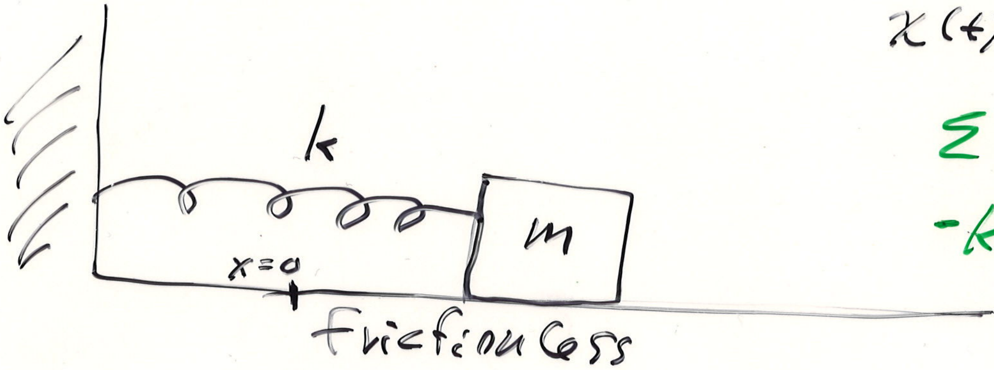
A trip down memory lane



$x(t)$, $v(t)$

$$\Sigma F_x = m a_x$$

$$-kx = m \frac{d^2x}{dt^2}$$

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0 \quad \text{Differential equation}$$

X $x(t) = 3t^2 + 5$

$$\frac{dx}{dt} = v(t) = 6t$$

$$\frac{d^2x}{dt^2} = a(t) = 6$$

$$6 + \frac{k}{m}\left(3t^2+5\right) = 0$$

must be true at all times

$$x(t) = \underline{A} \sin(\omega t + \underline{\varphi})$$

arbitrary

$$v(t) = \frac{dx}{dt} = A\omega \cos(\omega t + \varphi)$$

$$a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A\omega^2 \sin(\omega t + \varphi)$$

$$-A\omega^2 \sin(\omega t + \varphi) + \frac{k}{m} A \sin(\omega t + \varphi) = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

ω is angular frequency

units of $\frac{rad}{s}$ $\rightarrow$ MKS

---

f is linear frequency

unit of $Hz = \frac{1}{sec} = \frac{cycles}{sec} = \frac{rev}{sec}$

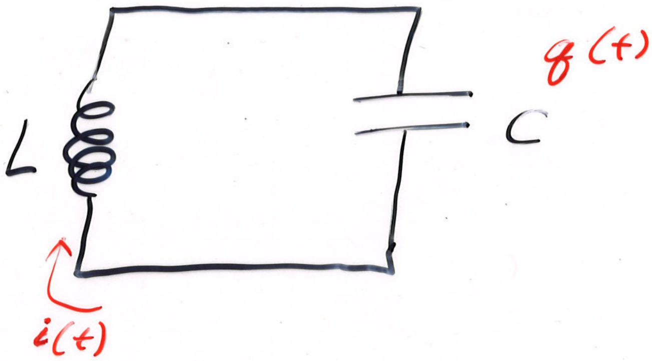
$$f = \frac{\omega}{2\pi}$$

'T' is period of oscillation

$$f = \frac{1}{T}$$

$$T = \frac{2\pi}{\omega}$$

Deja vu all over again



$$q = CV$$

$$V = \frac{q}{C}$$

$$i(t) = \frac{dq}{dt}$$

$$\frac{di}{dt} = \frac{d^2q}{dt^2}$$

Kirchhoff's Loop Rule

$$V_c + V_L = 0$$

$$\frac{q(t)}{C} + L\frac{di}{dt} = 0$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

solution: $q(t) = A\cos(\omega t + \varphi)$

$$i(t) = \frac{dq}{dt} = -A\omega\sin(\omega t + \varphi)$$

$$\frac{di}{dt} = \frac{d^2q}{dt^2} = -A\omega^2\cos(\omega t + \varphi)$$

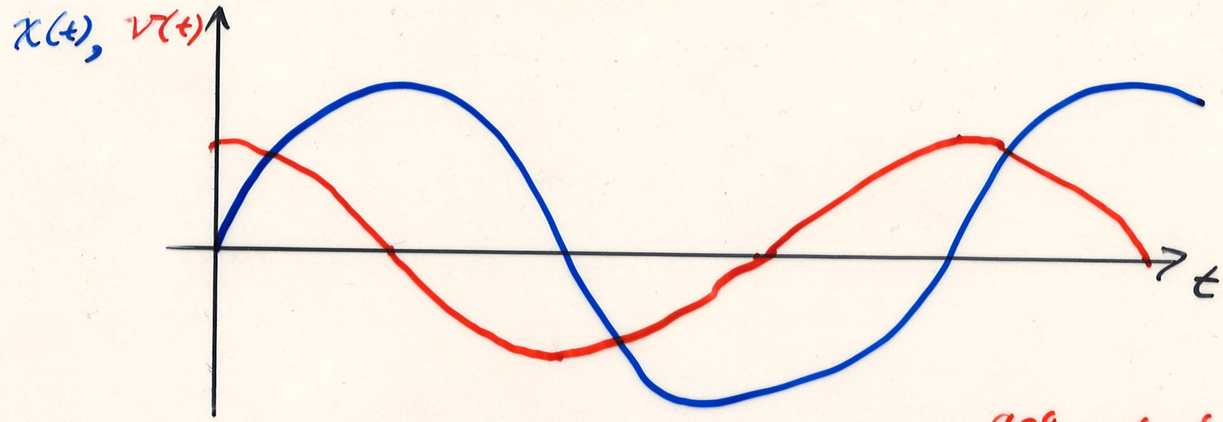
$$= -\omega^2 q(t)$$

$$\frac{d^2 q}{dt^2} + \frac{1}{LC} q = 0$$

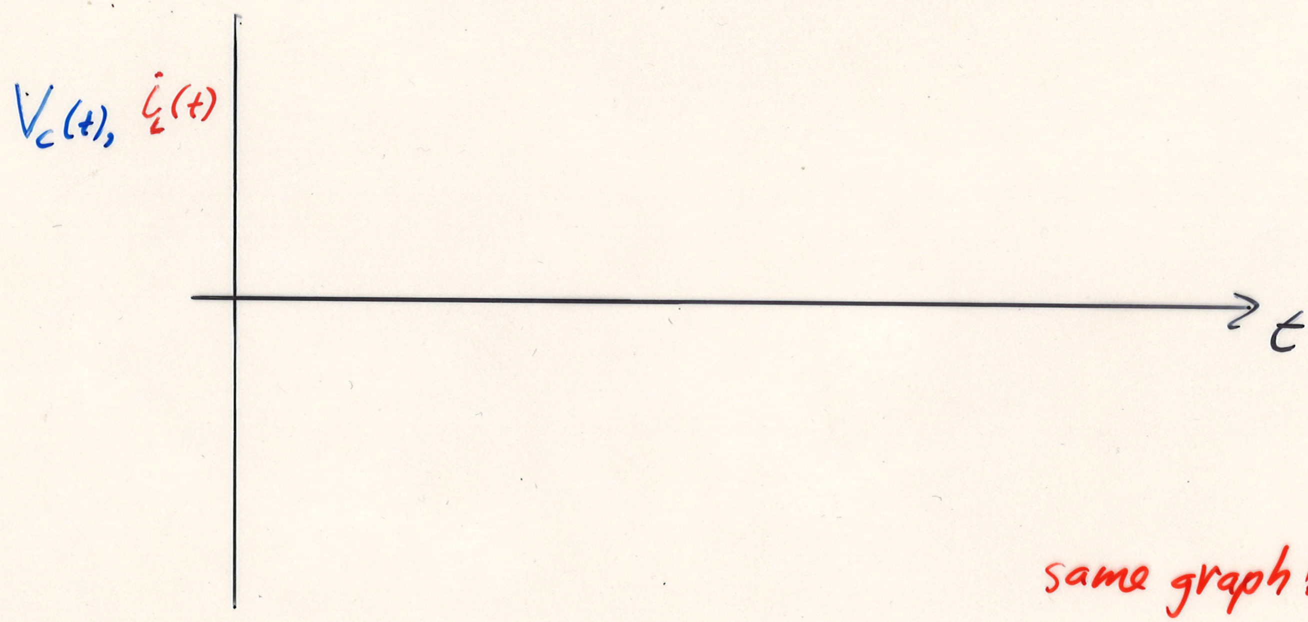
$$-\omega^2 q(t) + \frac{1}{LC} q(t) = 0$$

$$\Longrightarrow \omega = \sqrt{\frac{1}{LC}}$$ natural angular frequency

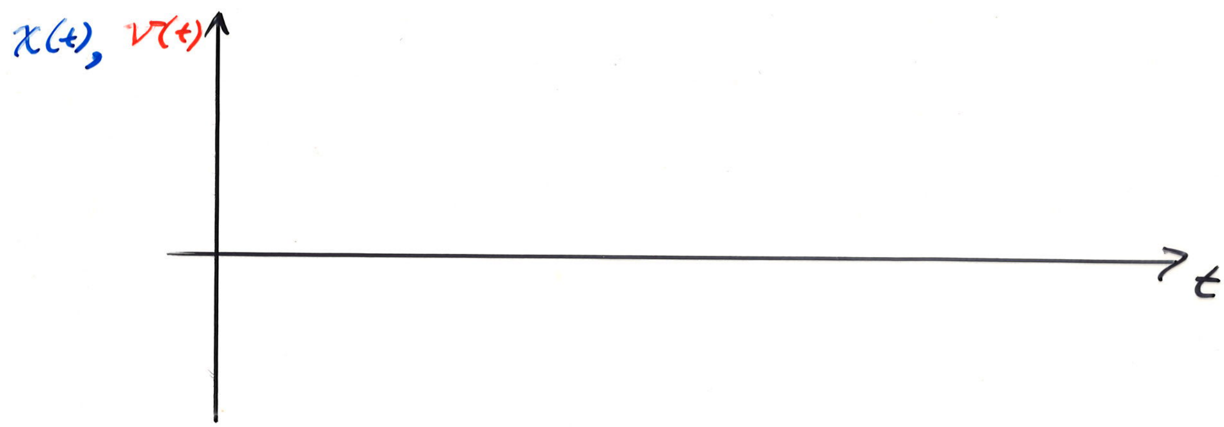
$[\omega] = \frac{1}{T}$ units $\frac{\text{rad}}{\text{s}}$

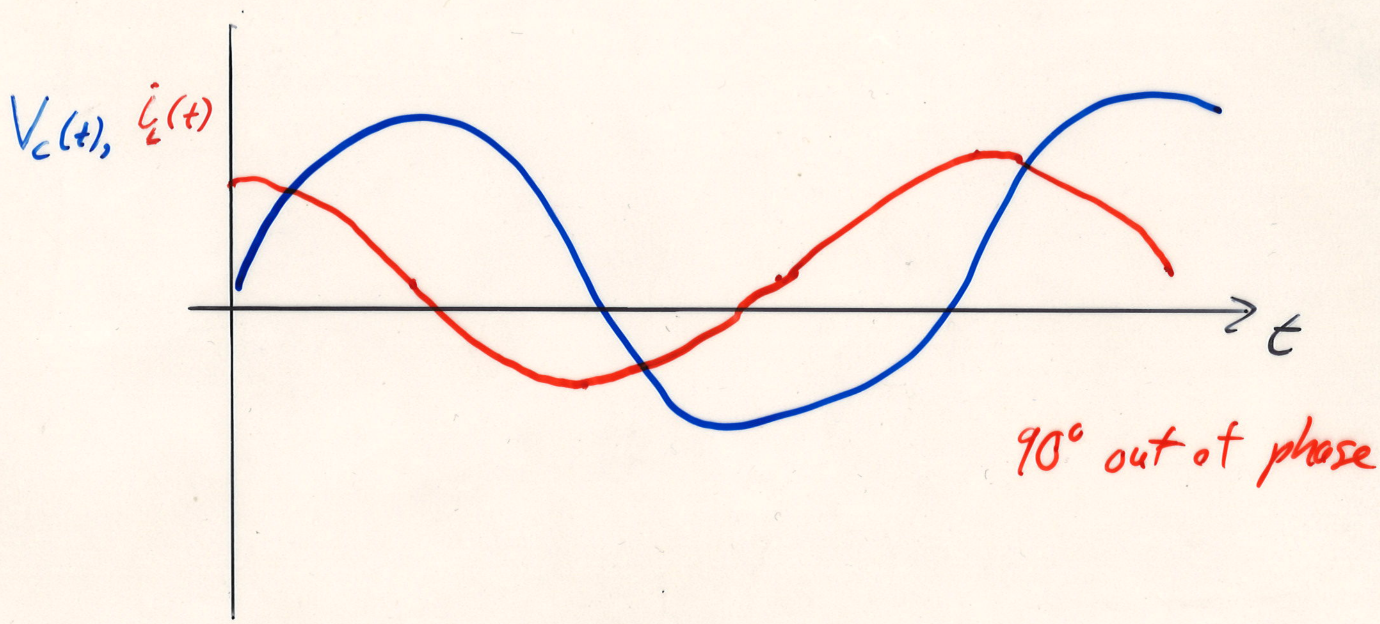


90° out of phase



same graph!


x(t), v(t)
t


Vc(t), iL(t)
t
90° out of phase

Conservation of Energy

Mech

$$E = K + U$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Conserved $\Rightarrow \frac{dE}{dt} = 0$ (E = constant)

$$0 = \frac{dE}{dt} = \frac{1}{2}m(2v)\underbrace{\frac{dv}{dt}}_{a} + \frac{1}{2}k(2x)\underbrace{\frac{dx}{dt}}_{v}$$ chain rule

$$0 = mva + kxv = v(ma + kx)$$

either $v = 0$ for all times rest

or $ma + kx = 0 = m\frac{d^2x}{dt^2} + kx$

oscillation

Elec

$$E_{total} = U_E + U_B$$
$$= \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2$$

Inductors store energy.
(so do capacitors!)

Capacitors store energy in the electric field:

$$U_E = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C}$$

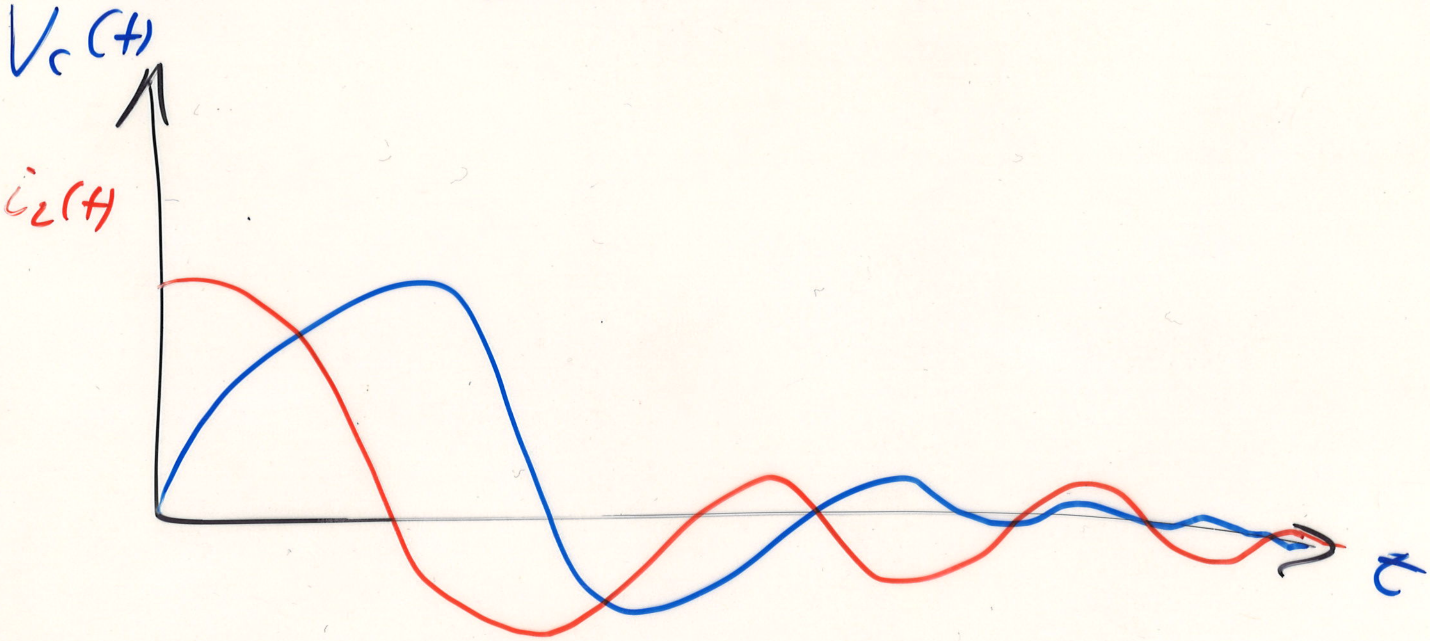
$\boxed{Q = CV}$

Inductors store energy in the magnetic field:

$$U_B = \frac{1}{2} L i^2$$

Resistance

If you add a resistor to an LC circuit then some energy from the electric and magnetic fields is converted into heat by the resistor. The oscillations die out exponentially.



Mechanical - Electrical Correspondences

x	q
$v = \frac{dx}{dt}$	$i = \frac{dq}{dt}$
m	L
k	$1/C$

large $k \leftrightarrow$ stiff spring $\leftrightarrow$ small capacitor

$m \uparrow$	$L \uparrow$	$\omega \downarrow$
$k \downarrow$	$C \uparrow$	$\omega \downarrow$

$\omega = \sqrt{\frac{k}{m}}$ $\qquad$ $\omega = \sqrt{\frac{1}{LC}}$

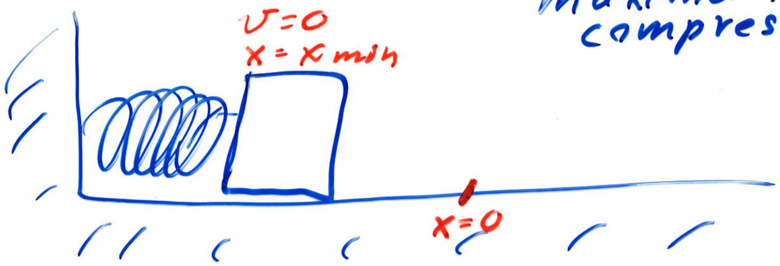
e.g. $L = 1\,mH$ $\quad$ $C = 1\,\mu f$

$\omega = 32{,}000 \frac{rad}{s}$

$f = \frac{\omega}{2\pi} = 5000 \frac{cycles}{s} = 5000\,Hz$

maximum compression

$t = 0$

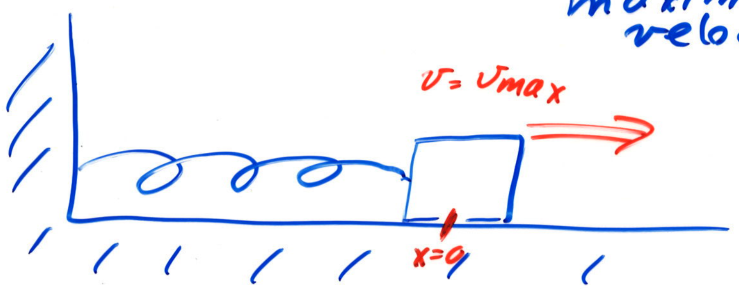


$U = \frac{1}{2} k \, x_{min}^2 = \frac{1}{2} k A^2$

$K = 0$

maximum velocity

$t = \frac{1}{4}$ cycle

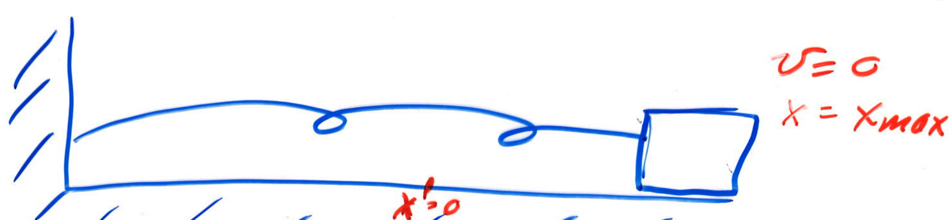


$U = 0$

$K = \frac{1}{2} m \, v_{max}^2 = \frac{1}{2} k A^2$

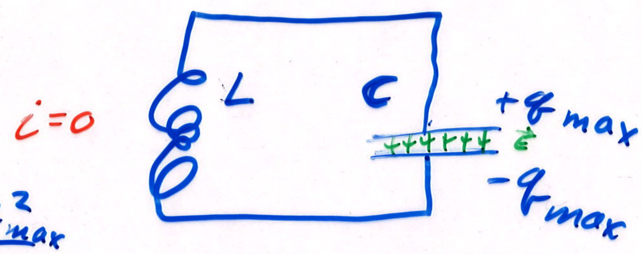
$t = \frac{1}{2}$ cycles

maximum extension



$U = \frac{1}{2} k \, x_{max}^2 = \frac{1}{2} k A^2$

$K = 0$

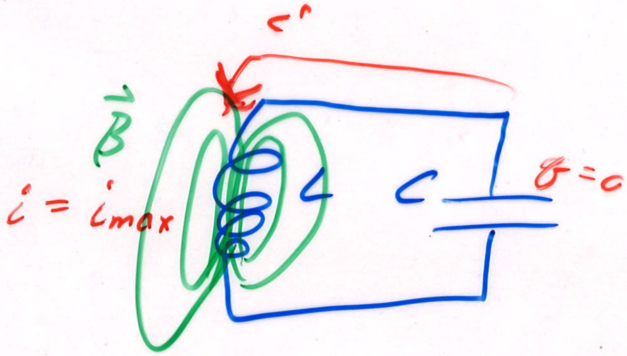


Capacitor fully charged (positive on the top plate)

$t=0$

$U_E = \frac{1}{2}\frac{q_{max}^2}{C}$

$U_B = 0$



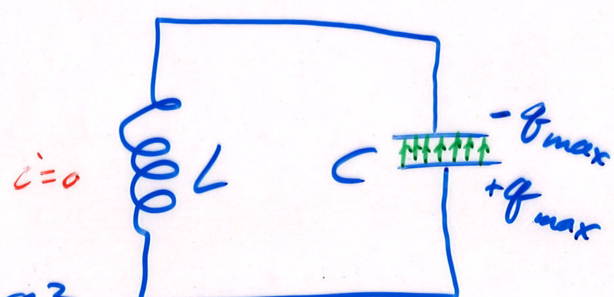
Capacitor discharged

$t = \frac{1}{4}$ cycle

$U_E = 0$

$U_B = \frac{1}{2} L i_{max}^2$

$t = \frac{1}{2}$ cycle



Capacitor fully charged (but positive on the bottom plate)

$U_E = \frac{1}{2}\frac{q_{max}^2}{C}$

$U_B = 0$

AC Circuits

(Alternating ~~Current~~ Voltage)

Consider a source of potential difference (seat of emf, voltage supply) whose voltage varies sinusoidally with time

$$V(t) = \underbrace{V_0}_{\text{maximum Voltage}} \sin(\omega t - \varphi)$$

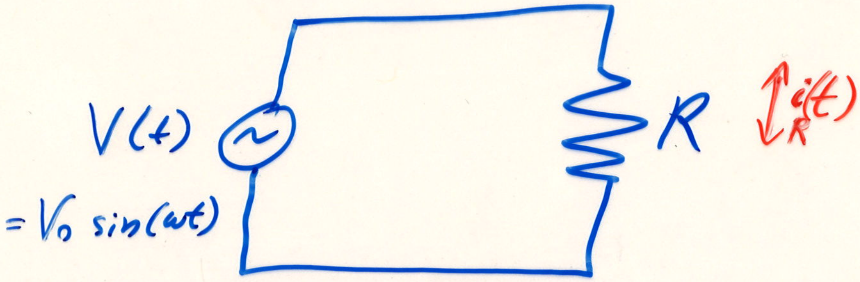
Put a resistor in a circuit with this Alternating Voltage. Demo

In the U.S. $V_0 = 163$ volts

$$f = 60\ Hz = 60\ \frac{\text{cycles}}{\text{sec}}$$

$$\omega = 2\pi f = 377\ \frac{\text{rad}}{\text{sec}}$$

A purely resistive circuit

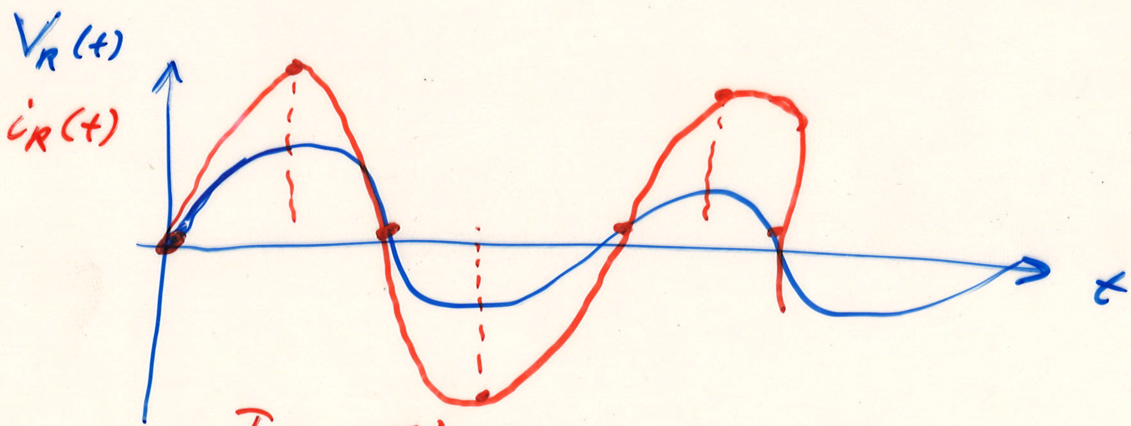


Voltage across the resistor:

$$V_R = V_0 \sin(\omega t)$$

Current through the resistor: $\quad V_{(t)} = i_{(t)} R$

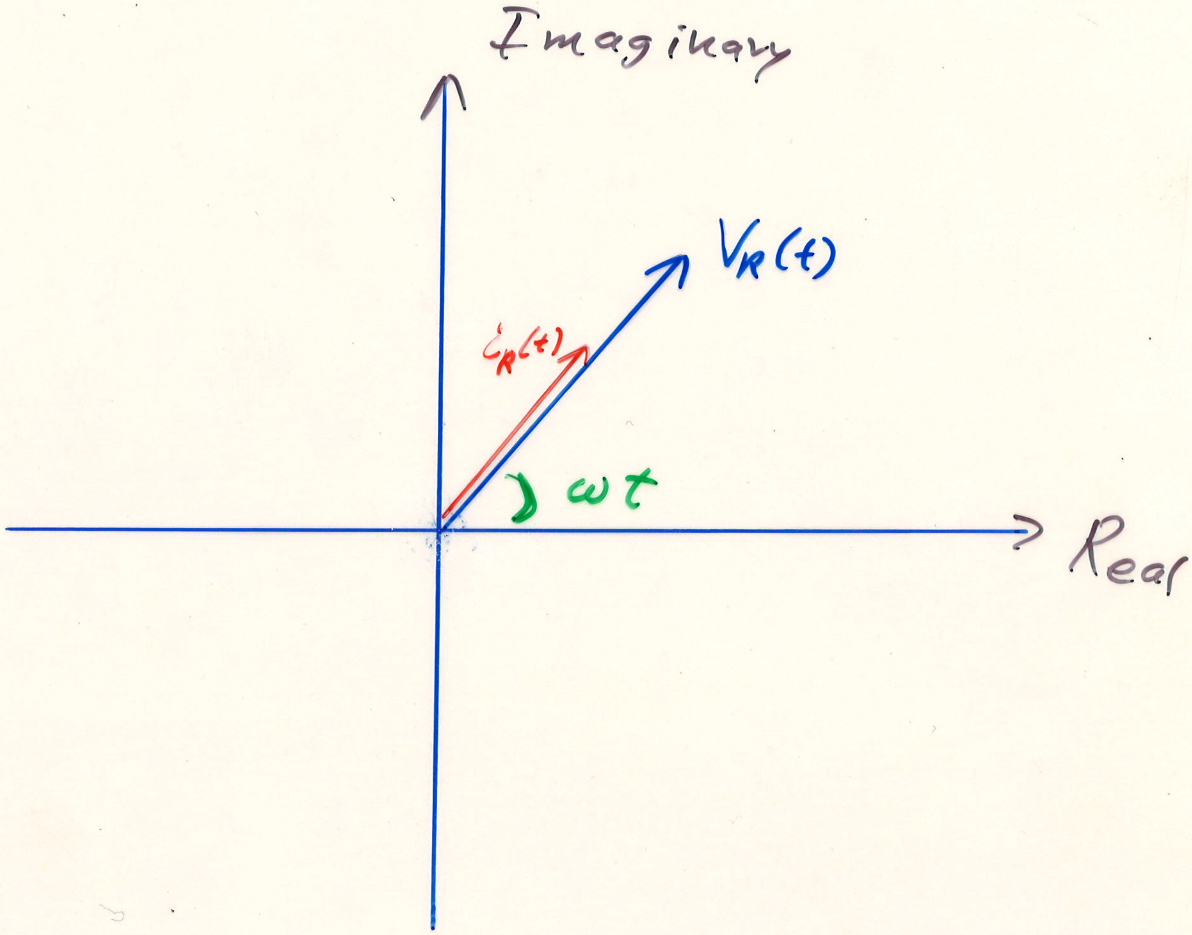
$$i_R = \frac{V_R}{R} = \frac{V_0 \sin(\omega t)}{R}$$



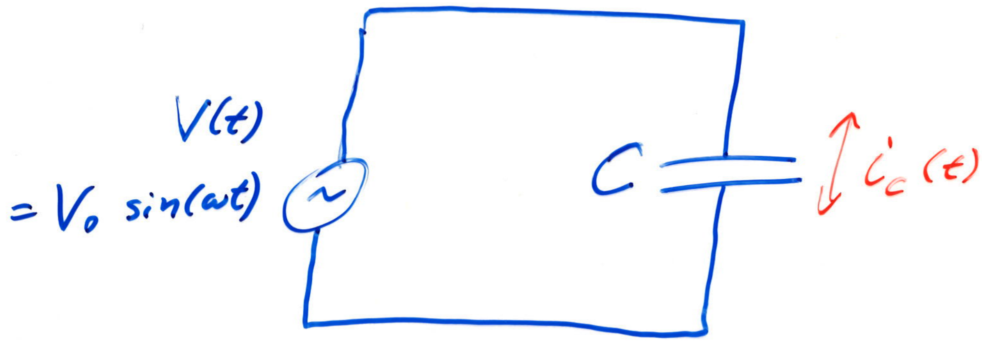
In phase

$i(t)$ and $V_R(t)$ in phase.

Phasors



A purely capacitive circuit



Voltage across the capacitor:

$$V_c(t) = V_0 \sin(\omega t)$$

$q(t) = C\,V_c(t)$

Charge on capacitor plate:

$$q_c(t) = C\,V_c(t) = C\,V_0 \sin(\omega t)$$

Current through the capacitor

$$i_c(t) = \frac{dq}{dt} = \omega C\,V_0 \cos(\omega t)$$

$$= \omega C\,V_0 \sin(\omega t + 90^\circ)$$

change to radians

This will look similar to the purely resistive result: $i_R = \frac{V_R}{R} = \frac{V_0}{R} \sin(\omega t)$

if we define the "capacitive reactance"

$$X_c \equiv \frac{1}{\omega C}$$

then $i_c = \omega C \, V_0 \sin(\omega t + 90°)$

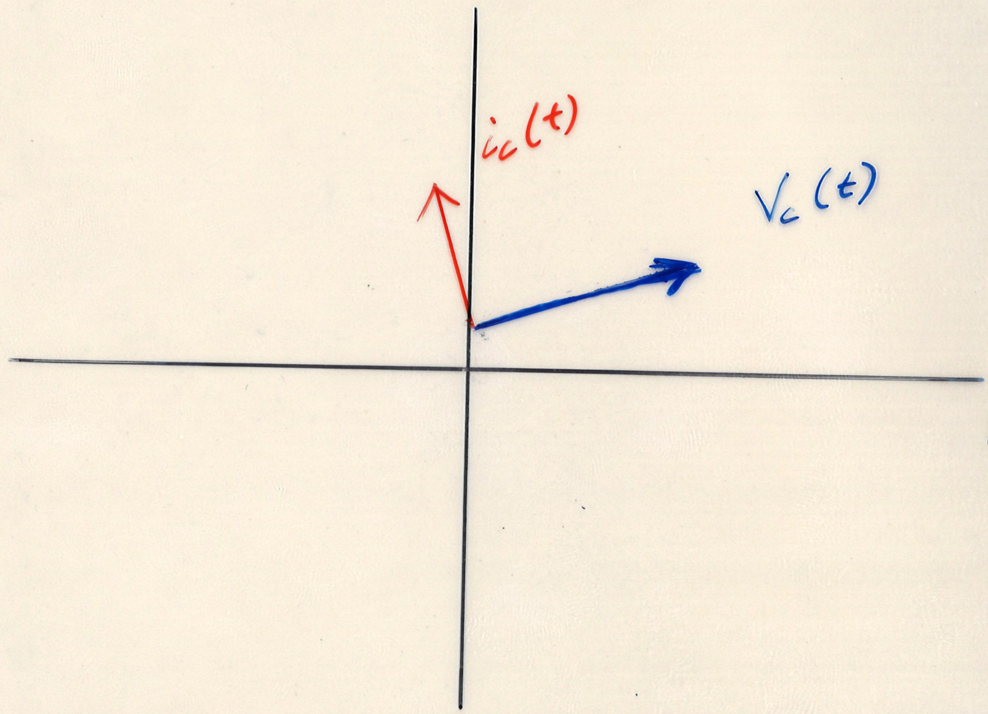
$$= \frac{V_0}{X_c} \sin(\omega t + 90°)$$

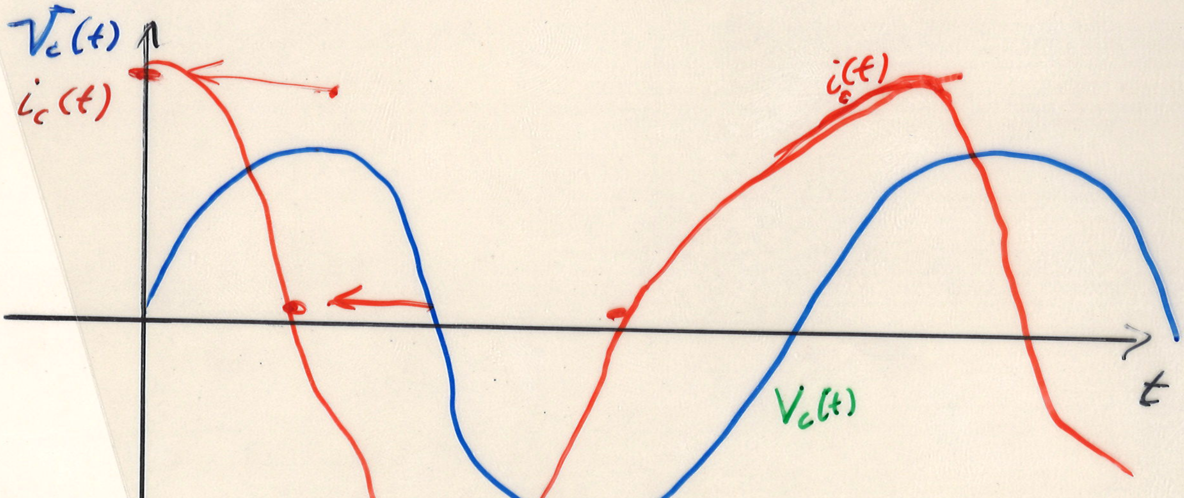
Think of the capacitive reactance as the "resistance" of a capacitor to alternating current flow.

Also, note that the current i_c is $90°$ <u>ahead</u> of the voltage V_c.

$\boxed{\text{ICE}}$

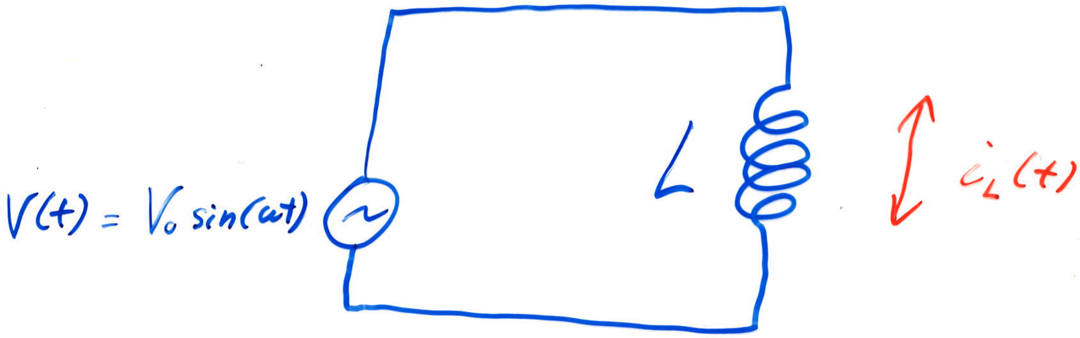
Phasors ready, Captain.





$i(t)$ leads $V_c(t)$ by 90°

A purely inductive circuit



Voltage across the inductor:

$$V_L(t) = V_0 \sin(\omega t)$$

For an inductor

$$V_L = L \frac{di}{dt} \qquad \frac{di}{dt} = \frac{V_0 \sin(\omega t)}{L}$$

Current through the inductor

$$i_L = \int \left(\frac{di}{dt}\right) dt = \frac{V_0}{L} \int \sin(\omega t)\, dt$$

$$= -\frac{V_0}{L\omega} \cos(\omega t)$$

$$= \frac{V_0}{L\omega} \sin(\omega t - 90^\circ)$$

We can make this resemble the purely resistive result $i_R = \frac{V_R}{R} = \frac{V_0}{R}\sin(\omega t)$

if we define the "inductive reactance"

$$X_L \equiv \omega L$$

then $i_L = \frac{V_0}{L\omega}\sin(\omega t - 90°)$

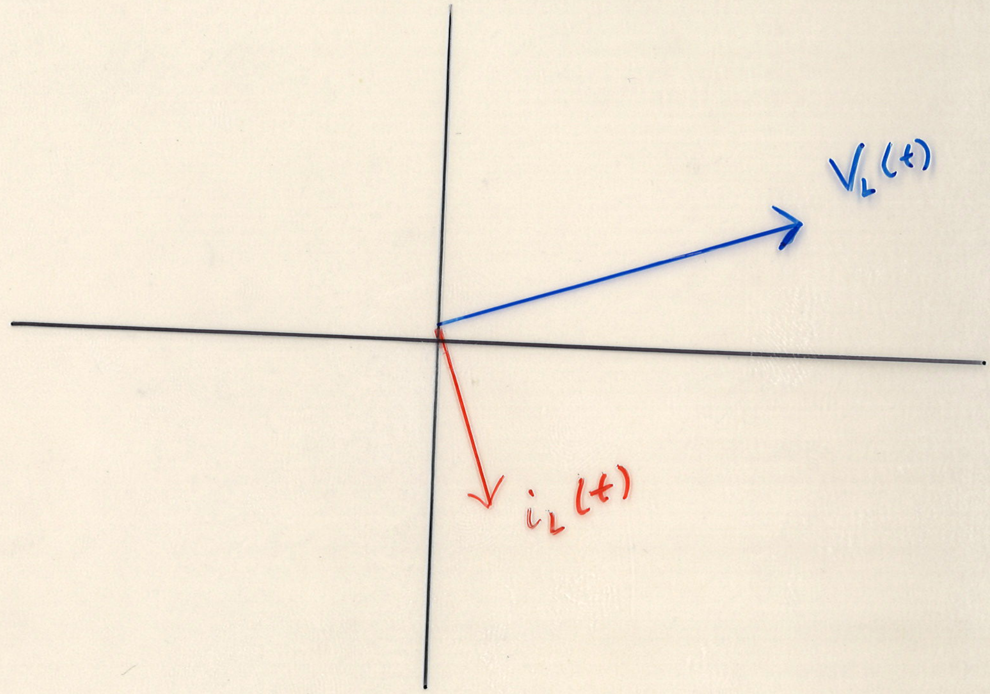
$$= \frac{V_0}{X_L}\sin(\omega t - 90°)$$

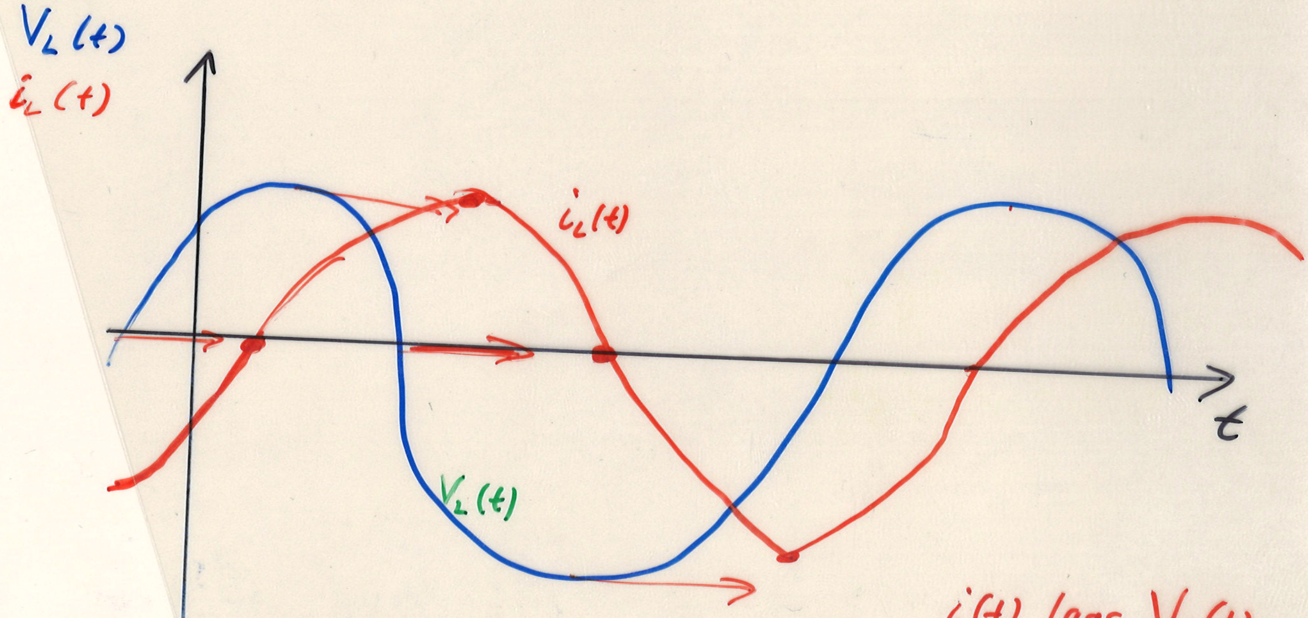
Think of the inductive reactance as the "resistance" of an inductor to alternating current flow.

Note that this time, the current i_L is 90° behind the voltage V_L.

$\boxed{ELI}$

Phasor Diagram





$i(t)$ lags $V_L(t)$ by 90°

Reactance

$$X_C = \frac{1}{\omega C}$$

This is small for large angular frequencies

A capacitor offers almost no "resistance" to high-frequency AC flow, but a capacitor offers infinite "resistance" to very low-frequency AC (that is, DC) flow.

$\omega = 0$

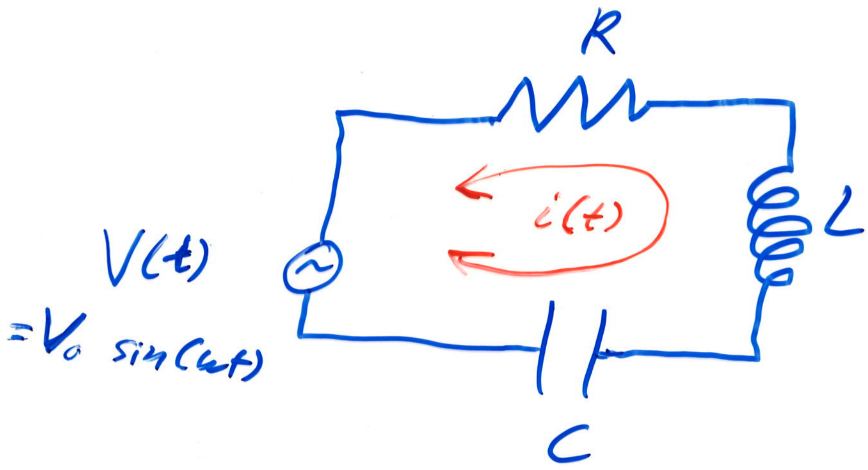
$$X_L = \omega L$$

This is large for large angular frequencies

For DC flow, an inductor looks like a resistance-less piece of wire.

However, an inductor offers considerable resistance to high-frequency AC flow.

A series RLC circuit



Kirchhoff's Loop Rule is valid at any instant.

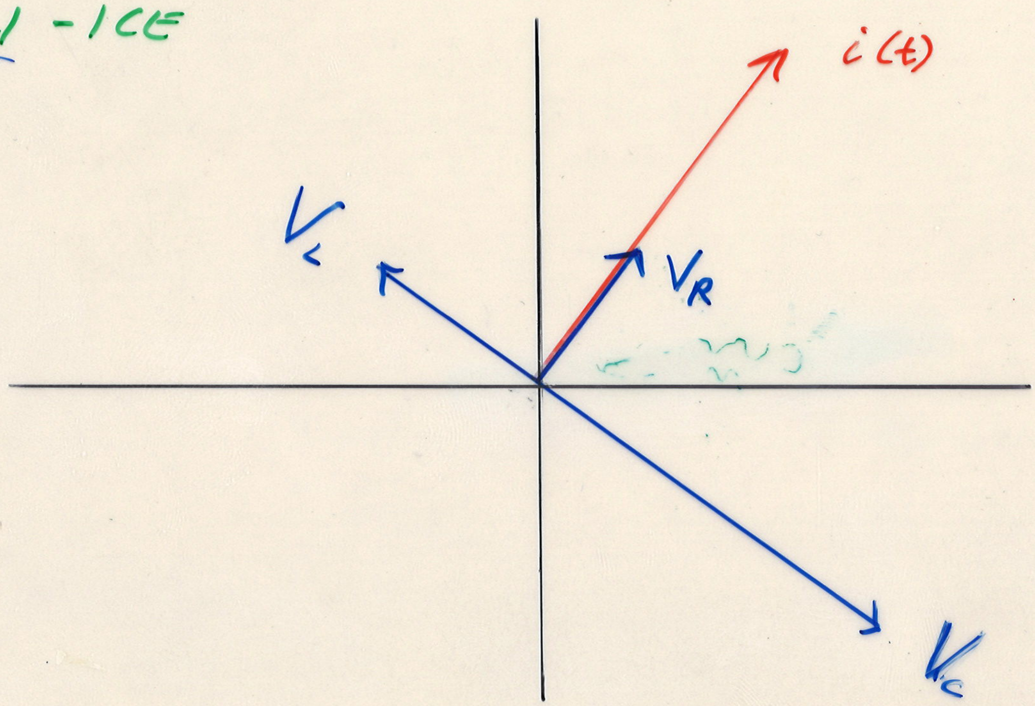
$$\underset{\text{supply}}{V(t)} - V_R(t) - V_L(t) - V_C(t) = 0$$

At t_1) $100 - 100 - 25 + 25 = 0$

At t_2) $0 - 0 - (-25) - 25 = 0$

Phasor Diagram

ELI - ICE



The three voltages : V_R, V_L, V_C add like vectors

$$V_0^2 = V_R^2 + (V_L - V_C)^2$$

$$= (iR)^2 + (iX_L - iX_C)^2$$

$$V_0 = i\sqrt{R^2 + (X_L - X_C)^2} \equiv iZ$$

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

is called the impedance of the AC circuit. It is frequency-dependent.

$$X_C = \frac{1}{\omega C}$$

$$X_L = \omega L$$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

Resonance occurs when Z is a minimum: $Z = R$

when $\omega L - \frac{1}{\omega C} = 0$ or $\omega L = \frac{1}{\omega C}$

or $\omega^2 = \frac{1}{LC}$ or $\omega = \sqrt{\frac{1}{LC}}$

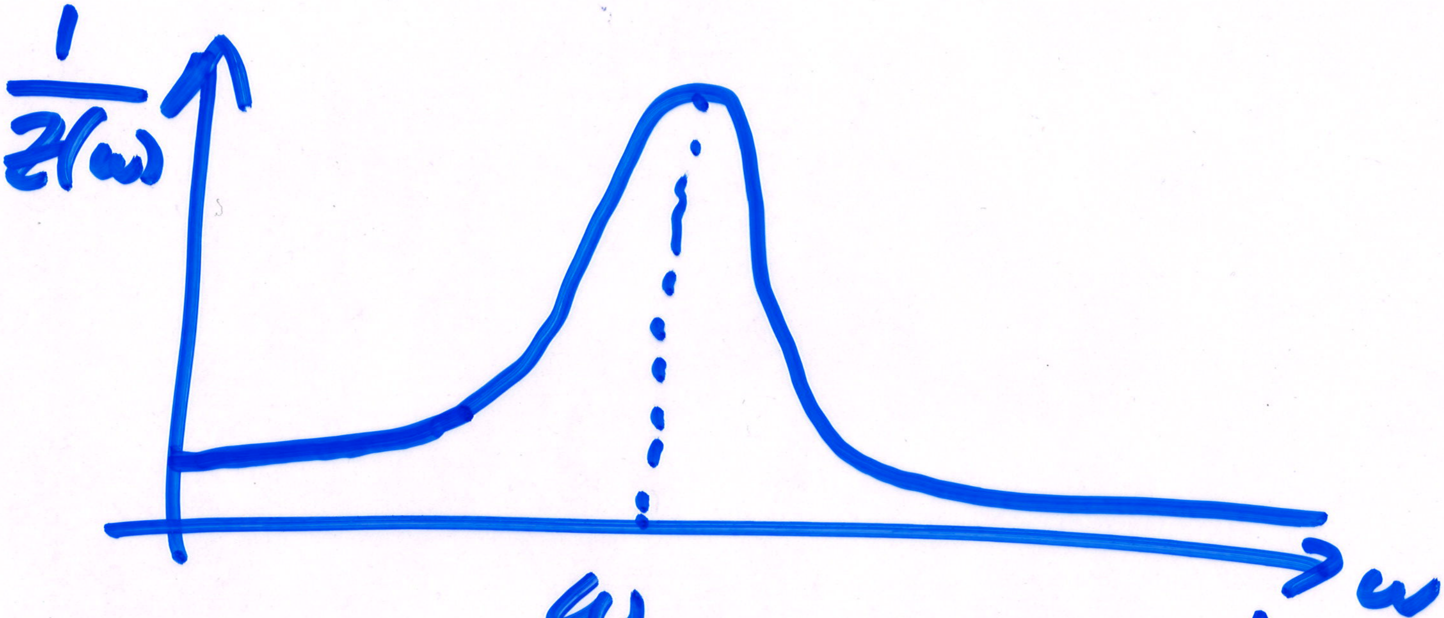
$Y = \frac{1}{Z}$ is called the

Admittance

$$Y = \frac{1}{\sqrt{R^2 + (X_C - X_L)^2}}$$

At resonance Z is a minimum

Y is a maximum



$$\omega_{resonance} = \frac{1}{\sqrt{LC}}$$

$$\underset{\text{at resonance}}{\underset{\uparrow}{Z}} = R$$

AC Problems

Find $i_{max} = \dfrac{V_{source}^{max}}{Z} = \dfrac{V_{source}^{max}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

$V_{R_{max}} = i_{max} R$ —

$i_{max} = \dfrac{V_{source}}{R}$

$V_{C_{max}} = i_{max} X_C = i_{max} \dfrac{1}{\omega C}$ —

$V_{L_{max}} = i_{max} X_L = i_{max} \omega L$ —

$P_{max} = i_{max}^2 R$

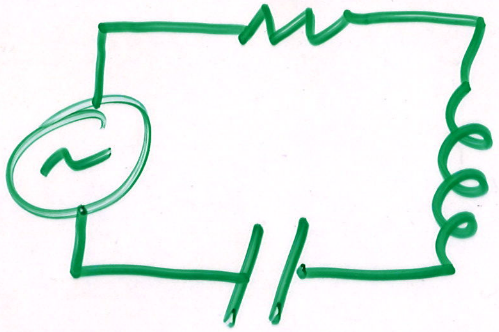
$P_{AVG} = \frac{1}{2} P_{max} = \frac{1}{2} i_{max}^2 R = (i_{rms})^2 R$

$i_{rms} = \dfrac{i_{max}}{\sqrt{2}}$

$V_o = V_{max} \sin(\omega t)$

$R = 20\,\Omega$

$V_o = (12\,V)\sin(100t)$



$V_{max} = 12\,V$

$L = 4\,mH$

$\omega = 100\,\frac{rad}{s}$

$C = 10\,nF$

milli $= 10^{-3}$

nano $= 10^{-9}$

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} =$$

$$Z = \sqrt{(20\,\Omega)^2 + \left[\left(100\,\frac{rad}{s}\right)(4\,mH) - \frac{1}{\left(100\,\frac{rad}{s}\right)(10\,nF)}\right]^2}$$

$$Z = 10^6\,\Omega = 1\,M\Omega$$

$$i_{max} = \frac{V_{max}}{Z} = \frac{12\,V}{1\,M\Omega} = 12\,\mu A$$

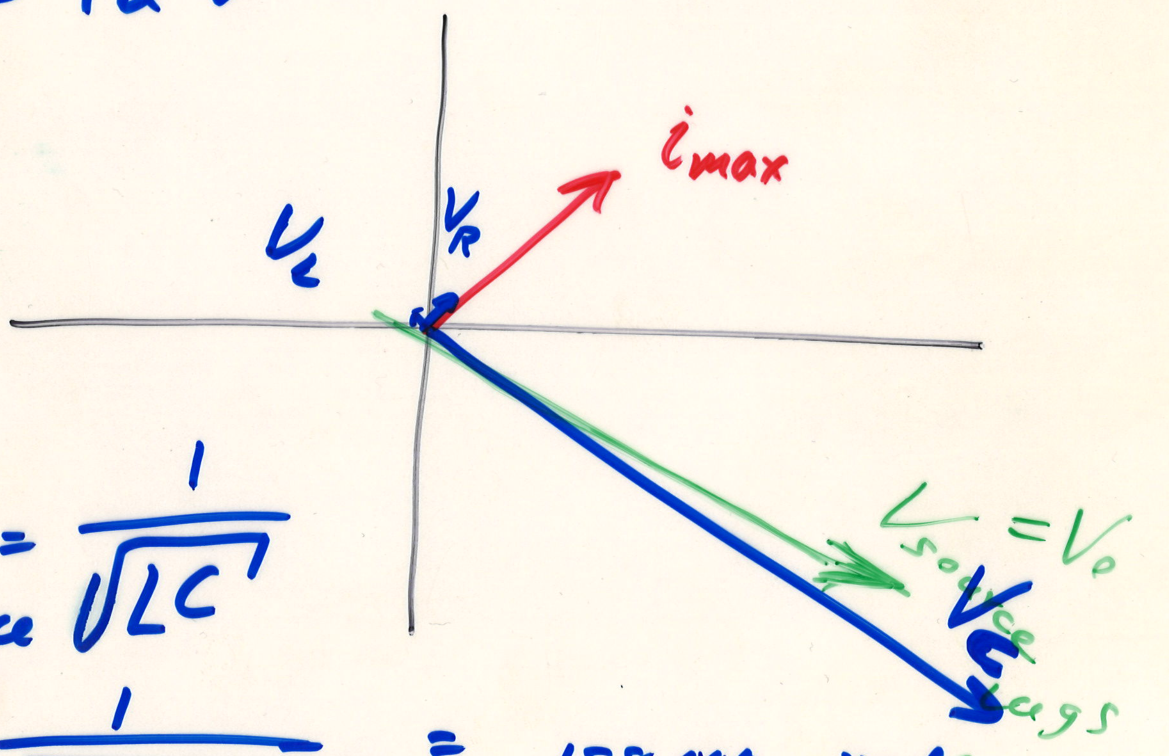
$$V_{R_{max}} = i_{max} \cdot R = (12\,\mu A)\, 20\,\Omega = 240\,\mu V$$

$$V_{L_{max}} = i_{max} X_L = i_{max}\, \omega L$$

$$= (12\,\mu A)\left(100\,\tfrac{rad}{s}\right)(4\,mH) = 4.8 \times 10^{-6}\, V$$

$$V_{C_{max}} = i_{max} X_C = \frac{i_{max}}{\omega C} = \frac{12\,\mu A}{\left(100\,\frac{rad}{s}\right)(10\,nF)}$$

$$\approx 12\, V$$



$$\omega_{Resonance} = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{4\,mH \cdot 10\,nF}} = \underline{158{,}000}\ \frac{rad}{s}$$

This circuit is capacitive

- $V_C > V_L$
- $X_C > X_L$
- V_S lags current i
- ω is below $\omega_{resonance}$

$100 \frac{rad}{s}$ $\qquad$ $158{,}000 \frac{rad}{s}$

$$\omega_{res} = \frac{1}{\sqrt{LC}}$$

AC Power

purely resistive

$$P_{inst} = [i(t)]^2 R = [i_{max} \sin(\omega t)]^2 R$$

$$= (i_{max})^2 R \sin^2(\omega t)$$

$$P_{AVG} = ?$$

Average value of $\sin^2(\omega t) = ?$

$$\langle \sin^2 \rangle = \frac{1}{2\pi}\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2\pi}\pi = 1/2$$

$$= \frac{1}{4\pi}\int_0^{4\pi} \sin^2\theta \, d\theta = \frac{1}{4\pi} 2\pi = \boxed{1/2}$$

$$P_{AVG} = \frac{1}{2}(i_{max})^2 R = \left(\frac{i_{max}}{\sqrt{2}}\right)^2 R \equiv i_{RMS}^2 R$$

$$i_{RMS} = \frac{i_{max}}{\sqrt{2}}$$

RMS → root mean square

$$i_{RMS} = \frac{i_{max}}{\sqrt{2}} \qquad V_{RMS} = \frac{V_{max}}{\sqrt{2}}$$

V_{RMS} = 120 volts AC

$V_{max} = \sqrt{2}\, V_{RMS} = \sqrt{2}\, 120$ volts = 170 volts

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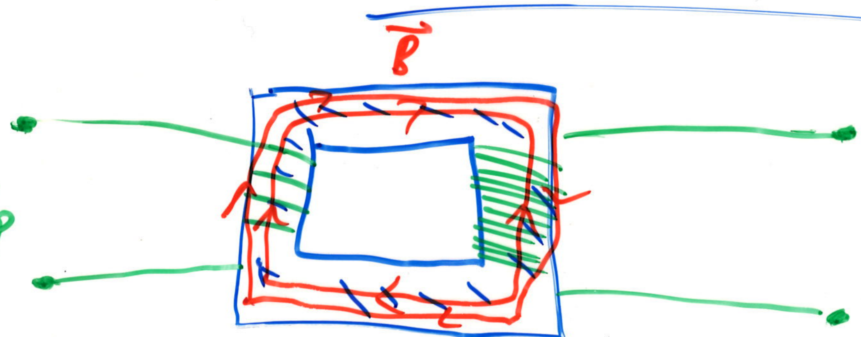
Transformer



V_p $V_s = 12\,kV$

120V

I_s = ~~1 Amp~~ 0.01 Amp

N_p turns
primary

N_s turns
secondary

I_p = ~~0.01 Amp~~ 1 Amp

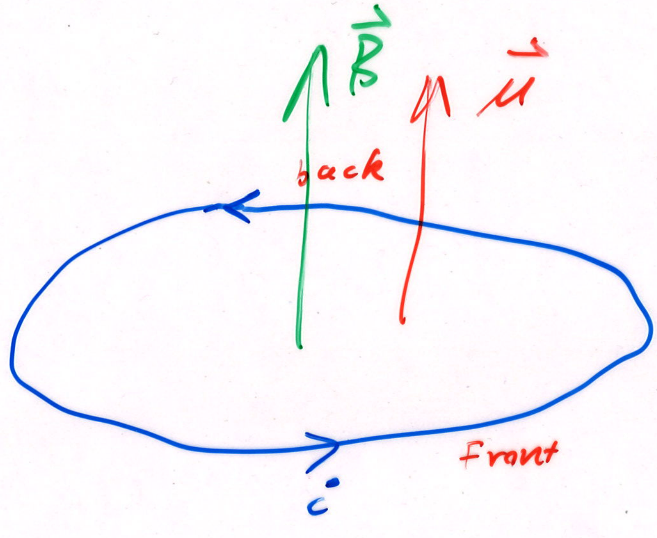
$\frac{\text{Energy}}{\text{time}}$ Conservation = Power Cons.

$$P_p = I_{p_{rms}} V_{p_{rms}} = P_s = I_{s_{rms}} V_{s_{rms}}$$

$$\frac{V_p}{N_p} = \frac{V_s}{N_s} \implies I_p N_p = I_s N_s$$

Magnetic Dipole Moment

For a flat current loop, the magnetic dipole moment is a vector $\vec{\mu}$ with magnitude $|\vec{\mu}| = i \cdot \text{Area}$ and direction given by the right-hand rule



If Area $\to 0$ and $i \to \infty$ with $i \cdot \text{Area} = \text{const.}$ we get a pure dipole magnetic field.