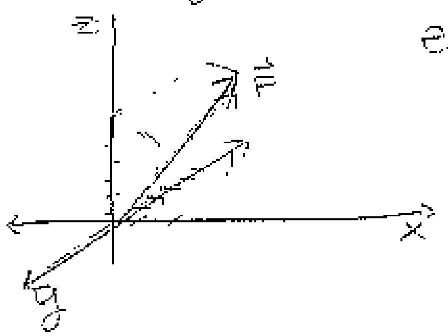
4321

http://www.physics.smu.edu/~scalise/p4321
~ weekly updated ~
Midterm & Final
labs 11 & 25 for Mathematica

Fourier Series

Analogy w/ vector decomposition



expand $\vec{F}$ in terms of components
Std. Basis:
$\{e_1, e_2, e_3\}$ or $\{e_x, e_y, e_z\}$

$$\vec{F} = \sum_{i=1}^{3} a_i \hat{e}_i = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

coefficients (a_i), basis vectors ($\hat{e}_i$)

$\therefore \vec{F} = 3\hat{e}_1 + 5\hat{e}_2 + 7\hat{e}_3$ (1)

Another Basis: orthonormal basis:
$\hat{u}_1 = (1, 0, -1)$ $\hat{u}_2 = (1, 1, 1)$ $\hat{u}_3 = (1, -2, 1)$
*orthogonal to one another & normalized (below)

$\hat{u}_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ $\hat{u}_2 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ $\hat{u}_3 = (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$

all ⊥ to one another, resulting "un"

9 relations:

$\hat{u}_1 \cdot \hat{u}_1 = 1$	$\hat{u}_2 \cdot \hat{u}_1 = 0$	$\hat{u}_3 \cdot \hat{u}_1 = 0$
$\hat{u}_1 \cdot \hat{u}_2 = 0$	$\hat{u}_2 \cdot \hat{u}_2 = 1$	$\hat{u}_3 \cdot \hat{u}_2 = 0$
$\hat{u}_1 \cdot \hat{u}_3 = 0$	$\hat{u}_2 \cdot \hat{u}_3 = 0$	$\hat{u}_3 \cdot \hat{u}_3 = 1$

when indices are the same $\delta_{ij} = 1$, otherwise $\delta_{ij} = 0$

$\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$ "Kronecker Delta" b's are just some more constant coefficients, like the as before... what we wanna solve for...

$$\vec{F} = \sum_{i=1}^{3} b_i \hat{u}_i = b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3$$

now find the b's... $\delta_{ji} = \hat{u}_j \cdot \hat{u}_i$

$$* \; \hat{u}_j \cdot \vec{F} = \sum_{i=1}^{3} b_i \underbrace{[\hat{u}_j \cdot \hat{u}_i]}_{\delta_{ji}} = \sum_{i=1}^{3} b_i \delta_{ji} = b_j \quad *$$

b/c $\delta_{ji} = 0$ unless $j=i$, in which case you get: $\hat{u}_j \cdot \vec{F} = 0 + 0 + b_j$

multiply in

∴ substitutes i in for j

coefficients from e_i basis

$$b_1 = \hat{u}_1 \cdot \vec{F} = \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right) \cdot (3,5,7) = \frac{-4}{\sqrt{2}} = \frac{3}{\sqrt{2}} + 0 - \frac{7}{\sqrt{2}}$$

$$b_2 = \hat{u}_2 \cdot \vec{F} = \frac{15}{\sqrt{3}}$$

$$\boxed{b_j = \hat{u}_j \cdot \vec{F}}$$

$$b_3 = \hat{u}_3 \cdot \vec{F} = 0$$

$$\therefore \vec{F} = -\frac{4}{\sqrt{2}}\hat{u}_1 + \frac{15}{\sqrt{3}}\hat{u}_2 + \cancel{0\hat{u}_3}$$

Basis vectors must be $\perp$, but not necessarily normal, (although advised)

$$|\vec{F}| \text{ in } \hat{e}_i : \sqrt{3^2\hat{e}_1\cdot\hat{e}_1 + 5^2\hat{e}_2\cdot\hat{e}_2 + 7^2\hat{e}_3\cdot\hat{e}_3}$$ * other terms are dropped b/c they're $\perp$

must equal:

$$|\vec{F}| \text{ in } \hat{u}_i : \sqrt{\tfrac{16}{2}\hat{u}_1\cdot\hat{u}_1 + \tfrac{225}{3}\hat{u}_2\cdot\hat{u}_2 + 0^2\hat{u}_3\cdot\hat{u}_3}$$

Now with functions:

Idea: Expand a function $F(t)$ in an "orthonormal" set of basis ~~vects~~ functions $\{u_i(t)\}$ infinite # of these functions

Analog of Dot Product: Dot Product: $\Sigma a_i b_i$

*normalize it!

$$u_i \cdot u_j = \langle u_i(t), u_j(t) \rangle = \frac{1}{N}\int_{t_1}^{t_2} u_i(t)\, u_j(t)\, dt = \delta_{ij}$$

handle separately

Fourier Basis Functions:

→ even functions

zero

$\{\, 1 = \cos(0\omega t),\ \cos(1\omega t),\ \cos(2\omega t),$ etc...

... $\sin(1\omega t),\ \sin(2\omega t),$ etc... $\}$, odd functions

together, this set is orthonormal & complete → can expand any function $F(t)$

* Homework on Fridays * ☺

$\langle 1 | 1 \rangle \Rightarrow$ dot product of 1 and 1

$= \frac{1}{N} \int_{t_1}^{t_1 + nT} 1 \cdot 1 \, dt = \frac{2}{nT} \int_0^T 1 \cdot 1 \, dt = 2$ → integrating over 1 period is plenty

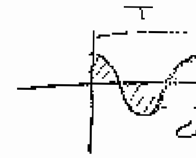
where T = period $T = \frac{2\pi}{\omega}$

ω = angular freq.

just make these 1 for simplicity

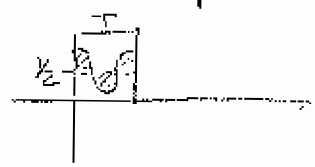
(*) case that gives us trouble, should've equaled 1

(*) $\langle 1 | \cos(\omega t) \rangle = \frac{2}{T} \int_0^T 1 \cdot \cos(\omega t) \, dt = 0$



cancels out

$\langle \cos(\omega t) | \cos(\omega t) \rangle = \frac{2}{T} \int_0^T \cos^2(\omega t) \, dt$



$= \frac{2}{T} \cdot \frac{1}{2} T = 1$ ✓

$\langle \cos(n\omega t) | \cos(p\omega t) \rangle = \delta_{np} \quad n, p \geq 1$

when same you get 1, 0 elsewhere.

$\langle \sin(n\omega t) | \sin(p\omega t) \rangle = \delta_{np}$

$\langle \cos(n\omega t) | \sin(p\omega t) \rangle = 0$, always

$$F(t) = \frac{a_0}{2} \cdot 1 + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]$$

* Now find a_n & b_n's ... like we did earlier to find b_1, multiply both sides by 1 and integrate $\frac{2}{T} \int_0^T \ldots dt$

lhs

$\langle 1 | F(t) \rangle = \frac{2}{T} \int_0^T 1 \cdot F(t) \, dt$ refer back to (*)

rhs: $\frac{2}{T} \int_0^T 1 \cdot \frac{a_0}{2} \cdot 1 \, dt + \sum_{n=1}^{\infty} \frac{2}{T} \int_0^T 1 \left[a_n \cos(n\omega t) + b_n \sin(n\omega t) \right]$

$\frac{2}{T} \int_0^T 1 \cdot F(t) \, dt = a_0$ $\frac{a_0}{2}$ is the time average of $F(t)$

multiply both sides by cos(pωt) and integrate $\frac{2}{T}\int_0^T \dots \, dt$ where $p \geq 1$

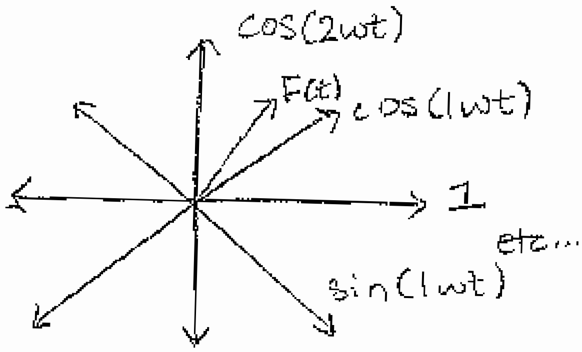
$$\langle \cos(p\omega t) | F(t) \rangle = \frac{a_0}{2} \underbrace{\langle \cos(p\omega t) | 1 \rangle}_{0} + \sum_{n=1}^{\infty} \Big[a_n \underbrace{\langle \cos(p\omega t) | \cos(n\omega t) \rangle}_{\delta_{np}} + b_n \underbrace{\langle \cos(p\omega t) | \sin(n\omega t) \rangle}_{0} \Big]$$

$$\sum_{n=1}^{\infty} a_n \delta_{np} = a_p$$

turns all n's to p's & removes Σ & δ ☺

$$\boxed{a_p = \frac{2}{T}\int_0^T \cos(p\omega t) F(t) \, dt}$$

$$\boxed{b_p = \frac{2}{T}\int_0^T \sin(p\omega t) F(t) \, dt}$$



infinite dimensions: to establish F(t)

* If F(t) is even: F(t) = F(−t), Then $b_p = 0 \ \forall_p$ (for all p's)

* If F(t) is odd: F(t) = −F(−t), then $a_p = 0 \ \forall_p$

If F(t) is neither even, nor odd, then you need all a_p's & b_p's ...

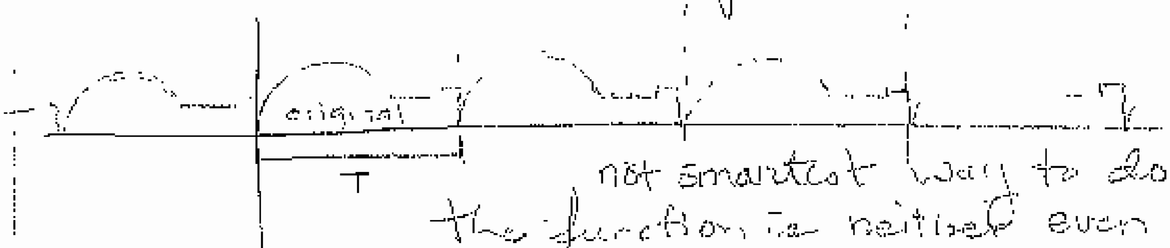
⇒ sines & cosines are periodic, ∴ the function F(t) to be expanded in <u>denumerably</u> many basis functions must therefore be periodic as well

① truly periodic ... heartbeat:

you can use more than 1 heartbeat to do a period, but be easy on yourself & keep it to 1

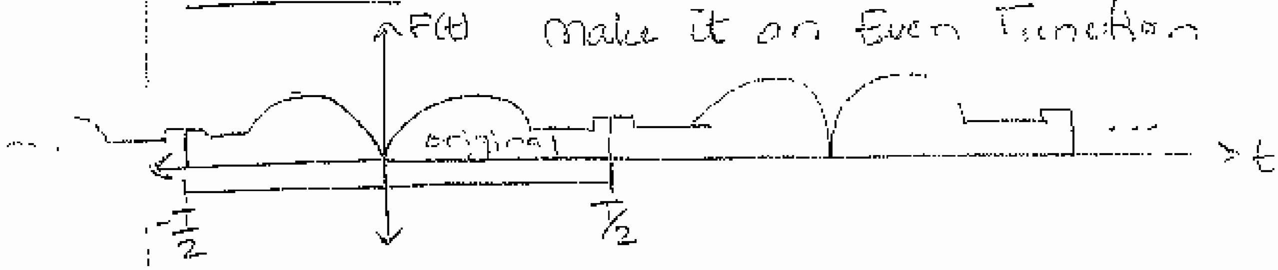
② F(t) could be defined on a finite time interval

& we don't care what it is outside of the range, so just pretend that it's periodic & expand to make life easier



not smartest way to do it b/c the function is neither even nor odd — so you need all a's & b's

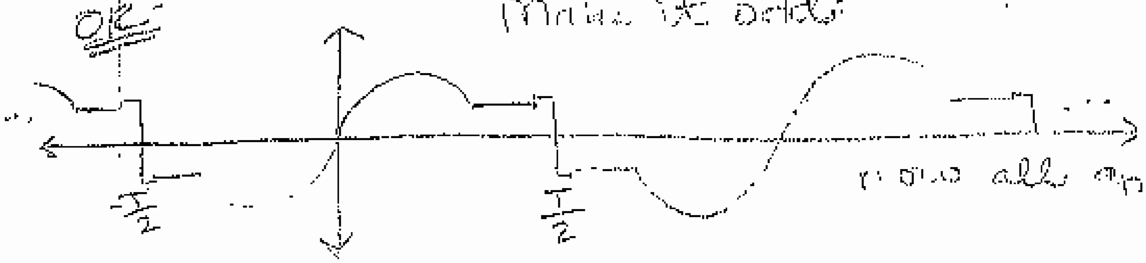
SMARTER WAY: make it an Even Function



Now F(t) is even & all b_n's are 0...

Make it odd:

OR:



now all a_n's are 0

Consider:

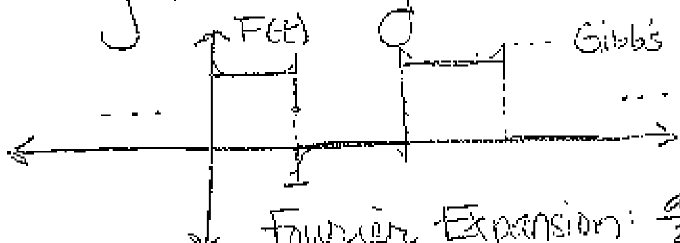
~What if the interval is not finite & F(t) is not periodic, although defined from $-\infty < t < \infty$

→ Then you need all frequencies (not a "denumerable" [illegible] distance) [illegible] the [illegible]

these become fourier integrals

SPS Thurs. 6:30pm;
Feb 11
w/ pizza ☺

discontinuous functions can be fudged into working this way:



... Gibb's phenomenon → no matter how precise you get, these discrepancies will still appear

Fourier Expansion: $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$

Show the completeness of a function:

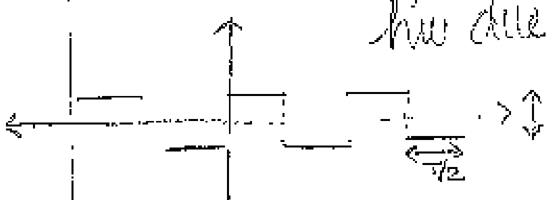
$\vec{F} = \sum_{n=1}^{3} b_n \hat{u}_n \qquad b_n = \hat{u}_n \cdot \vec{F}$ substitute

$\vec{F} = \sum_{n=1}^{3} (\hat{u}_n \cdot \vec{F}) \hat{u}_n \qquad \vec{F} = \sum_{n=1}^{3} (\vec{F} \cdot \hat{u}_n) \hat{u}_n$

$\vec{F} = \vec{F} \cdot \underbrace{\sum_{n=1}^{3} \hat{u}_n \cdot \hat{u}_n}_{\text{must eq. a identity matrix}}$

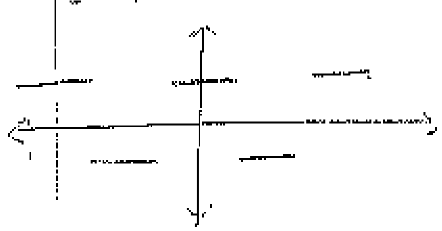
orthonormality
$\hat{u}_n \cdot \hat{u}_p = \delta_{np}$
→ orthonormality & completeness are rather the same, just different conditions

hw due: week from Friday ☺



50% duty cycle — on half the time

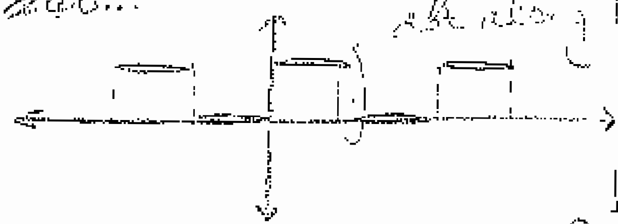
* putting x-axis there makes the function odd (like sin), so that all a_n's (including a_0) → 0



* putting y-axis here creates an even function → all b_n's are 0 + a_0 is 0 b/c that is the average

[illegible] add...

[illegible] how to [illegible] the fourier transform...



$a_1 \ldots a_\infty = 0$

$a_0 = \frac{h}{2}$

b/c it's basically an odd function (just moved down by a constant)

on $0 \le t \le T$ $\quad F(t) = \begin{cases} h & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} < t \le T \end{cases}$

"[illegible] the [illegible]"

$$a_0 = \underbrace{\langle 1 | F(t) \rangle}_{\text{dot product}} = \frac{2}{T}\int_0^T 1 \cdot F(t)\,dt$$

[illegible]

$$a_0 = \left[\int_0^{T/2} 1 \cdot h\,dt + \int_{T/2}^{T} 1 \cdot 0\,dt\right] = \frac{2}{T} h \frac{T}{2} = h \checkmark$$

$\frac{a_0}{2} =$ average value of function $\qquad \frac{a_0}{2} = \frac{h}{2}$ $\quad$ [illegible]

$$a_n = \langle \cos(n\omega t) | F(t) \rangle = \frac{2}{T}\int_0^T \cos(n\omega t) F(t)\,dt$$

$$a_n = \frac{2}{T}\left[\int_0^{T/2} \cos\left(\frac{2\pi n t}{T}\right) \cdot h\,dt + \int_{T/2}^{T} 0\,dt\right]$$

from cos; $\omega = \frac{2\pi}{T}$; $F(t)$ from $0 \to T/2$

$$a_n = \frac{2}{T} h \underbrace{\frac{T}{2\pi n}}_{\text{chain rule}} \sin\left(\frac{2\pi n t}{T}\right)\Big|_0^{T/2} = \frac{h}{\pi n}\left[\sin(n\pi) - \sin(0)\right]$$

$\sin(n\pi) = 0$, $\sin(0) = 0$

$\therefore$ all a_n's $= 0$

$$b_n = \langle \sin(n\omega t) | F(t) \rangle = \frac{2}{T}\int_0^T \sin(n\omega t) F(t)\,dt$$

$$b_n = \frac{2}{T}\left[\int_0^{T/2} \sin(n\omega t) \cdot h\,dt + 0\right]$$

$F(t)$ [illegible]

$b_n = -\frac{2}{T} h \frac{1}{n\omega} \cos(n\omega t)\Big|_0^{T/2} = -\frac{2}{T} h \frac{T}{2\pi n} \cos\left(\frac{2\pi n t}{T}\right)\Big|_0^{T/2}$

chain rule

$b_n = \frac{h}{n\pi}\left[\cos(0) - \cos(n\pi)\right] = \frac{h}{n\pi}\left[1-(-1)^n\right]$

$b_1 = \frac{h}{\pi}[2] = \frac{2h}{\pi}$ $\quad b_2 = \frac{h}{2\pi}[0] = 0$ & for all even values of n, $b_n = 0$

always gives you factor of 2 when n is odd

$\boxed{b_{n=odd} = \frac{2h}{n\pi}}$ & graph.

* moving origin vertically only changes a_0
* moving origin horizontally changes [illegible]

$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$

$F(t) = \frac{a_0}{2} + b_1 \sin(1\omega t) + b_3 \sin(3\omega t) + b_5 \sin(5\omega t) + \ldots$

$F(t) = \frac{h}{2} + \frac{2h}{\pi}\sin\left(\frac{2\pi t}{T}\right) + \frac{2h}{3\pi}\sin\left(\frac{6\pi t}{T}\right) + \frac{2h}{5\pi}\sin\left(\frac{10\pi t}{T}\right) + \ldots$

ω

Condensed Versions:

Sum over odds $\quad F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{odd\ n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2\pi n t}{T}\right)$ $\quad n = 2k+1$

sum over all $\quad F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)}\sin\left[\frac{2\pi(2k+1)t}{T}\right]$

so they're always odd

shifted from 0 over all values $\quad F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{j=1}^{\infty} \frac{1}{(2j-1)}\sin\left[\frac{2\pi(2j-1)t}{T}\right]$

* Take it out to enough terms to look like original function & you be the judge