<u>Kinematics</u> - the study of motion without regard to its cause.

Given the vector displacement function $\vec{x}(t)$, then

$$\vec{v}(t) = \frac{d\vec{x}}{dt} = \dot{\vec{x}} \quad \text{and} \quad \vec{a}(t) = \dot{\vec{v}} = \ddot{\vec{x}}$$

Given the acceleration vector function $\vec{a}(t)$, then

$$\vec{v}(t) = \underbrace{\int_{t'=t_0}^{t} \vec{a}(t')dt'}_{\substack{\text{function of } t, \\ \text{no } t' \text{ dependence left} \\ \text{after integrating}}} + \vec{v}(t_0)$$

t' = dummy integration variable
t_0 = arbitrary time origin
$\vec{v}(t_0)$ = <u>constant</u> initial velocity (initial condition).

e.g. $\sum_{j=1}^{n} j^3 = \sum_{k=1}^{n} k^3$ <u>j</u> and <u>k</u> are dummy indices

$$= 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$ function of n, not j or k.

$$\vec{x}(t) = \int_{t''=t_1}^{t} \vec{v}(t'')dt'' + \vec{x}(t_1)$$

t'' = dummy variable
t_1 = 2nd arbitrary time
$\vec{x}(t_1)$ = <u>constant</u> initial displacement vector.

$$\vec{x}(t) = \int_{t''=t_1}^{t} \left[\int_{t'=t_0}^{t''} \vec{a}(t')dt' + \vec{v}(t_0) \right] dt'' + \vec{x}(t_1)$$

$$= \int_{t''=t_1}^{t} \int_{t'=t_0}^{t''} \vec{a}(t')\,dt'dt'' + \vec{v}(t_0)[t-t_1] + \vec{x}(t_1)$$

2 arbitrary constant vectors for this second-order differential equation

e.g. constant acceleration in one dimension

$a = 9.8\ m/s^2$ down (choose up to be positive)

$a = -g$

$$v(t) = \int_{t'=t_0}^{t} (-g)\,dt' + v(t_0) = -g(t-t_0) + v(t_0)$$

$$x(t) = \int_{t''=t_1}^{t} v(t'')\,dt'' + x(t_1) = \int_{t''=t_1}^{t} \left[-g(t-t_0) + v(t_0)\right] dt'' + x(t_1)$$

$$x(t) = -g\left[\frac{t^2 - t_1^2}{2} - t_0(t-t_1)\right] + v(t_0)\left[t - t_1\right] + x(t_1)$$

Don't recognize this? Try $t_0 = 0 = t_1$

$$x(t) = -\tfrac{1}{2} g t^2 + v_0 t + x_0$$

<u>Dynamics</u> - the study of motion <u>and</u> its causes (forces).

Some comments on Newton's laws:

① A body remains at rest ($\vec{v} = 0$) or in uniform motion (constant speed in a straight line) unless acted upon by a force.

This laws <u>seems</u> to be a trivial case of the second law, but it is necessary — it <u>defines</u> an inertial reference frame.

② The net force acting on a body is the time rate of change of its momentum. Newton never wrote $\vec{F} = m\vec{a}$ – he knew that

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt} = \dot{m}\vec{v} + m\vec{a}$$

③ For every action there is an equal but opposite reaction.

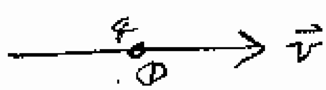
$$\vec{F}_{12} = -\vec{F}_{21}$$

on due to on due to

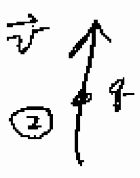
This law is true for central forces like gravity and electrostatics (act along the line connecting the two bodies), but it fails for velocity-dependent forces (like magnetism).

$$\vec{F}_{mag} = q\,\vec{v} \times \vec{B}$$

eg.



$F_{12}^{mag} = 0$ since B_2 vanishes at the position of q_1



but $F_{21}^{mag} \neq 0$ ① exerts a magnetic force on ②.

Newtonian Mechanics is an approximation	
	small particles ⟹ Quantum Mechanics
	high speeds ⟹ Special Relativity
	strong gravity ⟹ General Relativity.

What is Mass ?

Mass (inertia) is the resistance of a body to being accelerated.

How would you measure a mass where there is effectively no gravity?

- outer space far from any large body (no gravity)
- space shuttle in orbit } effectively no gravity
- freely falling elevator }

In the elevator $g = 9.8 \, m/s^2$ but everything inside is falling. In the space shuttle at 250 miles altitude $g = 86\%$ of surface gravity – hardly zero! But again, everything inside is falling together.

Consider the electric force on a charged particle:

$$F_{electric} = qE = \frac{kqQ}{R^2} = ma$$

Now consider the gravitational force on a massive particle:

$$F_{Grav} = mg = \frac{GmM}{R^2} = ma$$

The "m" on the right-hand side is the inertial mass – the resistance to acceleration. The "m" on the left is the gravitational mass – the coupling of matter to the gravitational field.

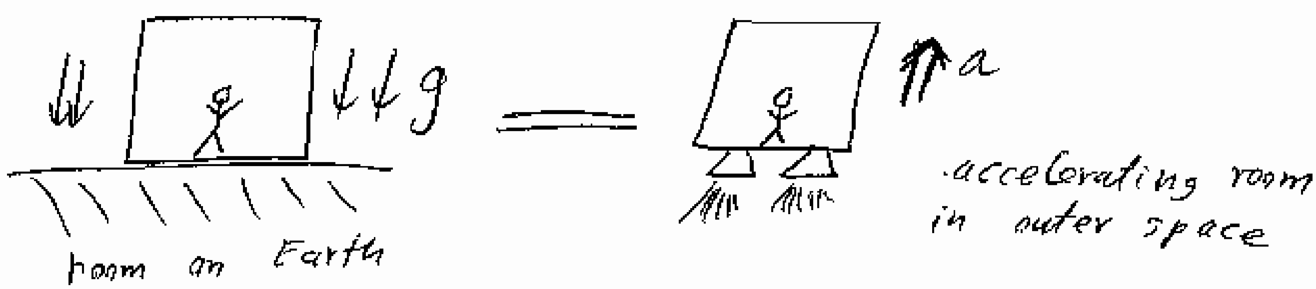
Experimentally, $m_{inertial} = m_{gravitational}$ to one part in a trillion. This equality is called the "Weak Principle of Equivalence." (WPE)

There is no mechanics experiment one can perform to distinguish a constant gravitational field and a uniform acceleration.

For a body in freefall

$$F = \underset{\substack{\uparrow \\ \text{grav}}}{m}g = \underset{\substack{\uparrow \\ \text{inertial}}}{m}a \implies a = g$$

All bodies free fall at the same rate (Galileo).



If WPE were not true, object of different mass would have different free-fall accelerations.

for pendula: $\omega = \sqrt{\frac{g}{l}}$ mass-independence relies on WPE.

Reference Frames

Two index notation : $\vec{v}_{p,f}$ velocity of particle p with respect to frame f.

Two rules : ① $\vec{v}_{a,b} = -\vec{v}_{b,a}$

② $\vec{v}_{a,\underbrace{b + \vec{v}_{b}}_{\text{same}},c} = \vec{v}_{a,c}$

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e.g. The airspeed of a plane is 200 m/s North (the compass heading - the direction in which the plane is pointing)

The ground speed of the plane is 250 m/s due NE.

What is the velocity of the wind with respect to the ground? (as measured by a ground-based weather station).

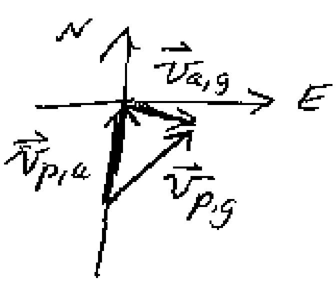
$$\vec{v}_{p,a} = 200\,\hat{j} \text{ m/s}$$

$$\vec{v}_{p,g} = \left(\frac{250}{\sqrt{2}}\hat{i} + \frac{250}{\sqrt{2}}\hat{j}\right) \text{ m/s}$$

$$\vec{v}_{a,g} = \vec{v}_{a,p} + \vec{v}_{p,g} = -\vec{v}_{p,a} + \vec{v}_{p,g}$$

$$= \left(-200\,\hat{j} + \frac{250}{\sqrt{2}}\hat{i} + \frac{250}{\sqrt{2}}\hat{j}\right) \text{ m/s} = (177\,\hat{i} - 23\,\hat{j}) \text{ m/s}$$

$$= (178 \text{ m/s at } 8° \text{ South of East})$$



Let the notation do the work. If you think too hard, you'll get it wrong!

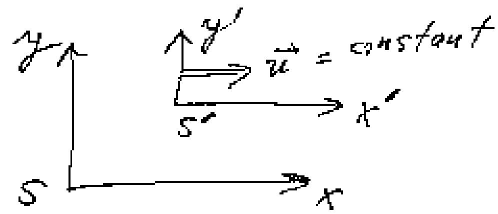
The Principle of Newtonian Relativity

There is no mechanics experiment that one can perform to decide if a frame is at rest (with respect to what ?!?)

There is no frame of absolute rest.

If Newton's Laws are valid in one frame of reference (an inertial reference frame), then they are also valid in any frame moving at a constant velocity (constant speed in a straight line) with respect to the first frame.

Galilean Invariance



$x = x' + ut$	$x' = x - ut$
$y = y'$	$y' = y$
$z = z'$	$z' = z$
$t = t'$	$t' = t$ ← time is absolute in Newton's world

$$v_x' = \dot{x}' = \dot{x} - u = v_x - u$$

v_x': particle velocity in frame S'

v_x: particle velocity in frame S

u: velocity of frame S' with respect to S

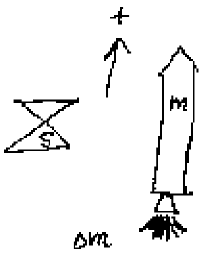
$$a_x' = \ddot{x}' = \ddot{x} = a_x \qquad \text{also} \quad a_y' = a_y, \quad a_z' = a_z$$

a_x': particle acceleration

$$m\vec{a}' = m\vec{a}$$
$$\vec{F}_{net}' = \vec{F}_{net}$$

Newton's 2^{nd} Law is unchanged (invariant) by a change of frame (S to S').

Rocket Equation – first with no gravity



$m_r(t)$ instantaneous mass of rocket (payload and unburnt fuel)

Δm_e small amount of mass lost in exhaust in time Δt.

$\uparrow \vec{v}_{r,s}(t)$ velocity of rocket w.r.t. station

$\downarrow \vec{v}_{e,s}(t)$ velocity of exhaust w.r.t. station

$\downarrow \vec{v}_{e,r}$ velocity of exhaust w.r.t. rocket – Assumed constant

Vectors: $\vec{v}_{e,s}(t) = \vec{v}_{e,r} + \vec{v}_{r,s}(t)$

y-components: $-v_{es}(t) = -v_{er} + v_{rs}(t)$

No external forces $\Rightarrow$ Linear momentum of the system (rocket and fuel) is conserved.

$$\vec{P}_{\substack{\text{system}\\\text{initial}}} = \vec{P}_{\substack{\text{system}\\\text{final}}} \qquad \text{final} = \text{after time } \Delta t$$

$$m_r(t)\, v_{rs}(t) = m_r(t+\Delta t)\, v_{rs}(t+\Delta t) - \Delta m_e\, v_{es}(t)$$

↑ should be $(t+\Delta t)$

$$m_r(t)\, v_{rs}(t) = \left[m_r(t) - \Delta m_e\right]\left[v_{rs}(t) + \Delta v_{rs}\right] - \Delta m_e\left[v_{er} - v_{rs}(t)\right]$$

but Δm_e is small so any error will be even smaller

$$\underline{m_r(t)\, v_{rs}(t)} = \underline{m_r(t)\, v_{rs}(t)} + m_r(t)\Delta v_{rs} - \underline{\underline{\Delta m_e\, v_{rs}(t)}} - \Delta m_e \Delta v_{rs}$$
$$- \Delta m_e\, v_{er} + \underline{\underline{\Delta m_e\, v_{rs}(t)}}$$

↑ negligible

$$m_r(t)\,\Delta v_{rs} = \Delta m_e\, v_{er}$$

calculus limit

$$m_r(t)\,dv_{rs} = dm_e\, v_{er}$$

$$dm_e = -dm_r$$

mass of exhaust came from the rocket

$$m_r(t)\,dv_{rs} = -dm_r\, v_{er}$$

$$dv_{rs} = -\frac{dm_r}{m_r(t)}\, v_{er}$$

integrate

$$\int_{v_{i,rs}}^{v_{f,rs}} dv_{rs} = -v_{er}\int_{m_i}^{m_f}\frac{dm}{m(t)}$$

$$v_{f,rs} - v_{i,rs} = -v_{er}\ln\left(\frac{m_f}{m_i}\right)$$

$$v_{f,rs} = v_{i,rs} + v_{er}\ln\left(\frac{m_i}{m_f}\right)$$

$$m\frac{dv_{rs}}{dt} = \underbrace{-\overset{\text{burn rate}}{\frac{dm}{dt}}\ \overset{\text{exhaust speed}}{v_{er}}}_{\text{thrust}}$$

Want: ① large v_{er}

② heavy fuel - large $\dot{m}$

③ large ratio $\frac{m_i}{m_f}$

Now with gravity

$$\sum \vec{F}_{external} = \frac{d\vec{p}_{system}}{dt}$$

y-components

$$F_{ext}\,dt = m_r\,dv_{rs} + dm_r\,v_{er}$$

$$-m_r g\,dt = m_r\,dv_{rs} + dm_r\,v_{er}$$

$$dv_{rs} = -v_{er}\frac{dm_r}{m_r} - g\,dt$$

$$\int_{v_{i,rs}}^{v_{f,rs}} dv_{rs} = -v_{er}\int_{m_i}^{m_f}\frac{dm_r}{m_r} - g\int_{t_i}^{t_f} dt$$

$$v_{f_{rs}} - v_{i_{rs}} = -v_{er} \ln\left(\frac{m_f}{m_i}\right) - g\,(t_f - t_i)$$

at launch
$v_i = 0$
$t_i = 0$
let $t_f \to t$

$$v_{f_{rs}} = v_{i_{rs}} + v_{er} \ln\left(\frac{m_i}{m_f}\right) - g\,(t_f - t_i)$$

$$v_{rs}(t) = v_{er} \ln\left(\frac{m_i}{m_f}\right) - gt$$

The thrust $v_{er}\frac{dm}{dt}$ must be greater than $m(t)g$ or the rocket will not leave the launch pad.

For $1g$ acceleration up, thrust $= 2m(t)g$.

Constant mass problems - one dimension $\quad F(x,\dot{x},t) = m\ddot{x}$

In general, the force F can depend on x, v, and t. If F depends on only one of x, v, or t then the equation of motion $F = m\ddot{x}$ can be solved analytically. In all other cases a numerical solution is required.

Case 1: $F = F(t)$

$$F(t) = ma = m\frac{dv}{dt} \implies F(t)\,dt = m\,dv$$

$$m\int_{v'=v_0}^{v(t)} dv' = \int_{t'=t_0}^{t} F(t')\,dt' \qquad \text{where } v_0 \equiv v(t_0)$$

$$m\left[v(t) - v_0\right] = \int_{t'=t_0}^{t} F(t')\,dt' \implies v(t) = v_0 + \frac{1}{m}\int_{t'=t_0}^{t} F(t')\,dt'$$

$$\frac{dx}{dt} = v_0 + \frac{1}{m}\int_{t'=t_0}^{t} F(t')\,dt'$$

$$dx = \left[v_0 + \frac{1}{m}\int_{t'=t_0}^{t} F(t')\,dt'\right] dt$$

$$\int_{x''=x_1}^{x(t)} dx'' = \int_{t''=t_1}^{t}\left[v_0 + \frac{1}{m}\int_{t'=t_0}^{t''} F(t')\,dt'\right] dt'' \qquad \text{where } x_1 \equiv x(t_1)$$

$$x(t) - x_1 = v_0\left[t - t_1\right] + \frac{1}{m}\int_{t''=t_1}^{t}\int_{t'=t_0}^{t''} F(t')\,dt'\,dt''$$

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e.g. $\quad F(t) = F_0 \sin(\omega t) \qquad\qquad F(t) = A + Bt + \frac{C}{t^2}$

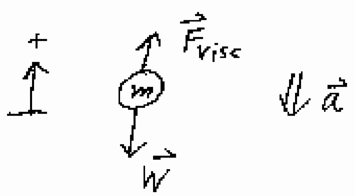
Case 2: $F = F(v)$

$$F(v) = ma = m\frac{dv}{dt} \implies dt = m\frac{dv}{F(v)}$$

Now integrate both sides and extract $v(t)$ from the right-hand side. This is as far as we can go in general, so we will proceed with a specific example.

Viscous Drag: $\vec{F}_{visc} = -b\vec{v}$

A particle is dropped from rest in oil under the influence of gravity



$$\Sigma F_y = ma_y \qquad b, v, g, m \text{ positive}$$

$$bv - mg = -m\frac{dv}{dt}$$

$$dt = \frac{dv}{g - \frac{b}{m}v}$$

$$\int_{t'=t_0}^{t} dt' = \int_{v'=v_0}^{v(t)} \frac{dv'}{g - \frac{b}{m}v'}$$

$$t - t_0 = -\frac{m}{b}\ln\left(g - \frac{b}{m}v'\right)\Bigg|_{v'=v_0}^{v(t)} = -\frac{m}{b}\ln\left[\frac{g - \frac{b}{m}v(t)}{g - \frac{b}{m}v_0}\right]$$

---

specialize: $t_0 = 0 \qquad v_0 = 0$

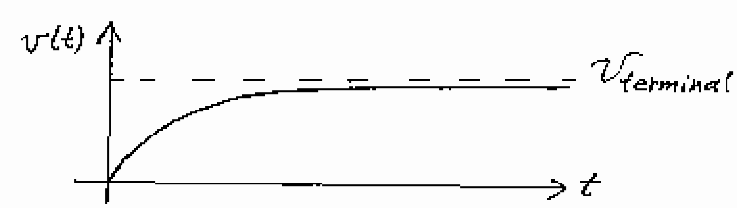
$$t = -\frac{m}{b}\ln\left[1 - \frac{b}{mg}v(t)\right] \implies v(t) = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$$

$v(0) = 0$

$v(\infty) = \frac{mg}{b} \equiv v_{terminal}$

$a(0) = g$

$a(\infty) = 0$



$b \to 0$ is the no-drag limit

$$\lim_{b\to 0}\left[\frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)\right] = gt \quad \text{(free-fall)}$$

Back to the general case: $t_0 \neq 0$, $v_0 \neq 0$

$$v(t) = \frac{dx}{dt} = \frac{mg}{b} - \left(\frac{mg}{b} - v_0\right)e^{-\frac{b}{m}(t-t_0)}$$

Integrate to get $x(t)$

$$\int_{x'=x_1}^{x(t)} dx' = \int_{t'=t_1}^{t} \left[\frac{mg}{b} - \left(\frac{mg}{b} - v_0\right)e^{-\frac{b}{m}(t'-t_0)}\right] dt'$$

$$x(t) - x_1 = \frac{mg}{b}(t-t_1) + \left(\frac{mg}{b} - v_0\right)\frac{m}{b}\left[e^{-\frac{b}{m}(t-t_0)} - e^{-\frac{b}{m}(t_1-t_0)}\right]$$

$$= v_{term}(t-t_1) + \left(v_{term} - v_0\right)\frac{m}{b}\left[e^{-\frac{b}{m}(t-t_0)} - e^{-\frac{b}{m}(t_1-t_0)}\right]$$

Homework: Aerodynamic Drag

$$\vec{F}_{aero} = -c v^2 \hat{v} = -c v \vec{v}$$

Case 3: $F = F(x)$

$$F(x) = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v$$

$$F(x)\,dx = mv\,dv$$

$$\int_{x'=x_0}^{x} F(x')\,dx' = m\int_{v'=v_0}^{v(x)} v'\,dv' = \frac{m}{2}(v')^2\Big|_{v'=v_0}^{v(x)} = \frac{m}{2}\left(v_{(x)}^2 - v_0^2\right)$$

where $v_0 \equiv v(x_0)$

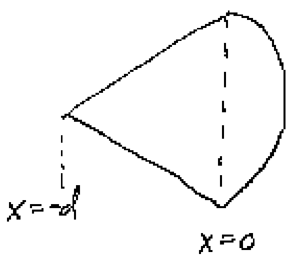
$$v(x) = \sqrt{\frac{2}{m}\int_{x'=x_0}^{x} F(x')\,dx' + v_0^2} = \frac{dx}{dt}$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\int_{x'=x_0}^{x} F(x')\,dx' + v_0^2}}$$

$$\int_{t''=t_1}^{t} dt'' = \int_{x''=x_1}^{x(t)} \frac{dx''}{\sqrt{\frac{2}{m}\int_{x'=x_0}^{x''} F(x')\,dx' + v_0^2}}$$

solve for $x(t)$

e.g. Arrow of mass m accelerated by a Hooke's law bow string.



$$F(x) = \begin{cases} -kx & \text{for } x<0 \\ 0 & \text{for } x>0 \end{cases}$$

so v = constant for $x>0$

$$F(x) = ma = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt} = m\frac{dv}{dx}v$$

$$-kx\,dx = mv\,dv$$

$$v_0 \equiv v(-d)$$

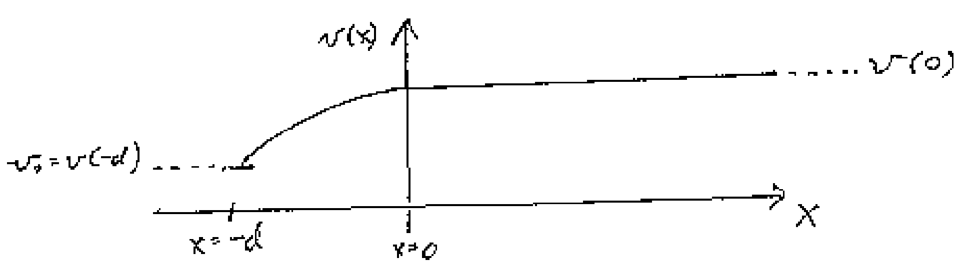
$$-k\int_{x'=-d}^{x} x'dx' = m\int_{v'=v_0}^{v(x)} v'dv'$$

$$\left.\frac{-k(x')^2}{2}\right|_{x'=-d}^{x} = \left.\frac{m(v')^2}{2}\right|_{v'=v_0}^{v(x)}$$

$$\frac{k}{2}\left(d^2-x^2\right) = \frac{m}{2}\left[v(x)^2 - v_0^2\right]$$

$$v(x) = \sqrt{\frac{k}{m}\left(d^2-x^2\right)+v_0^2} \quad \text{for } x<0$$

$$v(x) = v(0) = \sqrt{\frac{k}{m}d^2+v_0^2} \quad \text{for } x>0$$



$$v(x) = \sqrt{\frac{k}{m}(d^2 - x^2) + v_0^2} = \frac{dx}{dt} \qquad \text{for } x < 0$$

$$dt = \frac{dx}{\sqrt{\frac{k}{m}(d^2 - x^2) + v_0^2}}$$

$$\int_{t''=t_1}^{t} dt'' = \int_{x''=x_1}^{x(t)} \frac{dx''}{\sqrt{\frac{k}{m}(d^2 - x''^2) + v_0^2}}$$

Specialize : $t_0 = 0 = t_1 \qquad v_0 = 0 \qquad x_1 = -d$

$$t = \sqrt{\frac{m}{k}} \int_{x''=-d}^{x(t)} \frac{dx''}{\sqrt{d^2 - x''^2}} = \sqrt{\frac{m}{k}} \left[-\arccos\left(\frac{x''}{d}\right) \right]_{x''=-d}^{x}$$

$$t\sqrt{\frac{k}{m}} = \underbrace{\arccos(-1)}_{\pi} - \arccos\left(\frac{x}{d}\right)$$

$$x(t) = d \cos\left(\pi - t\sqrt{\frac{k}{m}}\right)$$

$$x(t) = -d \cos\left(t\sqrt{\frac{k}{m}}\right)$$

for $-d < x < 0$, $0 < t < \frac{T}{4}$

T = period of SHM if arrow did not release.

$$x(t) = (\text{constant}) \cdot t = v(0)\,t = d\sqrt{\frac{k}{m}}\,t \qquad \text{for } x > 0,\ t > \frac{T}{4}$$