.

A judicious choice will save us a lot of work!



$$\vec{P_{1i}} = m_1 \vec{v}_{1i} = (2kg)\, 4\hat{i} \ m/s$$

$$\vec{P_{2i}} = m_2 \vec{v}_{2i} = (1kg)\, 4\hat{j} \ m/s$$

$$\vec{P_i} = \vec{P_{1i}} + \vec{P_{2i}}$$

$$= (8\hat{i} + 4\hat{j}) \ kg \cdot m/s = \vec{P_f}$$

What is the final velocity?

$$m = 3kg \qquad \vec{v_f} = \frac{\vec{P_f}}{m}$$

$$\vec{P_f} = 3 v_{fx} \hat{i} + 3 v_{fy} \hat{j}$$

$$\vec{P_i} = \vec{P_f}$$

$$8\hat{i} + 4\hat{j} = 3 v_{fx} \hat{i} + 3 v_{fy} \hat{j}$$

$\hat{i}$ $\quad v_{fx} = \frac{8}{3} \ m/s = 2.66 \ m/s$

$\hat{j}$ $\quad v_{fy} = \frac{4}{3} \ m/s = 1.33 \ m/s$

How much mechanical energy is converted into heat?

$$K_i = \frac{1}{2} m_1 (v_{1i})^2 + \frac{1}{2} m_2 (v_{2i})^2$$

$$= \frac{1}{2}(2\,kg)(4\,m/s)^2 + \frac{1}{2}(1\,kg)(4\,m/s)^2$$

$$= 24\ J$$

$$K_f = \frac{1}{2}(m_1 + m_2)\, v_f^2 = \frac{1}{2}(3\,kg)\underbrace{(2.66^2 + 1.33^2)}_{\underbrace{v_{fx}^2 + v_{fy}^2}_{v_f^2}}$$

$$= 13.3\ J$$

So $24\,J - 13.3\,J = 10.7\ J$

are converted to heat.

Center of Mass

An idea we have been using for some time.

Example: Bowling Ball

Not a point particle, but we have been treating it like one.

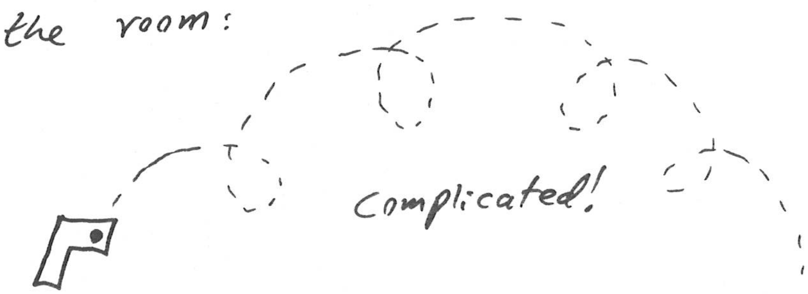
Justification for this?

We will see ...

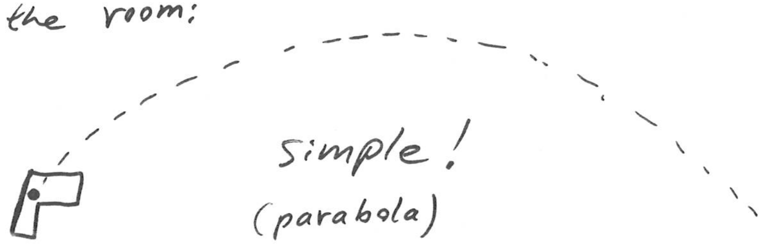
Demo

From Last Time

Trajectory of an arbitrary point in an extended body thrown across the room:



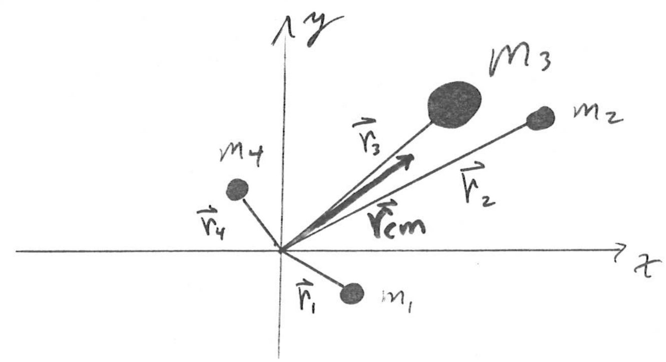
Trajectory of the center of mass of an extended body thrown across the room:



Definition of the center of mass of a set of particles:

$$\vec{r}_{cm} \equiv \frac{\sum_i m_i \vec{r}_i}{M_{total}} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots}{M_1 + M_2 + \cdots}$$

where M_{total} is the total mass of all the particles $= \sum_i m_i$

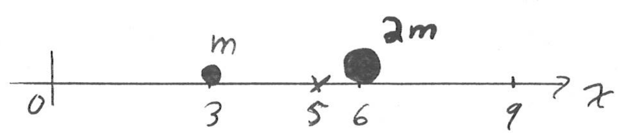


Idea: replace a lot of individual masses at many points by one big mass at one point.

$\vec{r}_{cm}$ is a vector quantity.

1-d example:

Two particles sit on the x-axis. One sits 3cm from the origin and has mass "m." The other sits 6 cm from the origin and has mass "2m." Where is the center of mass?



$$X_{cm} = \frac{\sum_i m_i x_i}{M_{total}} = \frac{m(3cm) + 2m(6cm)}{3 \cdot m}$$

$$= \frac{\cancel{m}(15cm)}{3\cancel{m}} = \boxed{5cm}$$

"Average" position

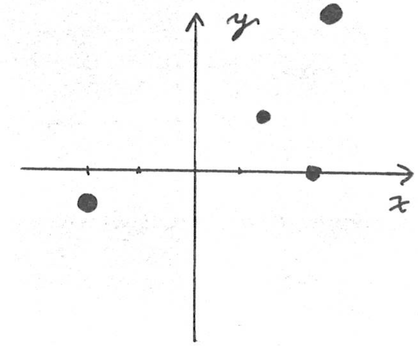
2-d example:

3 particles

1 kg at $(-2\hat{i} - \hat{j})$

2 kg at $(2\hat{i} + 0\hat{j})$

3 kg at $(2\hat{i} + 4\hat{j})$



$$x_{cm} = \frac{\sum_i m_i x_i}{\sum_i m_i}$$

$$= \frac{1kg(-2) + 2kg(2) + 3kg(2)}{1kg + 2kg + 3kg} = \boxed{\frac{8}{6}}$$

$$y_{cm} = \frac{\sum_i m_i y_i}{\sum_i m_i}$$

$$= \frac{1kg(-1) + 2kg(0) + 3kg(4)}{1kg + 2kg + 3kg} = \boxed{\frac{11}{6}}$$

$$\vec{r}_{cm} = \left(\frac{8}{6}\hat{i} + \frac{11}{6}\hat{j}\right)$$

How does the center of mass simplify physics?

Kinematics

$$\vec{r}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \vec{r}_i\right)$$

Take the time derivative of both sides.

$$\frac{d\vec{r}_{cm}}{dt} = \vec{v}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \frac{d\vec{r}_i}{dt}\right)$$

Velocity of the center of mass

$$\boxed{\vec{v}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \vec{v}_i\right)}$$

Individual velocities are "weighted" by the individual masses.

Example:

A 70 kg instructor throws a 0.5 kg ball at 35 m/s. How fast is the center of mass moving?

	Me	Ball
m_i	70 kg	0.5 kg
$\vec{v}_i$	0 m/s	35 m/s $\hat{\imath}$

$$\vec{v}_{cm} = \frac{70\,kg\,(0\,m/s) + 0.5\,kg\,(35\,m/s)\,\hat{\imath}}{70\,kg + 0.5\,kg}$$

$$= \boxed{0.25\,m/s}\ \hat{\imath}$$

"Average" vel. of system

Next step $\rightarrow$ Acceleration of the center of mass

$$\vec{v}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \vec{v}_i\right)$$

Take the time derivative of both sides.

$$\frac{d\vec{v}_{cm}}{dt} = \vec{a}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \frac{d\vec{v}_i}{dt}\right)$$

$$\boxed{\vec{a}_{cm} = \frac{1}{M_{total}}\left(\sum_i m_i \vec{a}_i\right)}$$

Important because we can now consider **Dynamics**

$$m_i \vec{a}_i = \vec{F}_i$$ total force on the ith particle

$$\vec{a}_{cm} = \frac{1}{M_{total}}\left(\sum_i \vec{F}_i\right)$$

$$M_{total}\,\vec{a}_{cm} = \sum_i \vec{F}_i$$ = Total Force on all particles

One more step and we will have our justification for using the center of mass.

$\sum_i \vec{F}_i$ is the sum of all forces acting on all particles.

This sum includes <u>internal</u> and <u>external</u> forces.

Simplify?

Consider just two particles.

$i = 1, 2$

$$\sum_i \vec{F}_i = \vec{F}_1 + \vec{F}_2$$

---

on m_1 by m_2

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{ext\,1}$$

internal

external

$$\vec{F}_2 = \vec{F}_{21} + \vec{F}_{ext\,2}$$

$$\vec{F}_1 + \vec{F}_2 = \vec{F}_{1\,ext} + \vec{F}_{2\,ext}$$

But!

Newton's third law relates $\vec{F}_{21}$ and $\vec{F}_{12}$.

$$\vec{F}_{21} = -\vec{F}_{12}$$

so $\vec{F}_{21} + \vec{F}_{12} = 0$ internal forces cancel!

$$\sum_i \vec{F}_i = \vec{F}_{ext\,1} + \vec{F}_{ext\,2} = \sum_i \vec{F}_{ext\,i}$$

This is true for: 2 particles

3 particles

⋮

∞ particles

Why?

Because for <u>each</u> pair of particles, the internal forces cancel.

$$\vec{F}_{47,89} = -\vec{F}_{89,47}$$

The total force acting on a system of particles equals the total <u>external</u> force acting on the system.

Where are we?

$$M_{total}\,\vec{a}_{cm} = \sum_i \vec{F}_{external_i} = \vec{F}_{\substack{total\\external}}$$

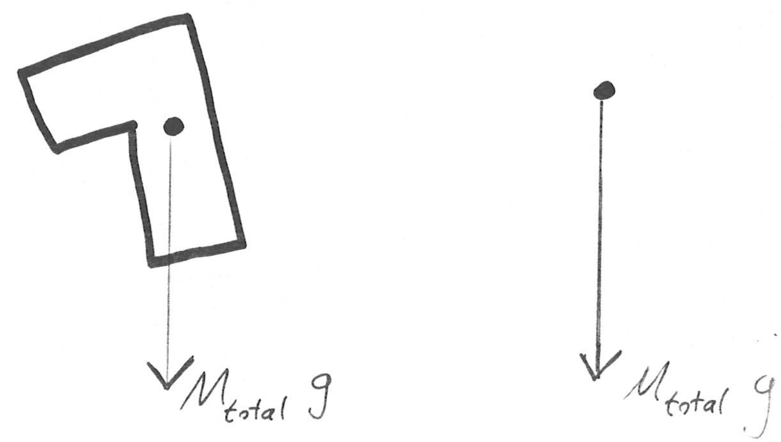
This is our justification for treating extended objects like point particles.

Internal forces don't matter!

A system of particles behaves as if it were a point particle with all of its mass (M_{total}) concentrated at the center of mass ($\vec{r}_{cm}$) acted on only by external forces, applied at the center of mass.

Example:

Free body diagram for an extended object thrown across the room.



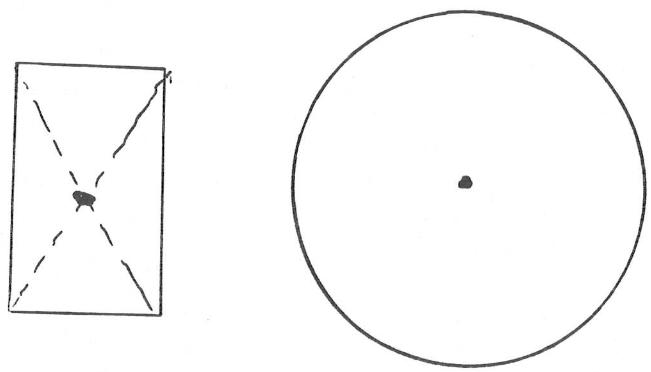
$$\vec{F}_{\substack{total\\ext}} = M_{total}\,\vec{a}_{cm}$$

Simplifies kinematics and dynamics greatly!

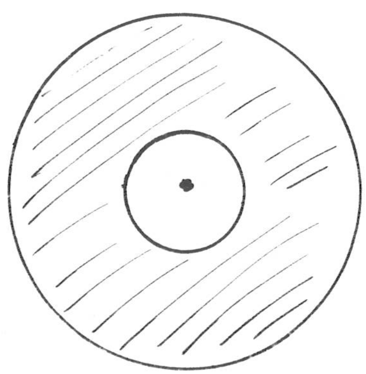
Demo ~

For symmetric objects, we don't even need to integrate.

Examples:

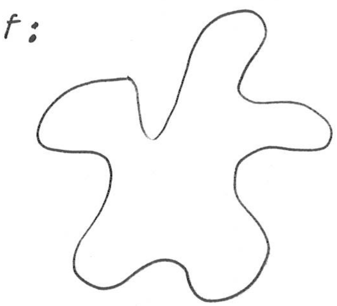


The center of mass need not be within the extended body.



What about shapes that cannot be integrated?

How can I find the center of mass of:



Suppose that I support an object from some arbitrary point.

Where does its C-of-M lie?

?

The C-of-M must lie directly below the support point.

Why?

The object behaves as if all of its mass is at the C-of-M.

The object will be in equilibrium when the C-of-M is at the lowest position possible. This is where the potential energy is a minimum.

How does this fact help us to find the C-of-M?

Demo^2