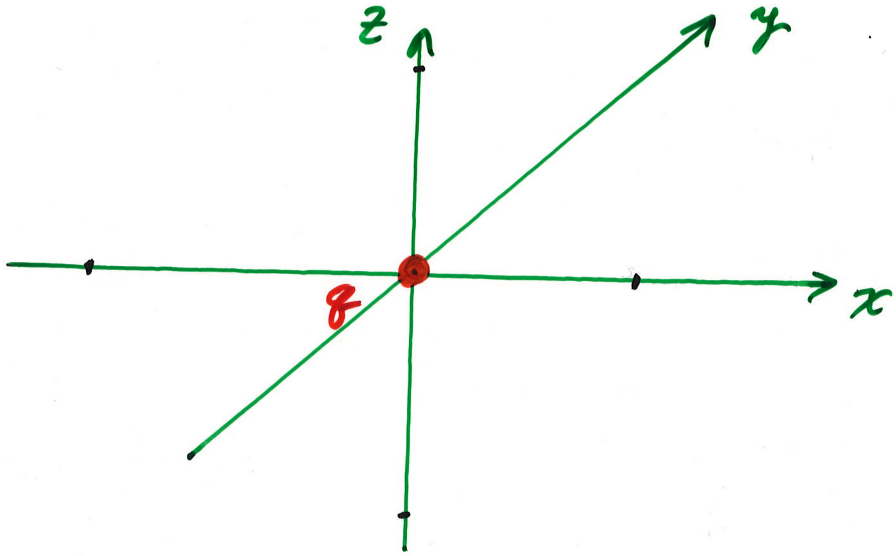
$\vec{E}$ field due to a positive point charge $\underline{q}$ at the origin:



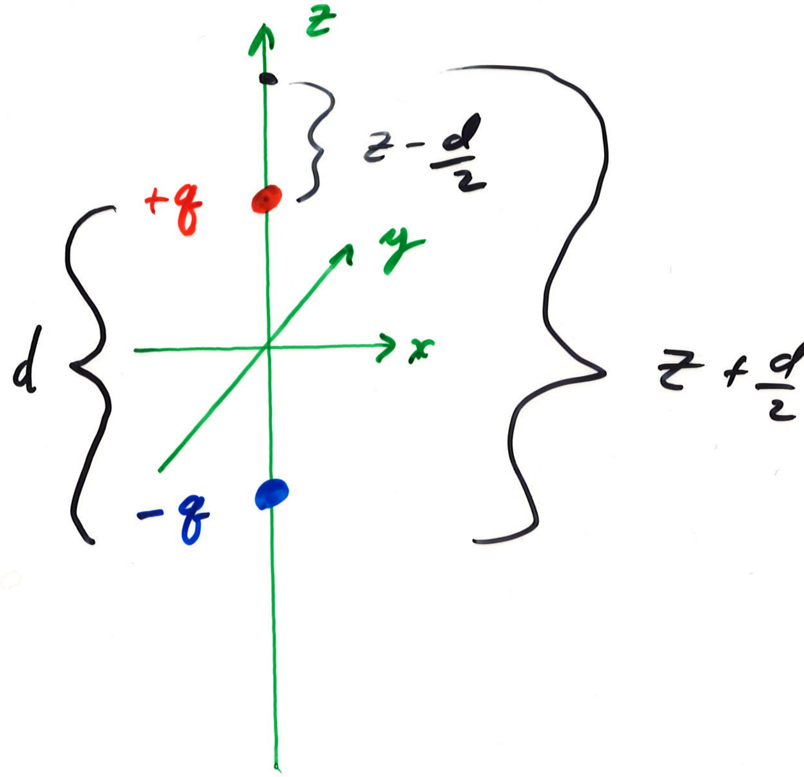
Force:

$$\underset{\substack{\text{on}\\ \text{test}\\ \text{charge}}}{\vec{F}}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q \;\; dq}{r^2} \hat{r}$$

Electric Field:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}$$

$\vec{E}$ field of an electric dipole (on axis):



$$\left(z-\frac{d}{2}\right)^2 = z^2\left(1-\frac{d}{2z}\right)^2$$

For points on the z-axis:

$$E(z) = E_{(+)} + E_{(-)}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{\left(z-\frac{d}{2}\right)^2} + \frac{1}{4\pi\epsilon_0}\frac{-q}{\left(z+\frac{d}{2}\right)^2}$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\left(1-\frac{d}{2z}\right)^{-2} - \left(1+\frac{d}{2z}\right)^{-2}\right]$$

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \frac{n(n-1)(n-2)x^3}{3!} + \cdots$$

$$|x| < 1$$

$$\left(1 - \frac{d}{2z}\right)^{-2} = 1 + (-2)\left(\frac{-d}{2z}\right) + \cdots$$

$$= 1 + \frac{d}{z} + \cdots$$

$$\left(1 + \frac{d}{2z}\right)^{-2} = 1 + (-2)\left(\frac{d}{2z}\right) + \cdots$$

$$= 1 - \frac{d}{z} + \cdots$$

$\vec{E}$ field of an electric dipole on the dipole axis:

$$E(z) = \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\left(1-\frac{d}{2z}\right)^{-2} - \left(1+\frac{d}{2z}\right)^{-2}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\left(1+\frac{d}{z}+\cdots\right) - \left(1-\frac{d}{z}+\cdots\right)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{q}{z^2}\left[\frac{2d}{z}+\cdots\right]$$

p is electric dipole moment

$$= \frac{1}{4\pi\epsilon_0}\frac{2(qd)}{z^3} + \cdots = \frac{1}{4\pi\epsilon_0}\frac{2p}{z^3} + \cdots$$

First term in an infinite series. This is a good approximation as long as

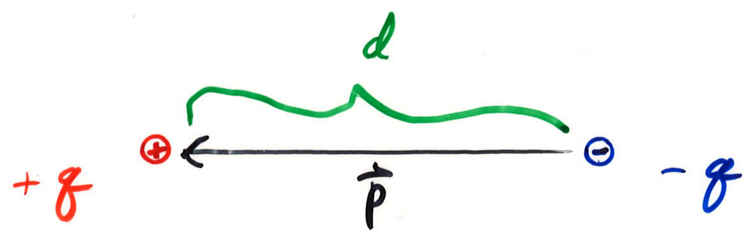
$z \gg d$

Electric Dipole Moment

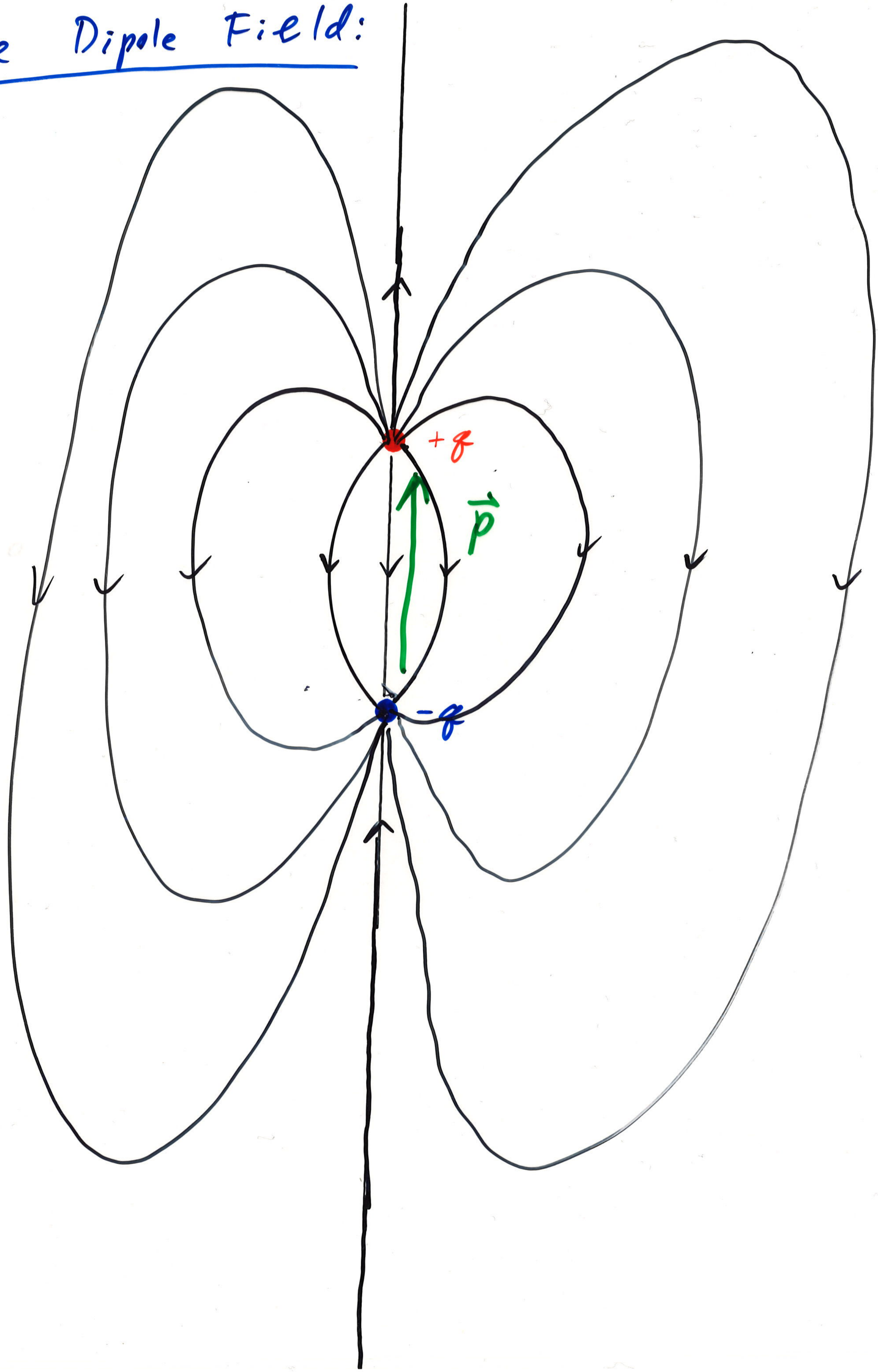
If two charges, each of magnitude $\underline{q}$, but of opposite sign are separated by a distance $\underline{d}$, that configuration has an electric dipole moment

$$|\vec{p}| = q\,d$$

The direction of $\vec{p}$ is from the negative charge to the positive one, Opposite to the $\vec{E}$ field.



The Dipole Field:



Mechanics

$\vec{F} = m\vec{a}$ Newton's 2nd Law

$$\vec{F} = m \frac{d^2\vec{r}}{dt^2}$$

Electricity + Magnetism

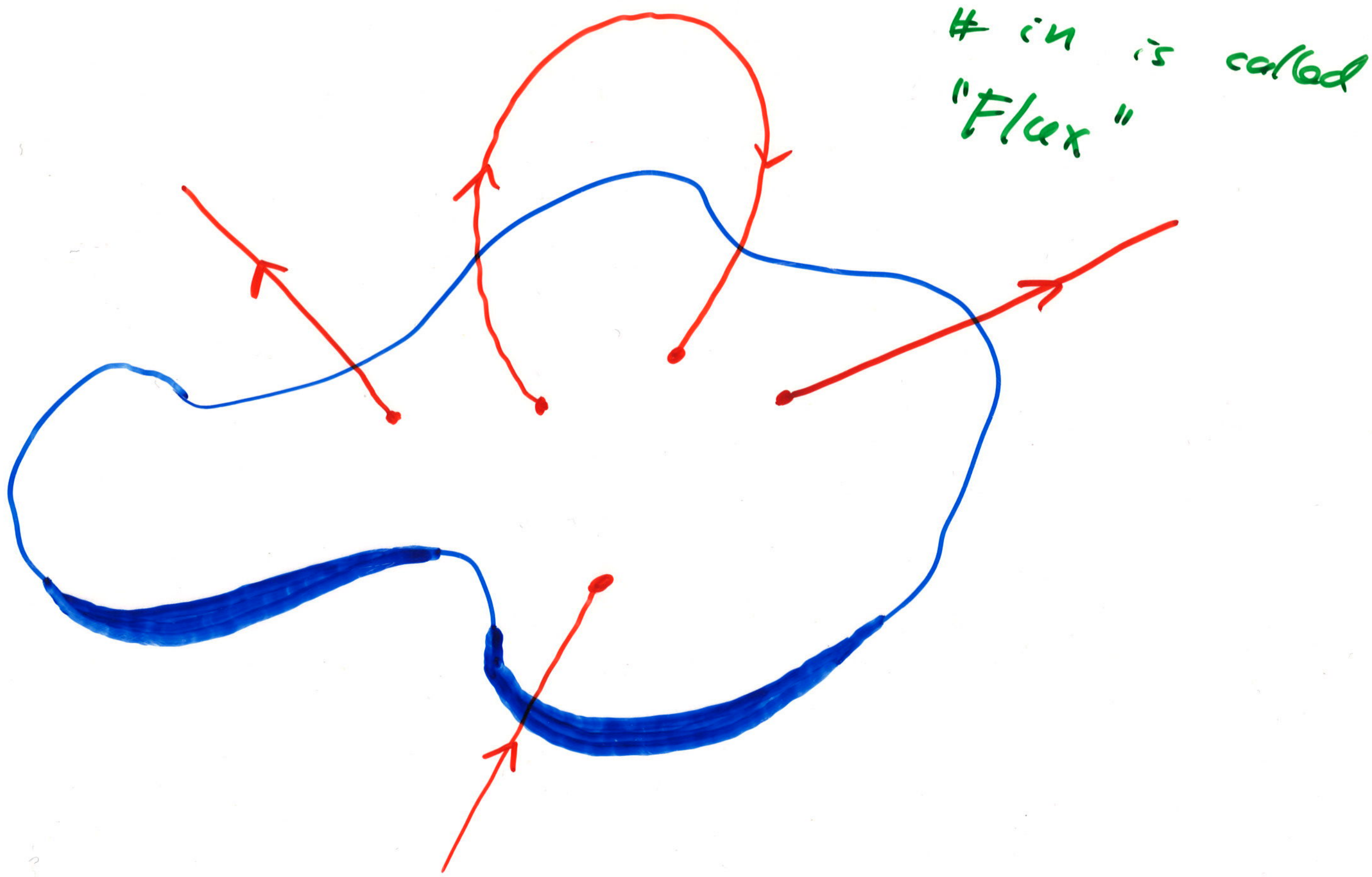
4 Maxwell's Equations

the first one that we will study is

⋮

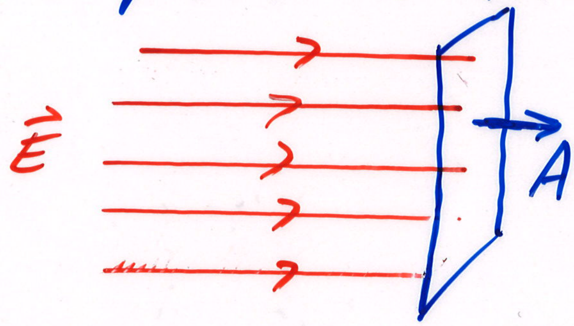
Gauss' Law

The number of electric field lines poking out through a closed surface minus the number of field lines poking into the surface tells you something about the net charge enclosed within that surface.



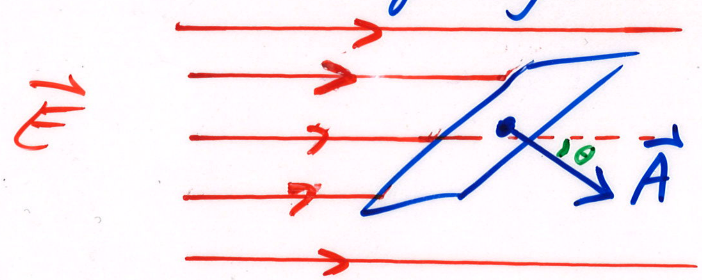
Special Cases

1) Constant $\vec{E}$ field perpendicular to a piece of flat Area:



number of field lines $\propto |\vec{E}|\, A$

2) Constant $\vec{E}$ field striking a piece of flat Area obliquely:



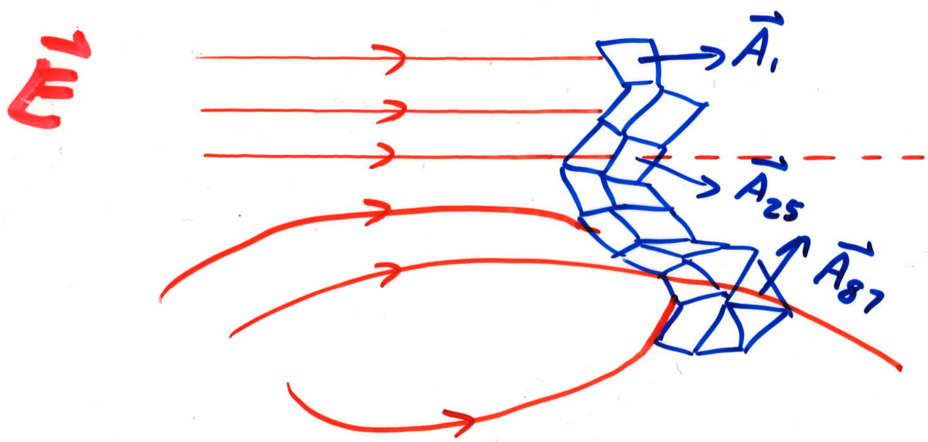
number of field lines <u>out</u> $\propto \vec{E} \cdot \vec{A}$

$= |\vec{E}|\,|\vec{A}| \cos\theta$

$\vec{A}$ points <u>out</u> of the closed surface

What if the electric field is <u>not</u> constant, and the piece of Area is <u>not</u> flat?

Break the surface into tiny regions which are <u>almost</u> flat and over which the $\vec{E}$ field does not vary considerably.



net number of field lines out

$$\propto \sum_{i=1}^{N} \vec{E}_i \cdot \vec{A}_i$$

To obtain a more accurate answer, let the size of each piece of Area shrink to zero while increasing the number N of pieces:

$$\lim_{N\to\infty} \sum_{i=1}^{N} \vec{E}_i \cdot \vec{A}_i = \iint \vec{E}(\vec{r}) \cdot \vec{dA}$$

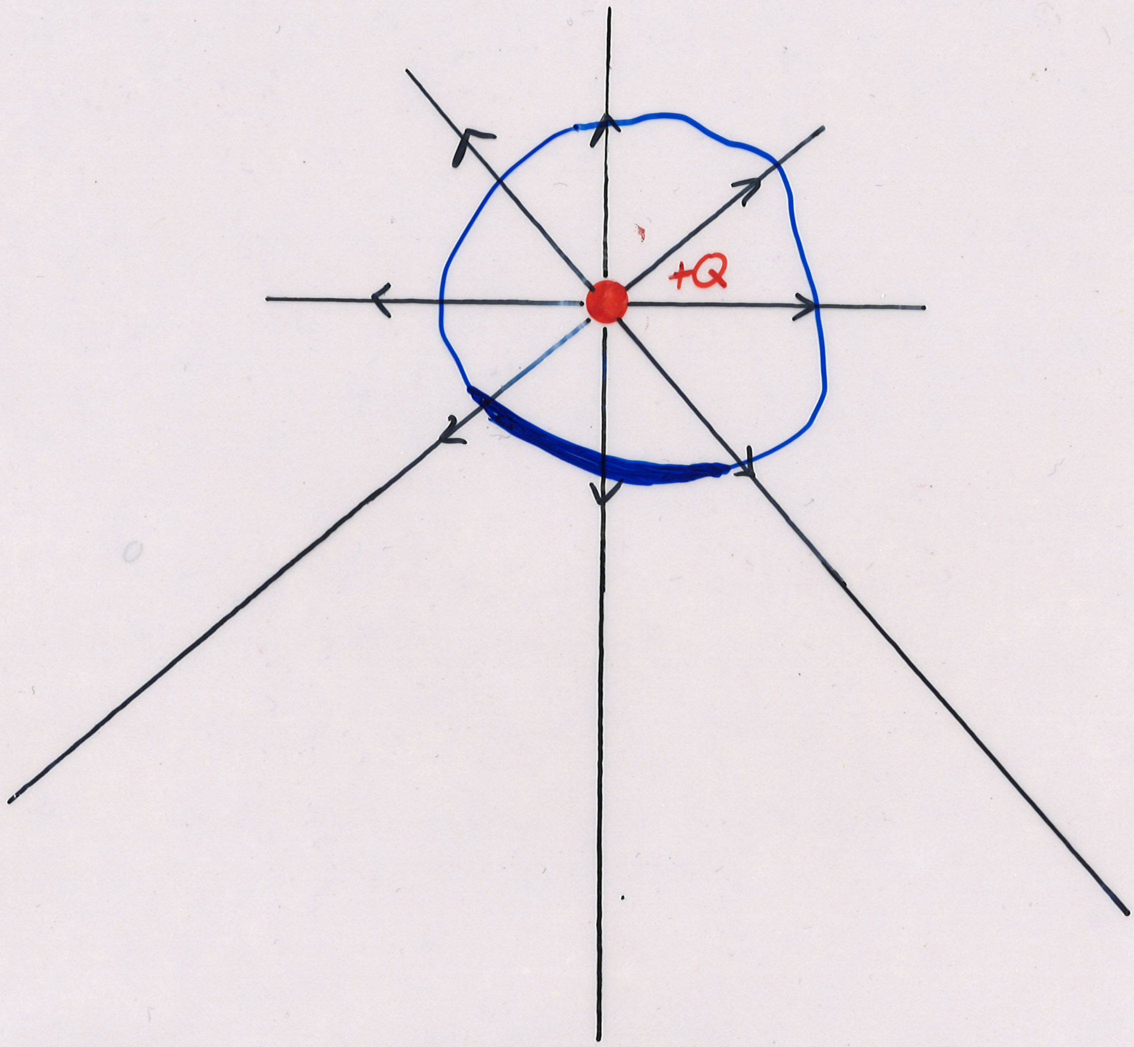
Gauss' Law

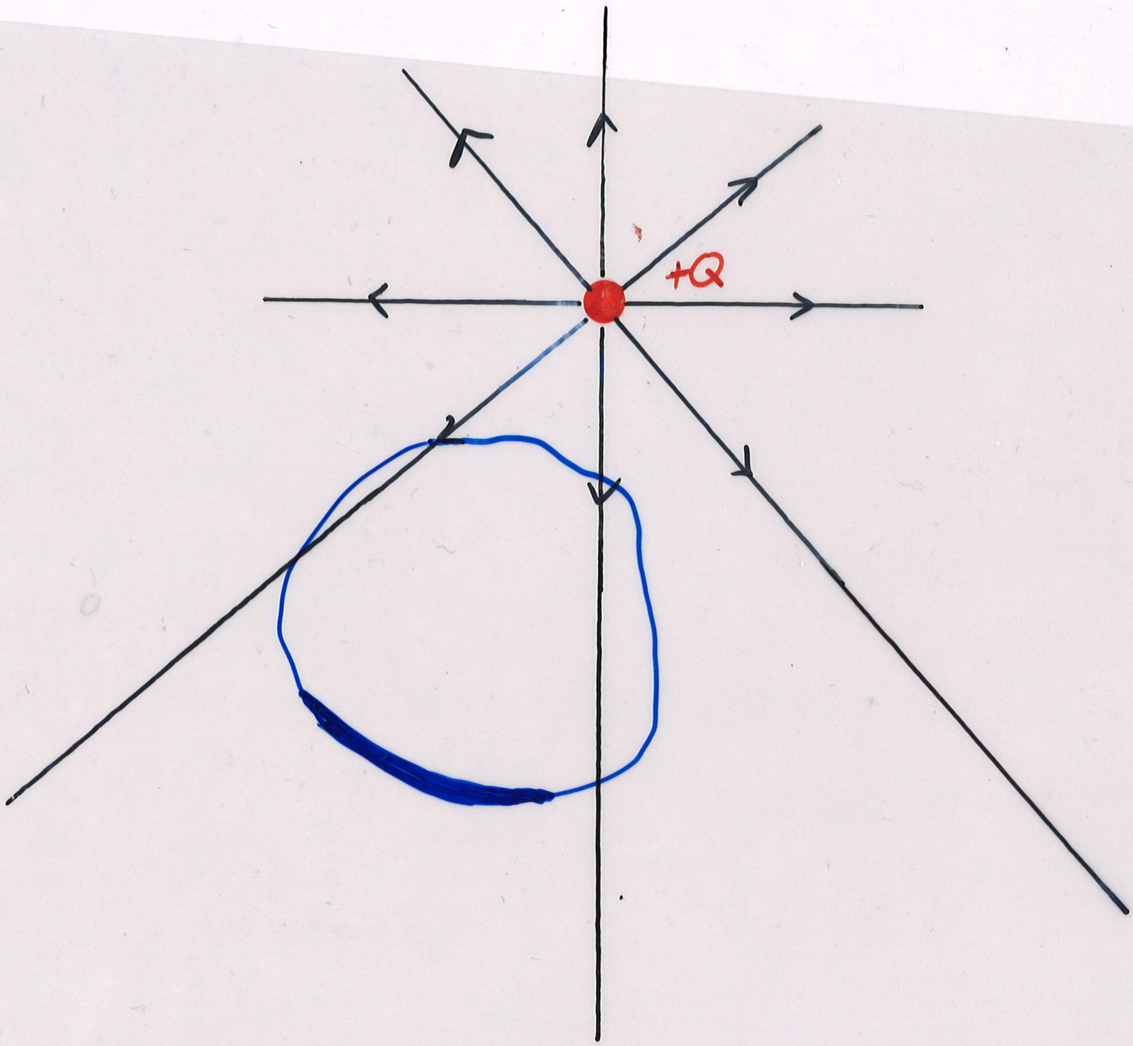
Flux

$$\oiint \vec{E}(\vec{r}) \cdot \vec{dA} = \frac{Q_{enc}}{\varepsilon_0}$$

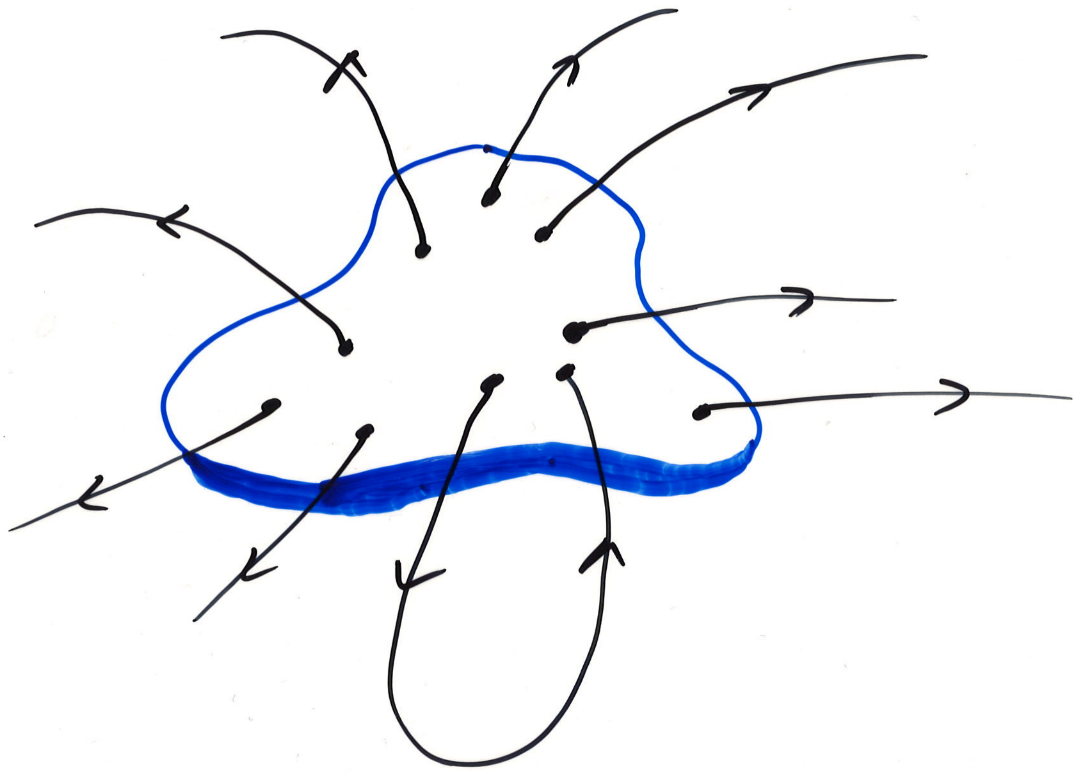
surface is closed

only the charge enclosed by the surface


+Q


+Q

Suppose that you couldn't see inside the <u>mathematical</u> surface.



What is inside?

<u>net</u> charge ^{+}Q inside.

$3Q$ and $-2Q$

or $47Q$ and $-46Q$

Gauss' Law: $\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$

and Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2} \hat{r}_{12}$

contain exactly the <u>same</u> physics.

Recall Newton's 2nd Law: $\vec{F} = m\vec{a}$

and Energy Conservation:

$$U_i + K_i = U_f + K_f$$

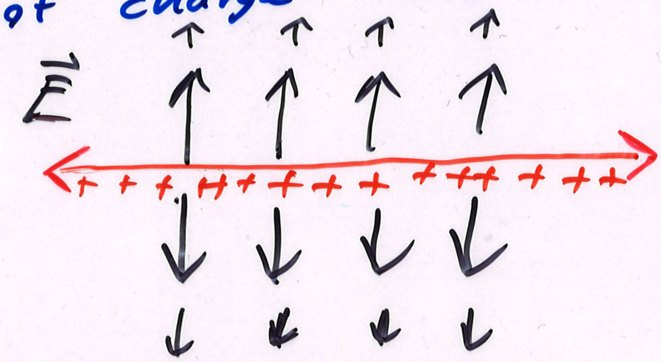
also contain the same physics.

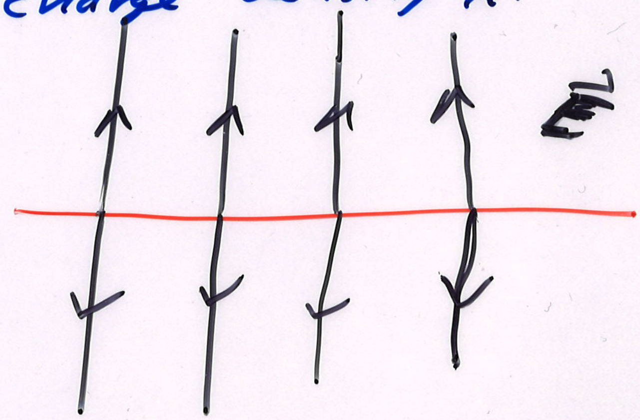
They are different methods for obtaining the same answer. Usually, one is easier than the other for a given problem.

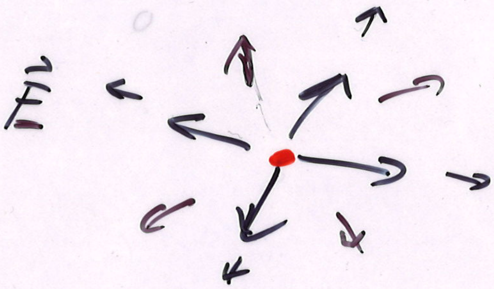
Gauss' Law is easy when the problem has a high degree of <u>symmetry</u>.

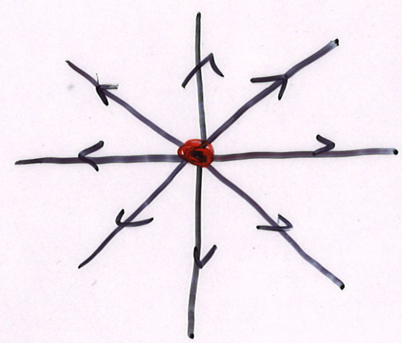
Derive the electric due to a positive point charge +Q.

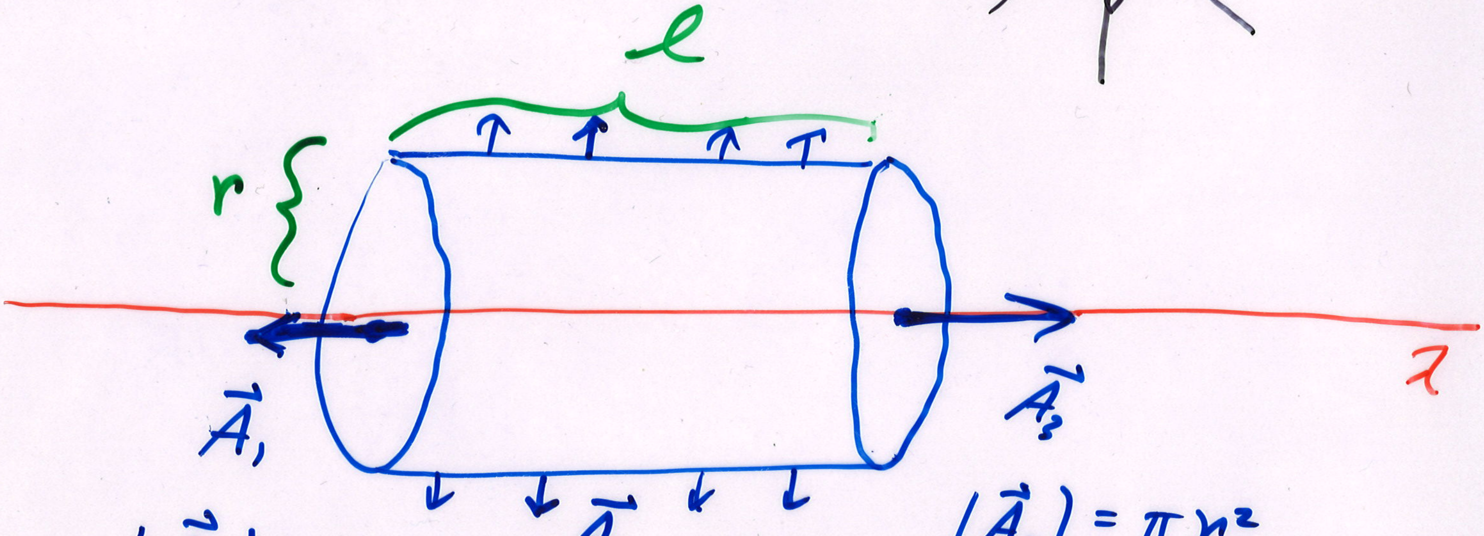
Use Gauss' Law to derive the electric field due to an <u>infinite</u> line of charge with linear charge density λ.











$|\vec{A}_1| = \pi r^2$

$|\vec{A}_3| = \pi r^2$

$|\vec{A}_2| = 2\pi r \ell$

$$\vec{E}(\vec{r}) \cdot \vec{A}_1 = 0 \qquad \vec{E}(\vec{r}) \cdot \vec{A}_3 = 0$$

$\vec{E}(\vec{r})$ is parallel to $d\vec{A}_2$ at all points.

$$\oint\!\!\oint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$= \iint_{A_1} \cancelto{0}{\vec{E} \cdot d\vec{A}_1} + \iint_{A_2} \vec{E} \cdot d\vec{A}_2 + \iint_{A_3} \cancelto{0}{\vec{E} \cdot d\vec{A}_3}$$

$$= |\vec{E}_{(r)}| \iint_{A_2} dA_2$$

$\vec{E} \cdot d\vec{A}_2 = |\vec{E}|\,|dA|_2$

$$= |\vec{E}_{(r)}| (2\pi r \ell) = \frac{Q_{enc}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}$$

elec field a distance r away from line

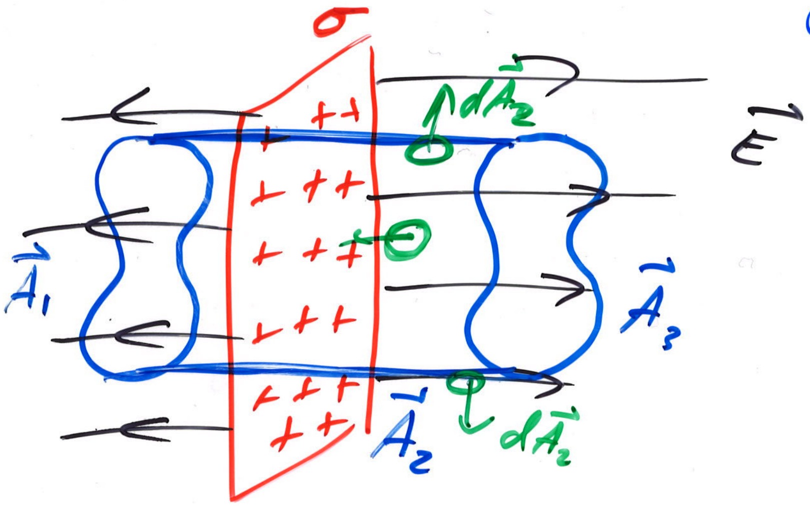
$$|\vec{E}(r)|\, 2\pi r \cancel{\ell} = \frac{\lambda \cancel{\ell}}{\epsilon_0}$$

$$|\vec{E}(r)| = \frac{\lambda}{2\pi\epsilon_0 r} = \frac{2\lambda}{4\pi\epsilon_0 r}$$

direction by symmetry away from wire

Use Gauss' Law to derive the electric field due to an infinite sheet of positive charge with constant surface charge density σ.

Gaussian pillbox



$$\vec{E}(\vec{r}) \cdot d\vec{A}_2 = 0$$

$$|\vec{A}_1| = |\vec{A}_3|$$

$$\oiint_S \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$\iint_{A_1} \vec{E} \cdot d\vec{A}_1 + \cancelto{0}{\iint_{A_2} \vec{E} \cdot d\vec{A}_2} + \iint_{A_3} \vec{E} \cdot d\vec{A}_3 = \frac{\sigma A_1}{\epsilon_0}$$

$$\iint_{A_1} E \, dA_1 + \iint_{A_3} E \, dA_3 = \frac{\sigma A_1}{\epsilon_0}$$

$$= E \iint_{A_1} dA_1 + E \iint_{A_3} dA_3$$

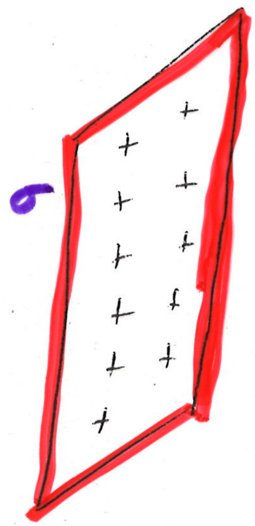
$$= 2E \cancel{A_1} = \frac{\sigma \cancel{A_1}}{\epsilon_0}$$

$$|\vec{E}| = \frac{\sigma}{2\epsilon_0}$$

$$= \frac{1}{4\pi\epsilon_0} 2\pi\sigma$$

$\vec{E} = \frac{\sigma}{2\epsilon_0}(-\hat{i})$ (left) $\qquad \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{i}$ (right)

To see that Gauss' Law is easier than Coulomb's Law for the case of an infinite sheet of charge, we will set up the Coulomb's Law calculation (but <u>not</u> solve it!)

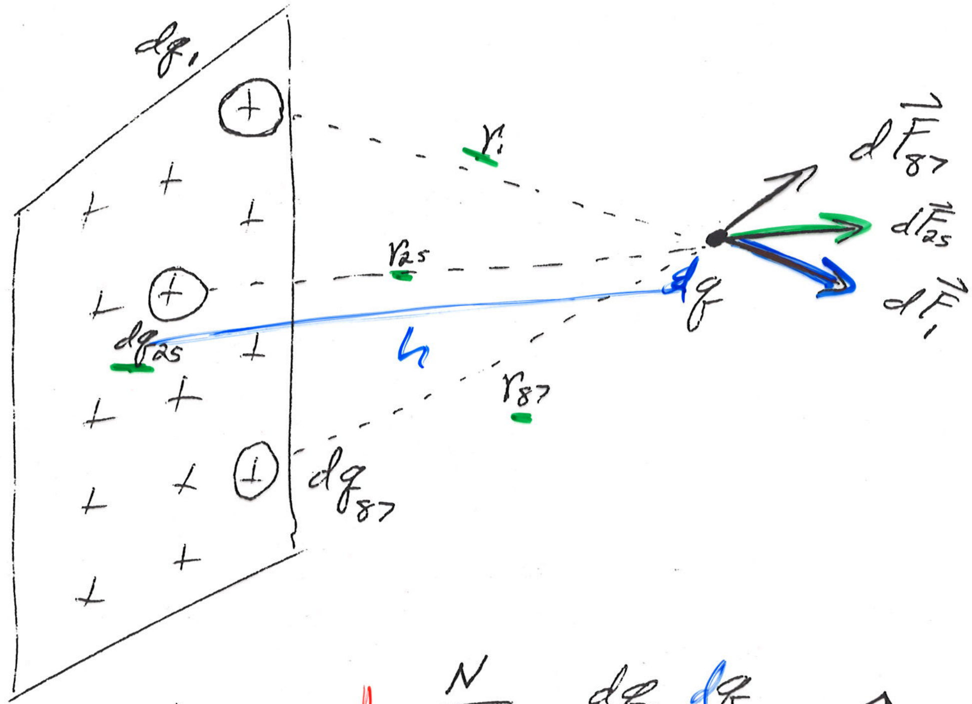


dq

positive infinitesimal test charge

1) Find the total vector force $\vec{F}$ acting on the test charge dq.

2) Divide by dq to get the electric field $\vec{E}$ (force per unit charge).



$$\vec{F}_{\substack{total\\ \text{on } dq}} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{dq_i\, dq}{(r_i)^2}\,\hat{r}_i$$

$$= \frac{dq}{4\pi\epsilon_0}\sum_{i=1}^{N} \frac{dq_i}{(r_i)^2}\,\hat{r}_i$$

$$\xrightarrow[N\to\infty]{} \frac{dq}{4\pi\epsilon_0}\iint \frac{dq}{r^2}\,\hat{r}$$

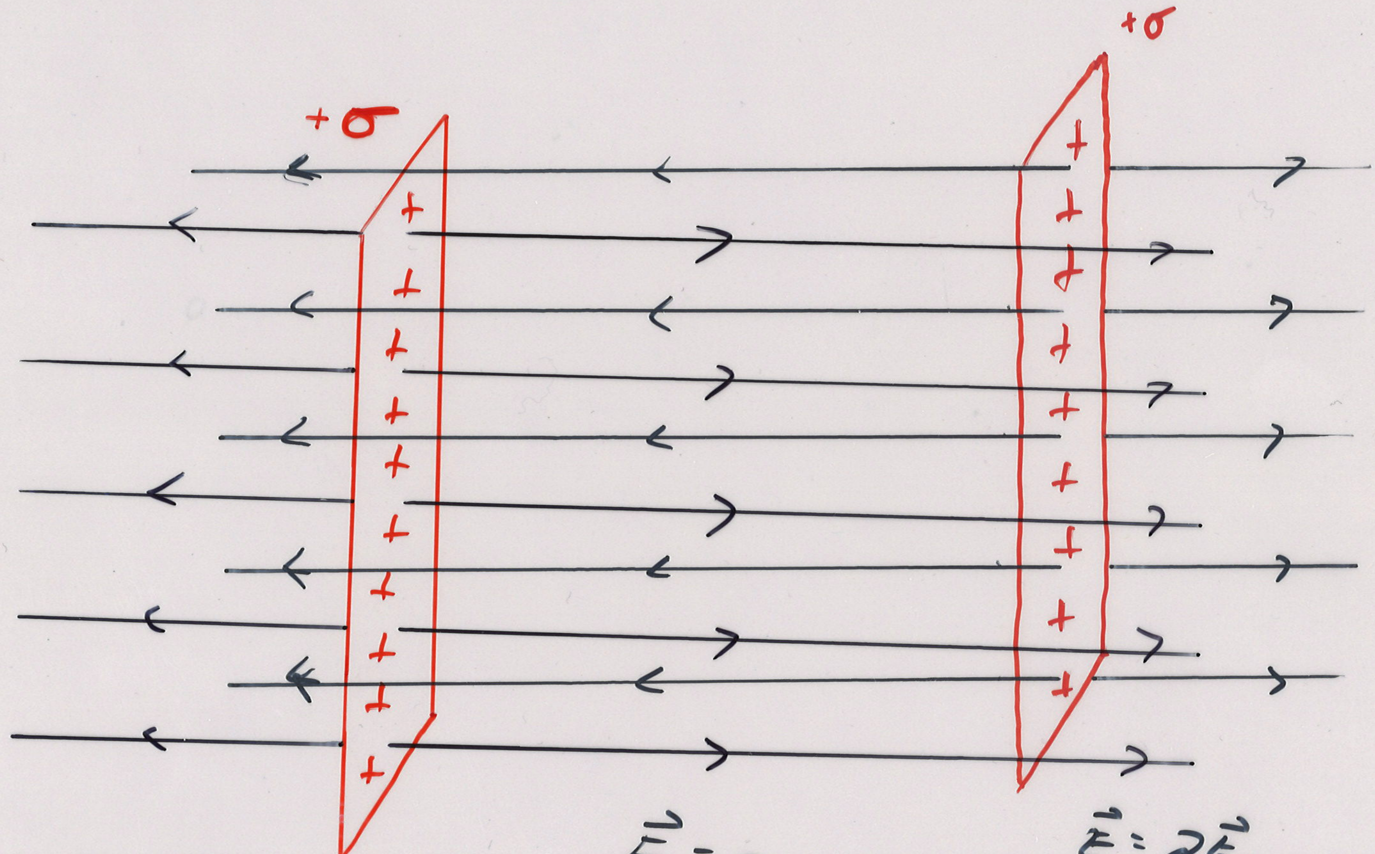
$$\boxed{dq = \sigma\, dx\, dy = \sigma\, dA}$$

$$= \frac{dq}{4\pi\epsilon_0}\int_{x=-\infty}^{\infty}\int_{y=-\infty}^{\infty} \frac{\sigma\, dx\, dy}{r^2}\,\hat{r}$$

Obviously Gauss' Law is much easier!

$$r^2 = h^2 + x^2 + y^2$$

The electric field due to many sheets of charge. Superposition

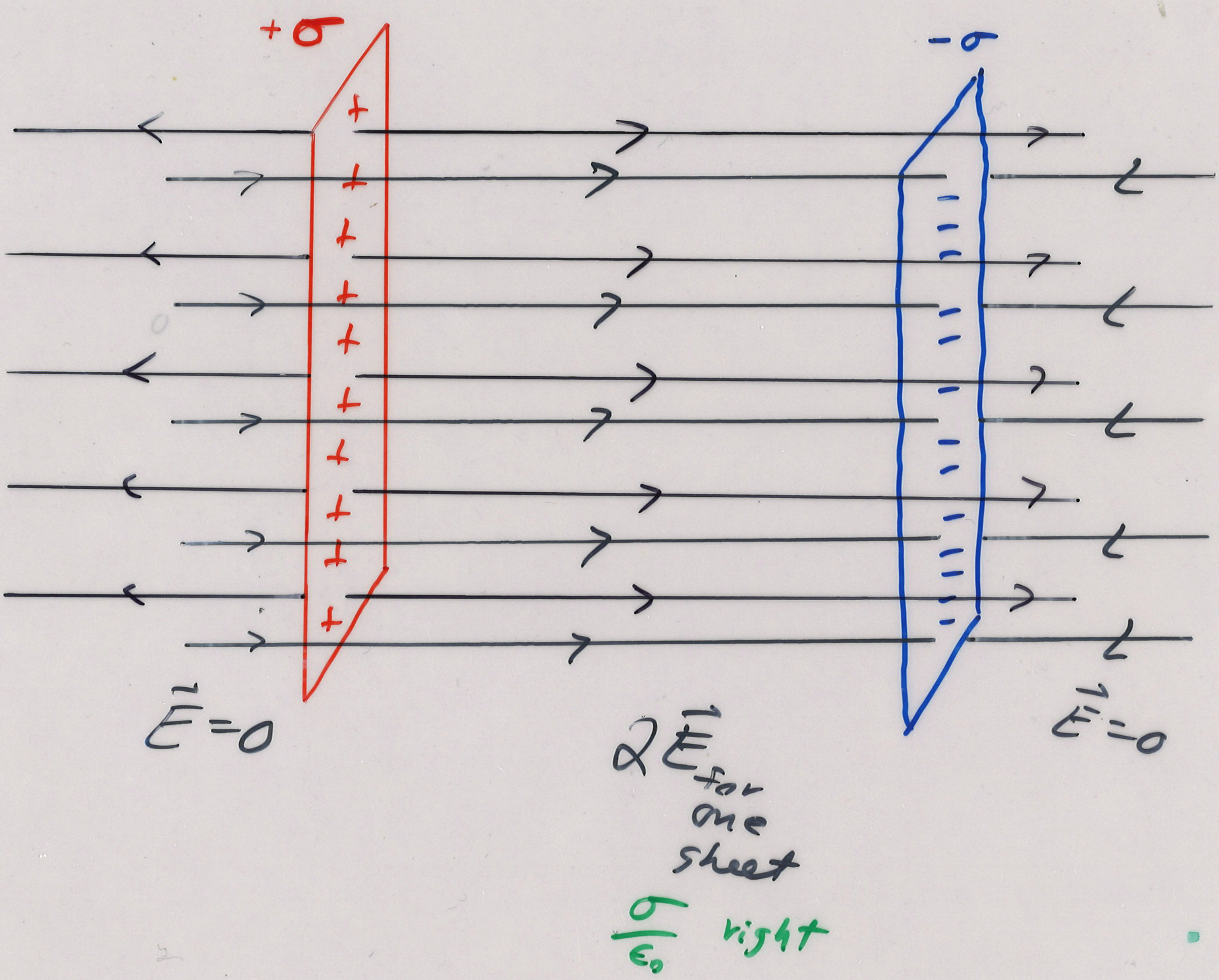


$\vec{E} = 2\vec{E}_{\text{for one}} = \frac{\sigma}{\epsilon_0}$ left

$\vec{E} = 0$

$\vec{E} = 2\vec{E}_{\text{for one}} = \frac{\sigma}{\epsilon_0}$ right

The electric field due to many sheets of charge. Superposition



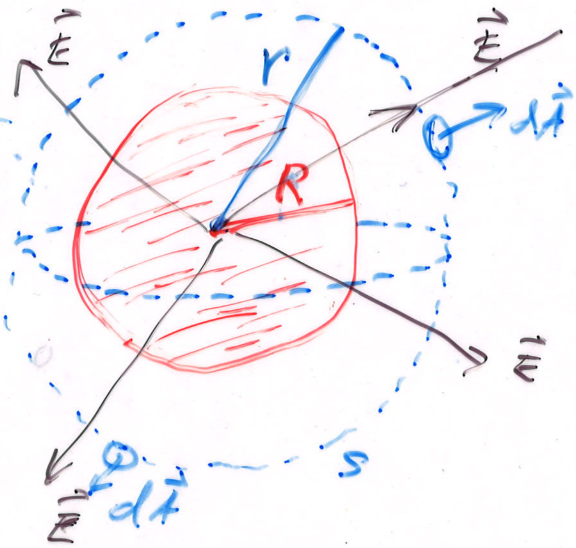
Consider a uniform ball of charge, radius $\underline{R}$, charge $\underline{Q}$. Find the electric field everywhere.

Gauss' Law:

$$\oint\!\!\!\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$\underline{r > R}$

outside



$$= \oint\!\!\!\oint_S E\,dA = \frac{Q}{\epsilon_0}$$

$$= E \oint\!\!\!\oint_S dA$$

$$= E\,4\pi r^2 = \frac{Q}{\epsilon_0}$$

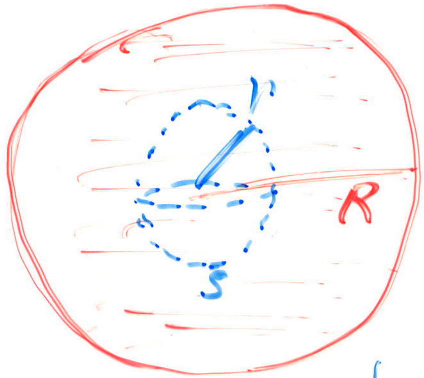
$$\boxed{\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}}$$

same $\vec{E}$ field as a point charge

a.k.a. Newton's Theorem

$\underline{r < R}$

inside



Gauss' Law

$$\oint_S \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

Volume charge density

$$\rho = \frac{Q}{\frac{4}{3}\pi R^3} = \text{constant}$$

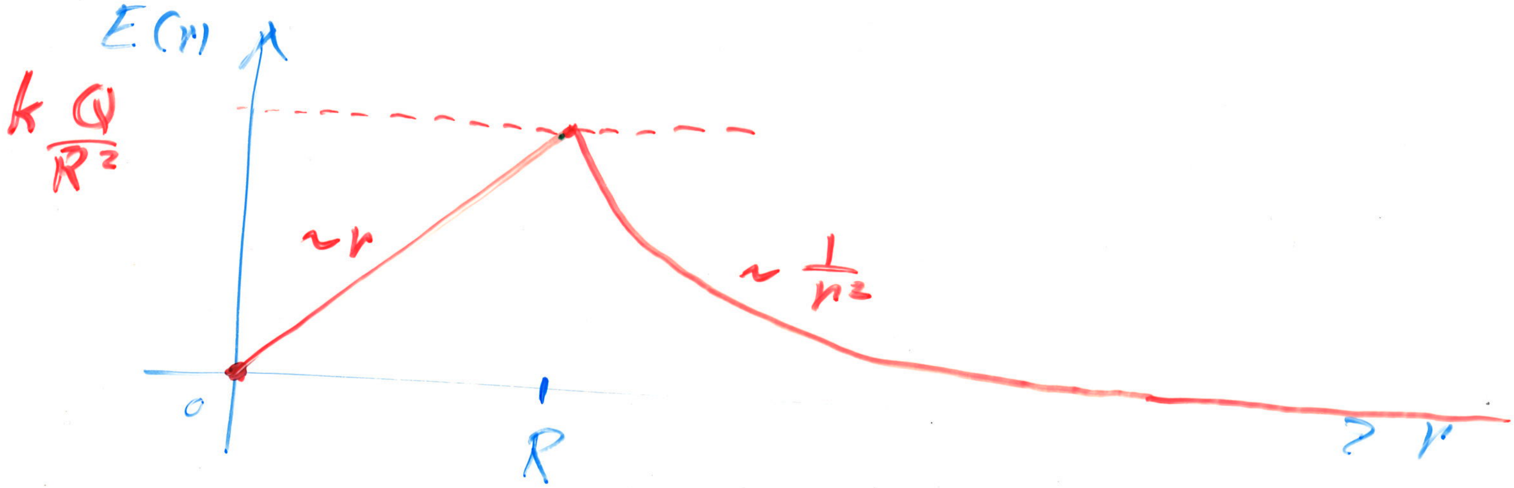
$$E\,4\pi r^2 = \frac{(\text{Volume of Gaussian sphere})\cdot \rho}{\epsilon_0}$$

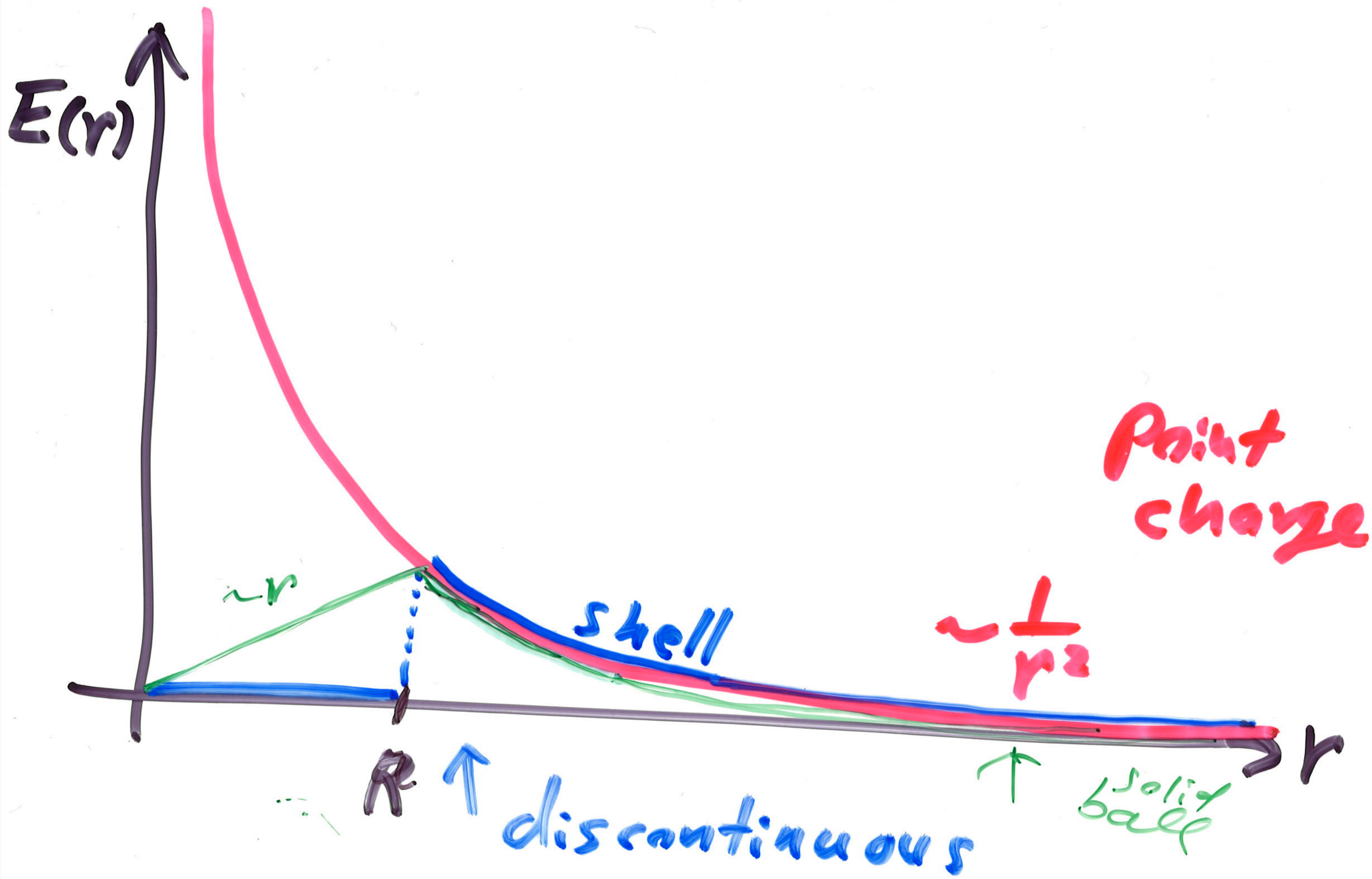
$$= \frac{\frac{4}{3}\pi r^3 \rho}{\epsilon_0}$$

$$= \frac{\cancel{\frac{4}{3}}\cancel{\pi} r^3 Q}{\cancel{\frac{4}{3}}\cancel{\pi} R^3 \epsilon_0}$$

$$E\,4\pi r^2 = \left(\frac{r^3}{R^3}\right)\frac{Q}{\epsilon_0}$$

$$\boxed{\vec{E}(r) = \frac{1}{4\pi\epsilon_0}\frac{rQ}{R^3}\hat{r}}$$




E(r)
Point charge
~r
Shell
$\sim \frac{1}{r^2}$
r
R
discontinuous
solid ball