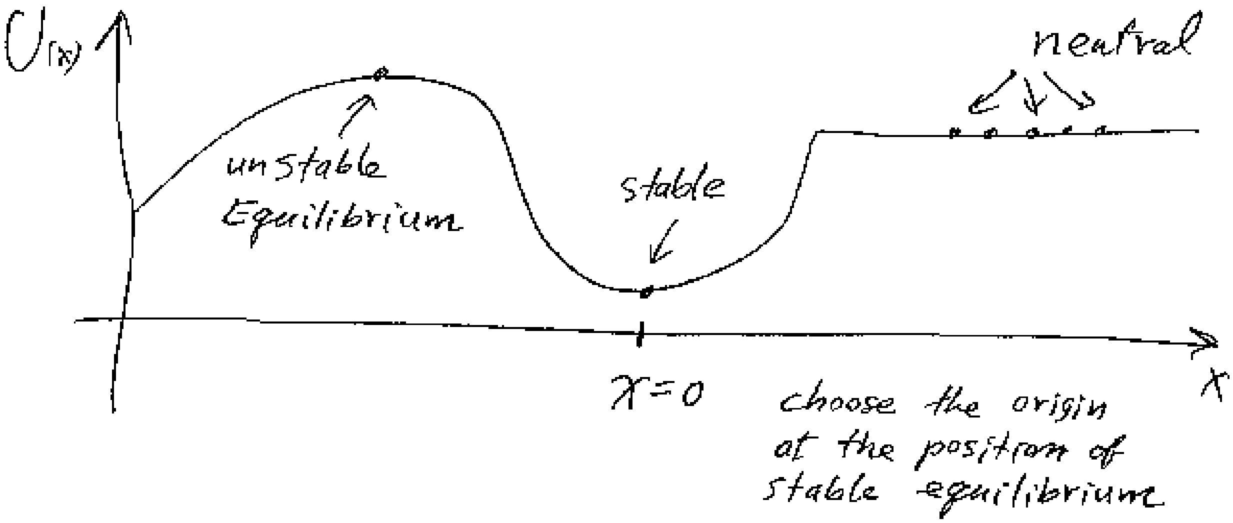
Oscillations

Consider a potential energy (that depends only on x for simplicity). Taylor expand about a stable equilibrium,

$$U(x) = U(0) + x\left.\frac{dU}{dx}\right|_{x=0} + \frac{x^2}{2!}\left.\frac{d^2U}{dx^2}\right|_{x=0} + \cdots$$



choose the origin at the position of stable equilibrium

Only changes in potential energy are measurable, so $U(x)$ is only defined up to a constant. Therefore, the first term above $U(0)$ can be set to zero.

At positions of equilibrium, the force vanishes. The force is $-\frac{dU}{dx}$ (or $-\vec{\nabla}U$ in more than one dimension). So the second term vanishes at equilibrium: $x\left.\frac{dU}{dx}\right|_{x=0} = 0$.

The first non-zero term is $\frac{x^2}{2!}\frac{d^2U}{dx^2}\Big|_{x=0}$ or $\frac{1}{2}kx^2$ where $k \equiv \frac{d^2U}{dx^2}\Big|_{x=0}$ is positive if the equilibrium is stable ($U(x)$ is concave up).

The force associated with this first approximation to the potential energy is

$$F = -\frac{d}{dx}\left[\frac{1}{2}kx^2\right] = -kx$$

which you will recognize as the ideal Hooke's law restoring force. A restoring force always brings the system back to equilibrium after displacements away from equilibrium.

Other terms can be added for more precision, but for small oscillations, the $\frac{1}{2}kx^2$ term will dominate. The resulting oscillations are called simple harmonic and can be solved exactly.

Why study these oscillations? They occur often in various branches of Physics and the resulting differential equation is linear in x, so ① there is hope of solving it and ② the solutions will superpose (linear combinations of solutions will also be a solution).

Mathematician Stanislaw Ulam said that studying non-linear science is like studying non-elephant zoology.

Newton's 2^{nd} Law:

$$\left.\begin{array}{l} F = ma \\ -kx = m\ddot{x} \end{array}\right\} \qquad \begin{array}{l} \ddot{x} + \frac{k}{m}x = 0 \\ \frac{d^2x(t)}{dt^2} + \frac{k}{m}x(t) = 0 \end{array}$$

This is a differential equation. The goal is to find a function $x(t)$ that makes the equation true for all times.

x is the function; t is the variable.

The differential equation is:

second order, linear, homogeneous, ordinary

Second order - the highest derivative is 2.

linear - the function and its derivatives ($x, \dot{x}, \ddot{x}$, etc) occur to at most the first power.

homogeneous - there is no term that does not depend on x.

ordinary - there is only one variable, t.

If the next term in the Taylor expansion of $U(x)$ were kept $\frac{x^3}{3!} \left.\frac{d^3U(x)}{dx^3}\right|_{x=0}$ the force term would be proportional to x^2 and the differential equation

$$\ddot{x}(t) + \frac{k}{m}x(t) + cx^2(t) = 0$$ is non-linear.

How do we solve the linear D.E.?

$$\ddot{x}(t) + \frac{k}{m}x(t) = 0$$

Try an exponential solution: $x(t) = Ae^{rt}$

The first derivative is $\dot{x}(t) = rAe^{rt}$

The second derivative is $\ddot{x}(t) = r^2Ae^{rt}$

$$\ddot{X}(t) + \frac{k}{m} X(t) = 0$$

$$r^2 A e^{rt} + \frac{k}{m} A e^{rt} = 0$$

The guess converts the D.E. into an <u>algebraic</u> equation for r.

$$\left(r^2 + \frac{k}{m}\right) A e^{rt} = 0$$

---

this implies that

$A = 0$ or $e^{rt} = 0$ or

$\left(r^2 + \frac{k}{m}\right) = 0$

e^{rt} is never zero. If $A = 0$, then this is the null solution $X(t) = 0$ for all time. Well, that solves the D.E., but it's not interesting. So

$r^2 + \frac{k}{m} = 0$ implies $r = \pm i \sqrt{\frac{k}{m}}$

With the definition $\omega_0 = \sqrt{\frac{k}{m}}$, the two solutions are $A_+ e^{+i\omega_0 t}$ and $A_- e^{-i\omega_0 t}$.

We expect two linearly independent solutions because the D.E. is second order. The arbitrary constants A_+ and A_- are constants of integration (you have to integrate twice to get from $\ddot{x}$ to x) and these are used to satisfy the boundary conditions, for example:

$x(0), v(0)$ or $X(0), X(t_1)$ or $v(0), v(t_1)$

So the complete solution to $\ddot{x}(t) + \frac{k}{m} x(t) = 0$ is

$$x(t) = A_+ e^{+i\omega_0 t} + A_- e^{-i\omega_0 t}$$

other equivalent forms are

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$x(t) = C\cos(\omega_0 t + \varphi)$$

$$x(t) = D\sin(\omega_0 t + \alpha)$$

These can all be transformed into the others.

Remember:

$$\cos\theta = \frac{e^{+i\theta} + e^{-i\theta}}{2}$$

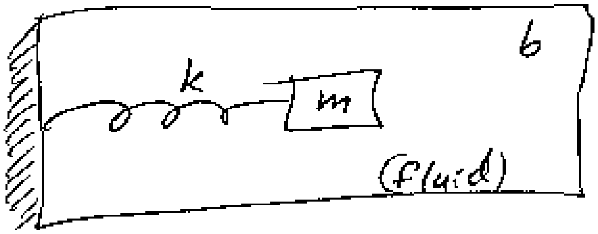
$$\sin\theta = \frac{e^{+i\theta} - e^{-i\theta}}{2i}$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Damped Oscillations

Assume a linear resistive force, like viscous drag

$F_{visc} = -b\vec{v} = -b\dot{x}$ b is positive



$$F = ma$$

$$-kx - bv = ma$$

$$-kx - b\dot{x} = m\ddot{x}$$

$$\ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = 0$$

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

$$\omega_0^2 \equiv \frac{k}{m}$$

$$\beta \equiv \frac{b}{2m}$$

Try an exponential solution again: $x(t) = Ae^{rt}$

The first derivative is $\dot{x}(t) = rAe^{rt}$

The second derivative is $\ddot{x}(t) = r^2 Ae^{rt}$

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

$$(r^2 + 2\beta r + \omega_0^2) A e^{rt} = 0$$

As before, we get an algebraic equation for r

$r^2 + 2\beta r + \omega_0^2 = 0$ has solutions $r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

So the complete solution to the D.E. is

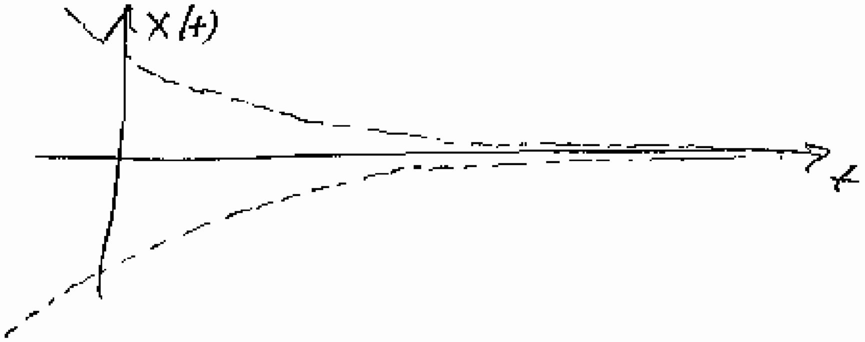
$$x(t) = A_+ e^{(-\beta + \sqrt{\beta^2 - \omega_0^2})t} + A_- e^{(-\beta - \sqrt{\beta^2 - \omega_0^2})t}$$

or

$$x(t) = e^{-\beta t}\left[A_+ e^{+\sqrt{\beta^2 - \omega_0^2}\,t} + A_- e^{-\sqrt{\beta^2 - \omega_0^2}\,t}\right]$$

damping envelope

$x \to 0$ as $t \to \infty$



There are three cases to consider;

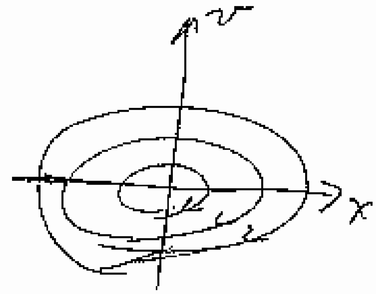
$\omega_0 > \beta$ so $\sqrt{\beta^2 - \omega_0^2}$ is imaginary : underdamped

$\omega_0 < \beta$ so $\sqrt{\beta^2 - \omega_0^2}$ is real : overdamped

$\omega_0 = \beta$ so $\sqrt{\beta^2 - \omega_0^2}$ is zero : critically damped

Phase Space v vs. x

Phase space is 2-dim for 1-dim moti

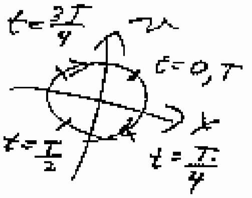


1-dim SHO: $x(t) = A\cos(\omega_0 t + \delta)$

$v(t) = -A\omega_0\sin(\omega_0 t + \delta)$ eliminate t

$$v(x) = \cancel{A\omega_0\sqrt{1-\frac{x^2}{A^2}}} = \pm\omega_0\sqrt{A^2-x^2}$$

$$\frac{x^2}{A^2} + \frac{v^2}{A^2\omega_0^2} = 1$$ ellipse

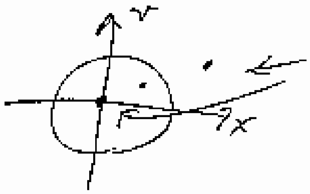


- A point in P.S. represents a possible state of the system (t=
- As time evolves, the curve is traced out clockwise.
- Phase paths never cross

initial conditions

(x_0, v_0) which way would system evolve?

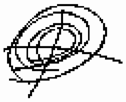
2^{nd} order D.E. $\Rightarrow$ solution is unique.



The system cannot exist in states off the phase p
in particular $x=0$, $v=0$ is disallowed

$x(t_1) = \frac{A}{2}$ $v(t_1) = \frac{\omega A}{2}$ impossible at same time

- different amplitudes A give different ellipses



- Area of ellipse : $S = \pi ab = \pi(A)(A\omega) = \frac{2\pi E}{\sqrt{km}}$ Area proportional to energy.

  a: semi minor, b: semi major

- closed curves in P.S. $\Rightarrow$ periodic, mechanical energy is conserved.
- do not confuse with Lissajous figures!

for 2-d oscillators
paths can cross
axes different
P.S. would be 4-dim.

Lecture #19

4 Nov 9[illegible]7

Undriven oscillations with damping (linear in velocity)

$m\ddot{x} = -kx - b\dot{x} \quad \Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$

try $x(t) = Ae^{rt} \quad \Rightarrow \quad r_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

<u>case 1</u> $\omega_0 > \beta$ underdamped

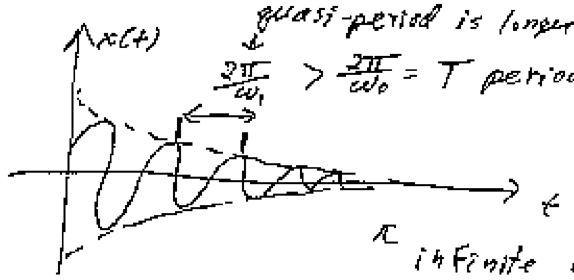
pure imaginary

define $\omega_1^2 \equiv \omega_0^2 - \beta^2 > 0$, $\sqrt{\beta^2 - \omega_0^2} = \pm i\omega_1$, $\omega_1 < \omega_0$

$$x(t) = A_1 e^{r_+ t} + A_2 e^{r_- t} = e^{-\beta t}\left[A_1 e^{+i\omega_1 t} + A_2 e^{-i\omega_1 t}\right]$$

$$= e^{-\beta t} A \cos(\omega_1 t + \delta)$$

$\frac{2\pi}{\omega_1}$ is the time between adjacent maxima (minima)



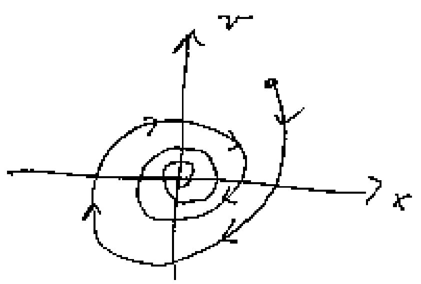
quasi-period is longer than T

$\frac{2\pi}{\omega_1} > \frac{2\pi}{\omega_0} = T$ period for undamped oscillation

infinite number of oscillations

Phase Diagram

infinite number of loops around origin



Mechanical energy is lost. → heat radiated away to ∞ in waves.

Case 2 $\omega_0 < \beta$ over damped

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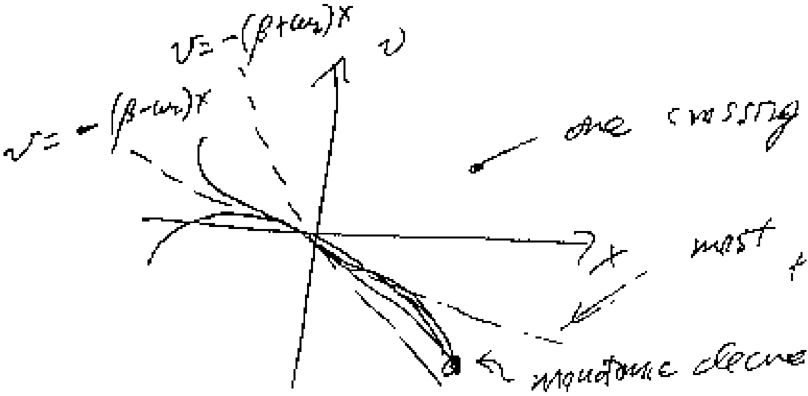
define $\omega_2^2 \equiv \beta^2 - \omega_0^2 > 0$

$$x(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} = e^{-\beta t}\left[A_1 \underset{\uparrow}{e^{+\omega_2 t}} + A_2 e^{-\omega_2 t}\right]$$

growing exponential, but $\beta > \omega_2$ kills it.

Phase Diagram



one crossing

most initial conditions follow this dashed line (A_1 term dominates for $t \to \infty$) unless $A_1 = 0$.

monotonic decrease to origin (no $x=0$ crossing)

Case 3 $\omega_0 = \beta$ critical damping

$r_+ = -\beta = r_-$ degerate roots of auxiliary equation

$A_1 e^{r_+ t}$ and $A_2 e^{r_- t}$ are no longer linearly independent solutions. Multiply by powers of t

why? $e^{\sqrt{\beta^2-\omega_0^2}\,t} \approx 1 + \sqrt{\beta^2-\omega_0^2}\,t + \cdots$ $\sqrt{\beta^2-\omega_0^2}$ small if $\beta \approx \omega_0$

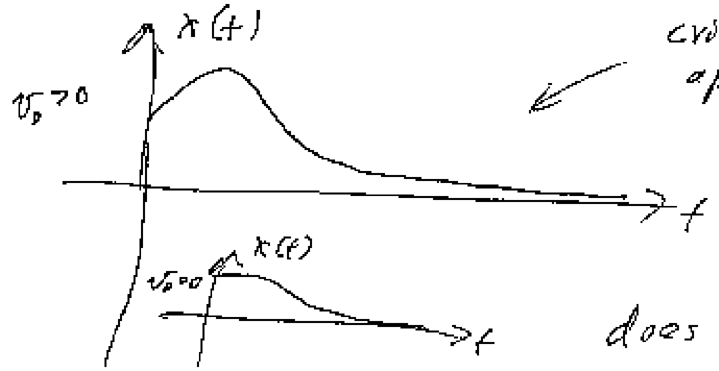
$$x(t) = A_1 e^{r_1 t} + t A_2 e^{r_2 t} = e^{-\beta t}(A_1 + t A_2)$$

verify 2nd solution ($A_1 e^{-\beta t}$ is of the form we guessed — guaranteed to work)

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$
$$\ddot{x} + 2\beta\dot{x} + \beta^2 x = 0$$

$$x = e^{-\beta t} t A_2$$
$$\dot{x} = e^{-\beta t} A_2 (1 - \beta t)$$
$$\ddot{x} = e^{-\beta t} A_2 (-2\beta + \beta^2 t)$$

$$e^{-\beta t} A_2 (-2\beta + \beta^2 t) + e^{-\beta t} A_2\, 2\beta (1 - \beta t) + e^{-\beta t} t A_2 \beta^2 \overset{?}{=} 0$$
$$-2\beta + \beta^2 t + 2\beta - 2\beta^2 t + t\beta^2 = 0 \checkmark$$

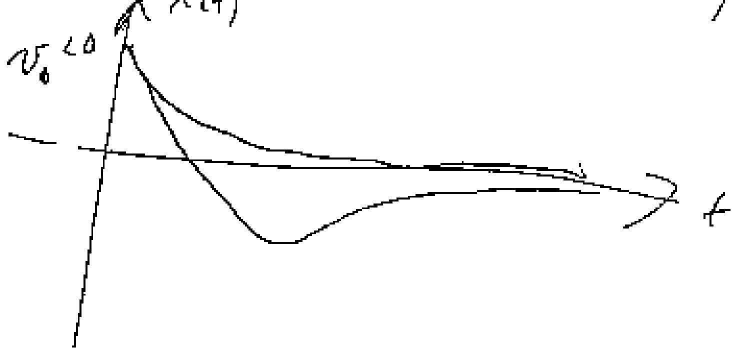


critically damped motion approaches x=0 faster than either underdamped or overdamped motion

once or zero

does not cross x = 0. an ∞ # of times

if β were any less it would cross x = 0. ∞ # of times



sinusoidally

$$m\ddot{x} = -kx - b\dot{x} + F_0 \cos(\underbrace{\omega t}) \quad \text{arbitrary driving frequency}$$

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos(\omega t) \qquad \beta = \frac{b}{2m} \text{ as before}$$

$$\mathcal{D}[x] = A\cos(\omega t) \qquad A = \frac{F_0}{m}$$

$\mathcal{D} = \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)$ operator.

solution = complementary function + particular solution

(solution to homogeneous eq.)
RHS = 0

no integration constants
steady state solution

transients - decay exp.

$$x(t) = x_c(t) + x_p(t)$$

$$\mathcal{D}[x_c(t)] = 0 \qquad \mathcal{D}[x_p(t)] = A\cos(\omega t)$$

We already know

$$x_c(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right]$$

under, critically or over-damped.

Tuning forks → boxes → Aluminum rod

Lecture #20

6 Nov 97

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)^{+\varphi}$$

$$x(t) = x_c(t) + x_p(t) \qquad \ddot{x}_c + 2\beta\dot{x}_c + \omega_0^2 x_c = 0$$

$$x_c(t) = e^{-\beta t}\left[A_1 e^{\sqrt{\beta^2-\omega_0^2}\,t} + A_2 e^{-\sqrt{\beta^2-\omega_0^2}\,t}\right] \left\{\begin{array}{l}\text{under-}\\ \text{critically-}\\ \text{over-damped}\end{array}\right.$$

Guess a form for the particular solution

$$x_p(t) = D\cos(\omega t \overset{+\varphi}{-} \delta)$$

response to forcing at frequency ω is oscillation at frequency ω. possible phase shift

D and δ are not arbitrary – they will be determined completely. Only A_1 and A_2 in complementary solution are up to you.

$$\dot{x}_p(t) = -\omega D\sin(\omega t - \delta)^{+\varphi}$$

$$\ddot{x}_p(t) = -\omega^2 D\cos(\omega t - \delta)^{+\varphi}$$

$$\ddot{x}_p + 2\beta\dot{x}_p + \omega_0^2 x_p = \frac{F_0}{m}\cos(\omega t)^{+\varphi}$$

$$D\Big[-\omega^2 \underset{\substack{\Downarrow\\ \cos(\omega t)\cos(\delta) + \sin(\omega t)\sin(\delta)}}{\cos(\omega t - \delta)} - 2\beta\omega \underset{\substack{\Downarrow\\ \sin(\omega t)\cos(\delta) - \cos(\omega t)\sin(\delta)}}{\sin(\omega t - \delta)} + \omega_0^2\cos(\omega t - \delta)\Big] = \frac{F_0}{m}\cos(\omega t)$$

$$\left\{\frac{F_0}{m} - D\left[(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta\right]\right\}\cos(\omega t)$$

$$- D\left[(\omega_0^2-\omega^2)\sin\delta - 2\omega\beta\cos\delta\right]\sin(\omega t) = 0 \qquad \text{for all } t$$

but sin(ωt) and cos(ωt) are linearly independent functions

$$\frac{F_0}{m} - D\left[(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta\right] = 0$$

$$D\left[(\omega_0^2-\omega^2)\sin\delta - 2\omega\beta\cos\delta\right] = 0 \qquad \Longleftarrow \text{ D drops out}$$

2nd eq. $\Rightarrow$ $$\tan\delta = \frac{\sin\delta}{\cos\delta} = \frac{2\omega\beta}{\omega_0^2-\omega^2}$$

$$\sin\delta = \frac{2\omega\beta}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

$$\cos\delta = \frac{\omega_0^2-\omega^2}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

1st eq. $\Rightarrow$ $$D = \frac{F_0/m}{(\omega_0^2-\omega^2)\cos\delta + 2\omega\beta\sin\delta}$$

$$D = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}$$

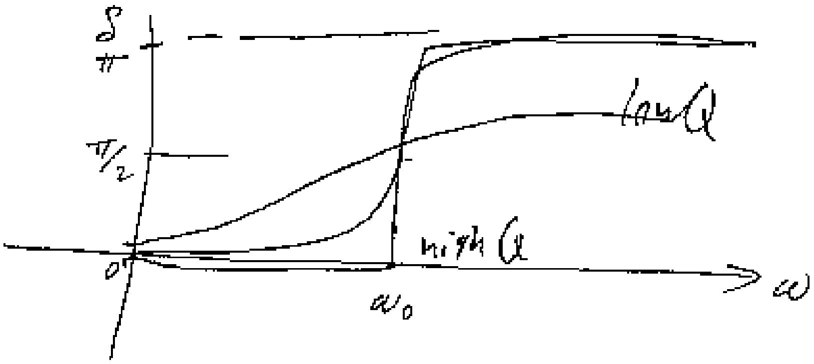
no damping $\beta = 0$
$D \to \infty$ as $\omega \to \omega_0$

Particular solution :

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+4\omega^2\beta^2}}\cos\left[\omega t \overset{+\varphi}{-} \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2-\omega^2}\right)\right]$$

no free parameters.

δ is the phase shift between input $F_0\cos(\omega t)$ and output (response) $x_p(t) = D\cos(\omega t - \delta)$

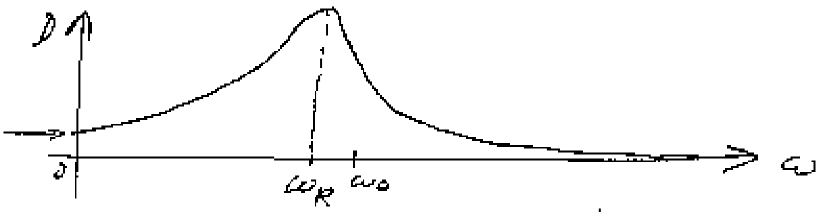


$\delta = 0$ no phase shift at $\omega \approx 0$ (D.C.) → response follows input
$\delta = \pi/2$ at $\omega = \omega_0$ (natural frequency without damping)
$\delta = \pi$ at $\omega \gg \omega_0$ ——→ response opposes input

Resonance

$\frac{F_0}{\omega_0^2 m} = \frac{F_0}{k}$

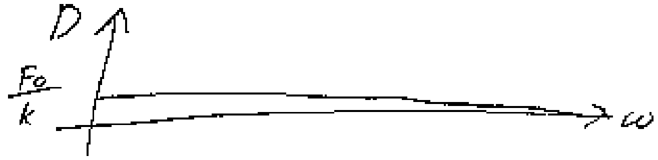
D.C. amplitude



$\left.\frac{dD}{d\omega}\right|_{\omega=\omega_R} = 0 \quad \Longrightarrow \quad \omega_R = \sqrt{\omega_0^2 - 2\beta^2}$ $\boxed{HW}$

$\omega_R < \omega_0$

For $\beta > \frac{\omega_0}{\sqrt{2}}$ there is no resonance (ω_R is imaginary)



Quality Factor $Q \equiv \frac{\omega_R}{2\beta}$

Foucault pendulum - many oscillations over several days

high Q oscillator.

tuning forks - 10^4

Not all physical quantities peak at the same frequency

Amplitude resonace frequency = $\omega_R = \sqrt{\omega_0^2 - 2\beta^2}$

= Potential energy resonance freq. ← depends on Amp^2

Kinetic energy resonance freq. $\omega_T = \omega_0$ $\boxed{HW}$

Velocity resonance freq. $\omega_v = ?$

Breit - Wigner ?

Simple Harmonic Oscillator (2-dim)

case 1 same spring constant in both directions

$$\vec{F} = -k\vec{r} \qquad F_x = m\ddot{x} = -kx \qquad \omega_0 \equiv \sqrt{\frac{k}{m}}$$

$$F_y = m\ddot{y} = -ky$$

$$x(t) = A\cos(\omega_0 t + \alpha)$$

$$y(t) = B\cos(\omega_0 t + \beta)$$

shape of the curve $y(x)$? eliminate time t

$$t = \frac{\cos^{-1}\left(\frac{x}{A}\right) - \alpha}{\omega_0}$$

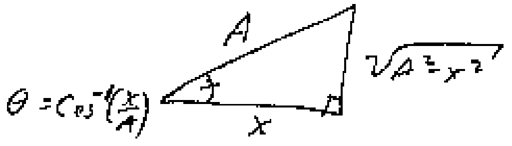
sub into $y(t)$

$$y(x) = B\cos\left[\omega_0 \frac{\cos^{-1}\left(\frac{x}{A}\right) - \alpha}{\omega_0} + \beta\right] = B\cos\left[\cos^{-1}\left(\frac{x}{A}\right) + \beta - \alpha\right]$$

$$\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$y(x) = B\cos\left[\cos^{-1}\left(\frac{x}{A}\right)\right]\cos(\beta - \alpha) - B\sin\left[\cos^{-1}\left(\frac{x}{A}\right)\right]\sin(\beta - \alpha)$$

$$\cos\left[\cos^{-1}\left(\frac{x}{A}\right)\right] = \frac{x}{A}$$



$$\sin\left[\cos^{-1}\left(\frac{x}{A}\right)\right] = \sin[\theta] = \frac{\text{opp}}{\text{hyp}} = \frac{\sqrt{A^2-x^2}}{A}$$

$$y(x) = \frac{B}{A}\cdot x\cos\underbrace{(\beta - \alpha)}_{\delta} - \frac{B}{A}\sqrt{A^2-x^2}\sin\underbrace{(\beta - \alpha)}_{\delta}$$

$$y(x) = \frac{B}{A}x\cos\delta - \frac{B}{A}\sqrt{A^2-x^2}\sin\delta$$

square both sides

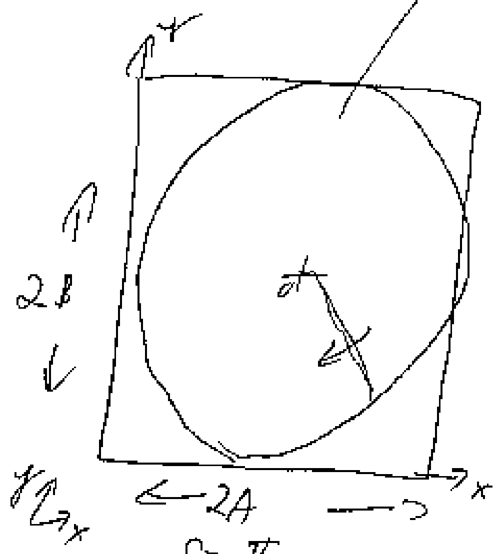
$$\Rightarrow \quad B^2x^2 - 2ABxy\cos\delta + A^2y^2 = A^2B^2\sin^2\delta$$

general quadratic form for

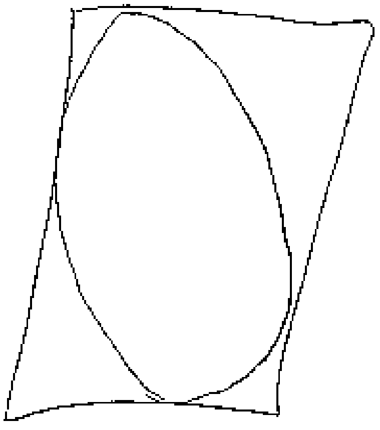
ellipse:

circle for A=B

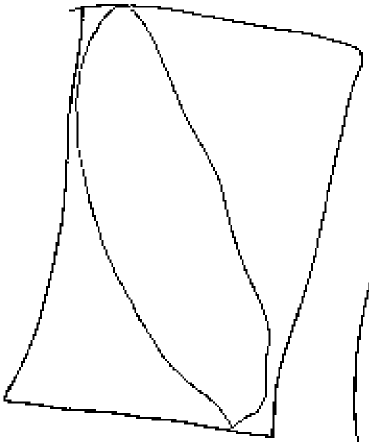
diagonal line is a degenerate ellipse



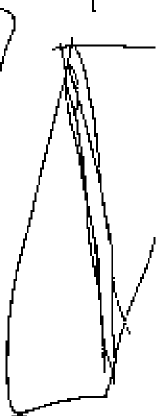
$\delta = \frac{\pi}{2}$



$\delta = 0, 2\pi$



A_{max} when B_{max}



$\delta = \pi$

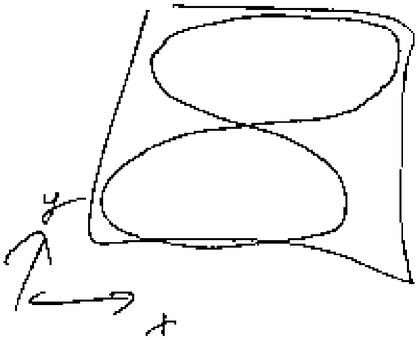
Case 2 - different spring constant for x and for y directions

$F_x = m\ddot{x} = -k_1 x$ $\qquad x(t) = A\cos(\omega_1 t + \alpha)$ $\qquad \omega_1 \equiv \sqrt{\frac{k_1}{m}}$

$F_y = m\ddot{y} = -k_2 y$ $\qquad y(t) = B\cos(\omega_2 t + \beta)$ $\qquad \omega_2 \equiv \sqrt{\frac{k_2}{m}}$

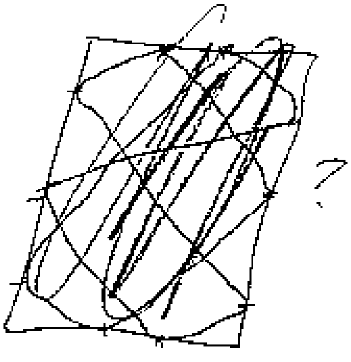
Lissajous figures.

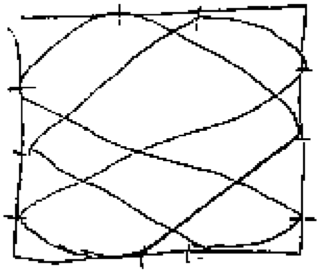
effect of Amplitude
frequency ratio
phase



$\omega_x \to \omega_y$

$\omega_x = 2\omega_y$



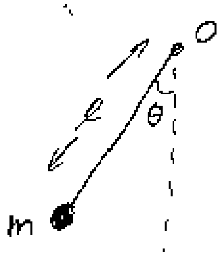


$\omega_x = \frac{3}{2}\omega_y$

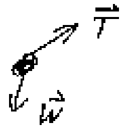
Lecture #18 centrifugal demo 30 Oct 97

Simple Pendulum (all mass in point)

$\theta < 0 \Rightarrow \ddot{\theta} > 0$
restoring torq



free-body



$$\Sigma \tau_0 = I_0 \alpha$$

$$T \cdot 0 - W l \sin\theta = m l^2 \ddot{\theta}$$

$$\ddot{\theta} = \frac{-g}{l} \sin\theta \qquad \text{not SHM}$$

but for small angles $\sin\theta \approx \theta$

$$\ddot{\theta} = \frac{-g}{l} \theta \qquad \text{looks like}$$

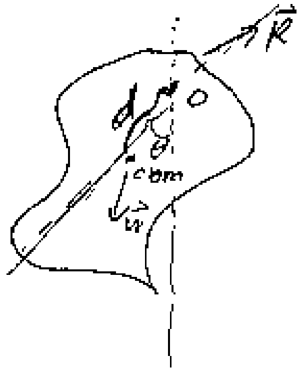
$$\ddot{x} = -\omega_0^2 x \qquad \Rightarrow \omega_0 = \sqrt{\frac{g}{l}}$$

Hooke's law spring is SH for both small <u>and</u> large Amplitudes
Pendula are SH only for small θ $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$ (θ in radia

whenever $\frac{\theta^3}{3!} \ll \theta$ $\pm 5^\circ$

Aside: the simple ~~pendulum~~ can be solved <u>without</u> the approximation

Physical Pendulum



$$\Sigma \tau_0 = I_0 \alpha$$

$$R \cdot 0 - W d \sin\theta = I_0 \alpha$$

$$-mgd \sin\theta = I_0 \ddot{\theta}$$

$$\ddot{\theta} \approx -\frac{mgd}{I_0} \theta \qquad \Rightarrow \omega_0 = \sqrt{\frac{mgd}{I_0}}$$