Cartesian unit vectors

displacement: $\vec{r} = x\hat{e}_1 + y\hat{e}_2 + z\hat{e}_3 = \sum_n x_n \hat{e}_n$

velocity: $\vec{v} = \frac{d\vec{x}}{dt} = \frac{d}{dt}\left(\sum_n x_n \hat{e}_n\right)$

$$= \sum_n \left[\left(\frac{dx_n}{dt}\right)\hat{e}_n + x_n \cancelto{0}{\frac{d\hat{e}_n}{dt}} \right] \quad \text{product rule}$$

but the Cartesian unit vectors are constant

$$\vec{v} = \sum_n \left(\frac{dx_n}{dt}\right)\hat{e}_n = \sum_n v_n \hat{e}_n$$

Time derivatives occur so often, they are denoted

$$\vec{v} = \dot{\vec{r}} \qquad \vec{v} = \sum_n \dot{x}_n \hat{e}_n \qquad v_i = \dot{x}_i$$

---

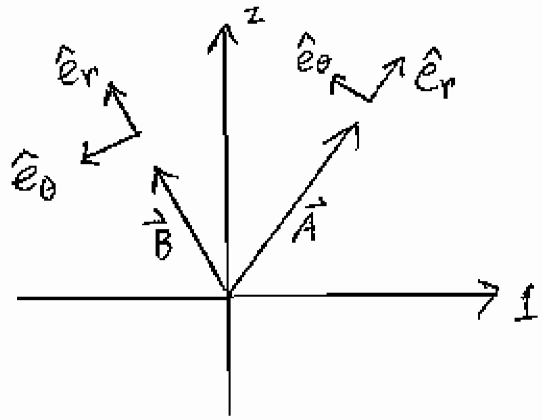
acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} = \ddot{\vec{r}}$

$$\vec{a} = \frac{d}{dt}\sum_n \left(\frac{dx_n}{dt}\right)\hat{e}_n = \sum_n \left(\frac{d^2x_n}{dt^2}\right)\hat{e}_n = \sum_n a_n \hat{e}_n$$

If the Cartesian unit vectors are so simple, why use anything else?

For circular problems, they are very much the hard way!

Two-dimensional Polar Unit Vectors



$\hat{e}_r$ and $\hat{e}_\theta$ are not constant in direction like Cartesian unit vectors $\hat{e}_1$ and $\hat{e}_2$.

$\hat{e}_r$ points along the vector, radially outward.

$\hat{e}_\theta$ points in the direction of increasing θ, that is, counterclockwise.

$$\hat{e}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{e}_1 + y\hat{e}_2}{\sqrt{x^2+y^2}} \qquad \hat{e}_r \cdot \hat{e}_r = 1$$

$\hat{e}_\theta$ must be perpendicular to $\hat{e}_r$, $\hat{e}_\theta$ must have length 1, and $\hat{e}_\theta$ must point counterclockwise.

$$\hat{e}_\theta = \frac{-y\hat{e}_1 + x\hat{e}_2}{\sqrt{x^2+y^2}}$$

$$\hat{e}_\theta \cdot \hat{e}_\theta = 1$$

$$\hat{e}_\theta \cdot \hat{e}_r = 0$$

It is true that $\dot{\vec{r}} = \vec{v}$, but $\dot{r} \neq v$.

$$r \equiv \sqrt{x^2+y^2}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d}{dt}(x^2+y^2)^{1/2} = \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}}$$ (check this)

$$\vec{r} = x\hat{e}_1 + y\hat{e}_2$$

$$\dot{\vec{r}} = \vec{v} = \dot{x}\hat{e}_1 + \dot{y}\hat{e}_2$$

$$|\vec{v}| = \sqrt{\dot{x}^2+\dot{y}^2}$$ clearly, this is not $\dot{r}$

$\dot{r}$ means take the magnitude of $\vec{r}$ first, then the time derivative.

v means take the time derivative of $\vec{r}$, then the magnitude.

Marion is confusing on this point in 1-32, page 47.

$$\theta \equiv \arctan\left(\frac{y}{x}\right)$$

$$\dot{\theta} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{x\dot{y}-y\dot{x}}{x^2} = \frac{x\dot{y}-y\dot{x}}{x^2+y^2}$$

$\hat{e}_r$ and $\hat{e}_\theta$ are not constant.

$$\dot{\hat{e}}_r = \frac{d}{dt}\hat{e}_r = \frac{d}{dt}\left(\frac{x\hat{e}_1 + y\hat{e}_2}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2}{\sqrt{x^2+y^2}} + (x\hat{e}_1 + y\hat{e}_2)\frac{d}{dt}\left(\frac{1}{\sqrt{x^2+y^2}}\right)$$ chain rule

$$= \frac{\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2}{\sqrt{x^2+y^2}} - \frac{(x\hat{e}_1 + y\hat{e}_2)(x\dot{x}+y\dot{y})}{(x^2+y^2)^{3/2}}$$

$$= \frac{(\dot{x}\hat{e}_1 + \dot{y}\hat{e}_2)(x^2+y^2) - (x\hat{e}_1 + y\hat{e}_2)(x\dot{x}+y\dot{y})}{(x^2+y^2)^{3/2}}$$

$$= \frac{\cancel{x^2\dot{x}\hat{e}_1} + x^2\dot{y}\hat{e}_2 + y^2\dot{x}\hat{e}_1 + \cancel{y\dot{y}\hat{e}_2} - (\cancel{x^2\dot{x}\hat{e}_1} + xy\dot{x}\hat{e}_2 + xy\dot{y}\hat{e}_1 + \cancel{y^2\dot{y}\hat{e}_2})}{(x^2+y^2)^{3/2}}$$

$$= \frac{x^2\dot{y}\hat{e}_2 + y^2\dot{x}\hat{e}_1 - xy\dot{x}\hat{e}_2 - xy\dot{y}\hat{e}_1}{(x^2+y^2)^{3/2}} = \frac{(-y\hat{e}_1 + x\hat{e}_2)(x\dot{y} - y\dot{x})}{(x^2+y^2)^{3/2}}$$

$$= \frac{(-y\hat{e}_1 + x\hat{e}_2)}{\sqrt{x^2+y^2}}\,\frac{(x\dot{y}-y\dot{x})}{x^2+y^2} = \hat{e}_\theta\dot{\theta}$$

Similarly

$$\dot{\hat{e}}_\theta = \frac{d}{dt}\hat{e}_\theta = \frac{d}{dt}\left(\frac{-y\hat{e}_1 + x\hat{e}_2}{\sqrt{x^2+y^2}}\right) = -\hat{e}_r\dot{\theta}$$

Also useful:

$$\hat{e}_r = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$$

$$\hat{e}_\theta = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$$

Velocity and Acceleration in 2-dim Polar Coord's

Displacement: $\vec{r} = r\hat{e}_r$ (no $\hat{e}_\theta$ component)

Velocity: $\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{e}_r) = \dot{r}\hat{e}_r + r\dot{\hat{e}}_r$

$$= \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = v_r\hat{e}_r + v_\theta\hat{e}_\theta$$

$v_r = \dot{r}$ = radial velocity

$v_\theta = r\dot{\theta}$ = tangential velocity

Acceleration: $\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta)$

$$\vec{a} = \ddot{r}\hat{e}_r + \dot{r}\dot{\hat{e}}_r + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}\dot{\hat{e}}_\theta$$

$$= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta + r\dot{\theta}(-\dot{\theta}\hat{e}_r)$$

$$= (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

$$= a_r\hat{e}_r + a_\theta\hat{e}_\theta$$

$a_r = \ddot{r} - r\dot{\theta}^2$ = radial (centripetal) acceleration

$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ = tangential acceleration

Derivatives of Scalar and Vector Functions in Cartesian Coordinates

Let $\Phi(x,y,z)$ be a scalar function of the coordinates x, y, z

e.g. $\Phi(x,y,z) = x^2 + y\sin(z)$

Φ could represent the temperature in a room.

Let $\vec{A}(x,y,z)$ be a vector function of x,y,z.

$$\vec{A}(x,y,z) = \begin{pmatrix} A_1(x,y,z) \\ A_2(x,y,z) \\ A_3(x,y,z) \end{pmatrix}$$

each component of $\vec{A}$ depends on x, y, and z.

e.g. $\vec{A} = \hat{e}_1\, xy^2 + \hat{e}_2 \sin(z) + \hat{e}_3\, 4 = \begin{pmatrix} xy^2 \\ \sin(z) \\ 4 \end{pmatrix}$

The Gradient of a scalar function is

$$\vec{\nabla}\Phi = \hat{e}_1 \frac{\partial\Phi}{\partial x} + \hat{e}_2 \frac{\partial\Phi}{\partial y} + \hat{e}_3 \frac{\partial\Phi}{\partial z}$$

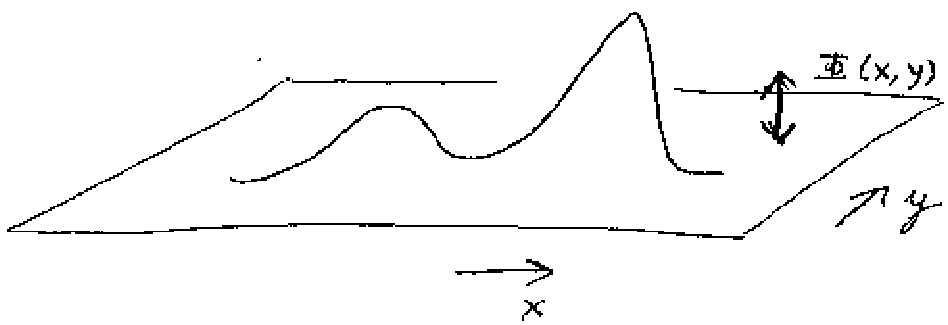
e.g. if $\Phi = x^2 + y\sin(z)$

then $\vec{\nabla}\Phi = \hat{e}_1\, 2x + \hat{e}_2 \sin(z) + \hat{e}_3\, y\cos(z) = \begin{pmatrix} 2x \\ \sin(z) \\ y\cos(z) \end{pmatrix}$

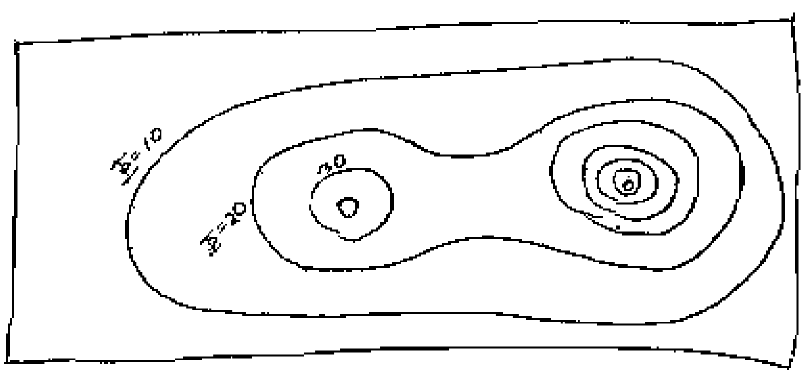
Notice that Φ is a scalar, but $\vec{\nabla}\Phi$ is a vector.

Physically, the gradient points in the direction of steepest increase of the function.

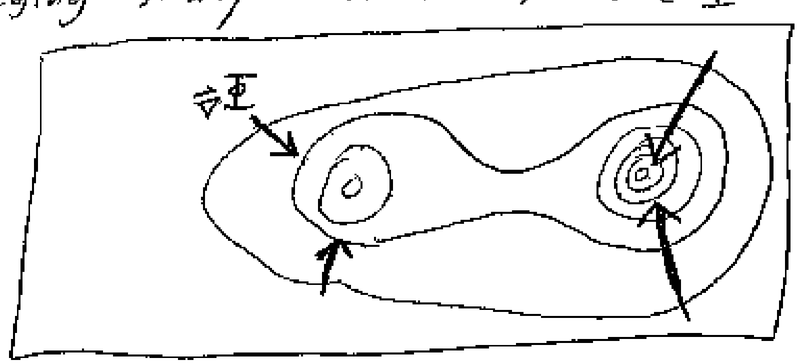
e.g. Let Φ be the height of terrain as a function of latitude x and longitude y.

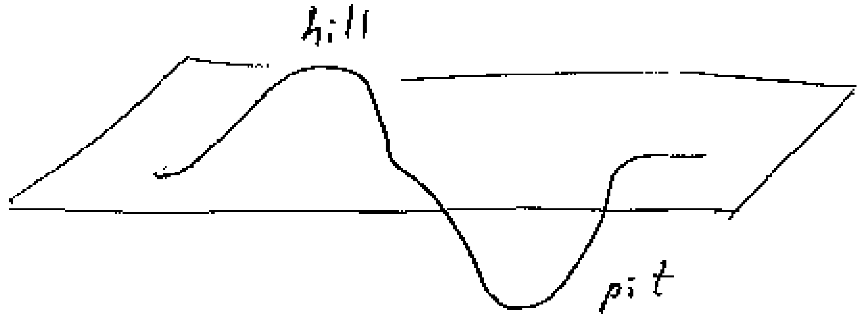


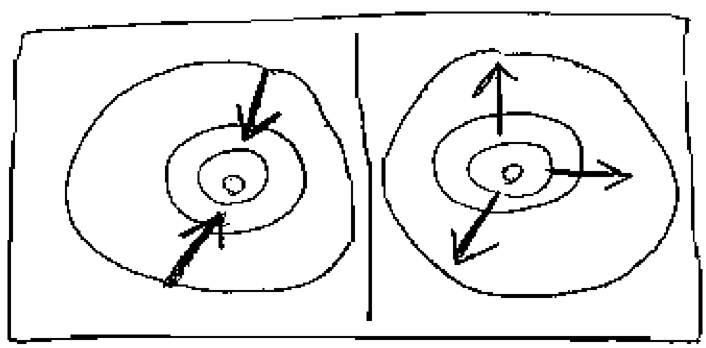
A topographic map shows curves of constant height, that is, curves on which $\Phi(x,y) =$ constant.



The gradient points uphill. It is zero at the peaks. It is small in magnitude where Φ is changing slowly — it is large where Φ changes quickly.







<u>Laplacian</u> $\nabla^2 \Phi = (\vec{\nabla} \cdot \vec{\nabla}) \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$

eg. $\Phi = x^2 + y \sin(z)$

$$\nabla^2 \Phi = 2 + 0 + (-y \sin(z)) = 2 - y \sin(z)$$

Φ is a scalar and $\nabla^2 \Phi$ is also a scalar function.

In mechanics $\nabla^2 \Phi = 0$ describes surfaces of minimal area subject to some boundary conditions. (soap bubble films on wire frames).

In E+M $\nabla^2 \Phi = 0$ describes voltage (or potential) that minimizes the energy in a configuration of charges. The two problems are <u>completely</u> analogous.

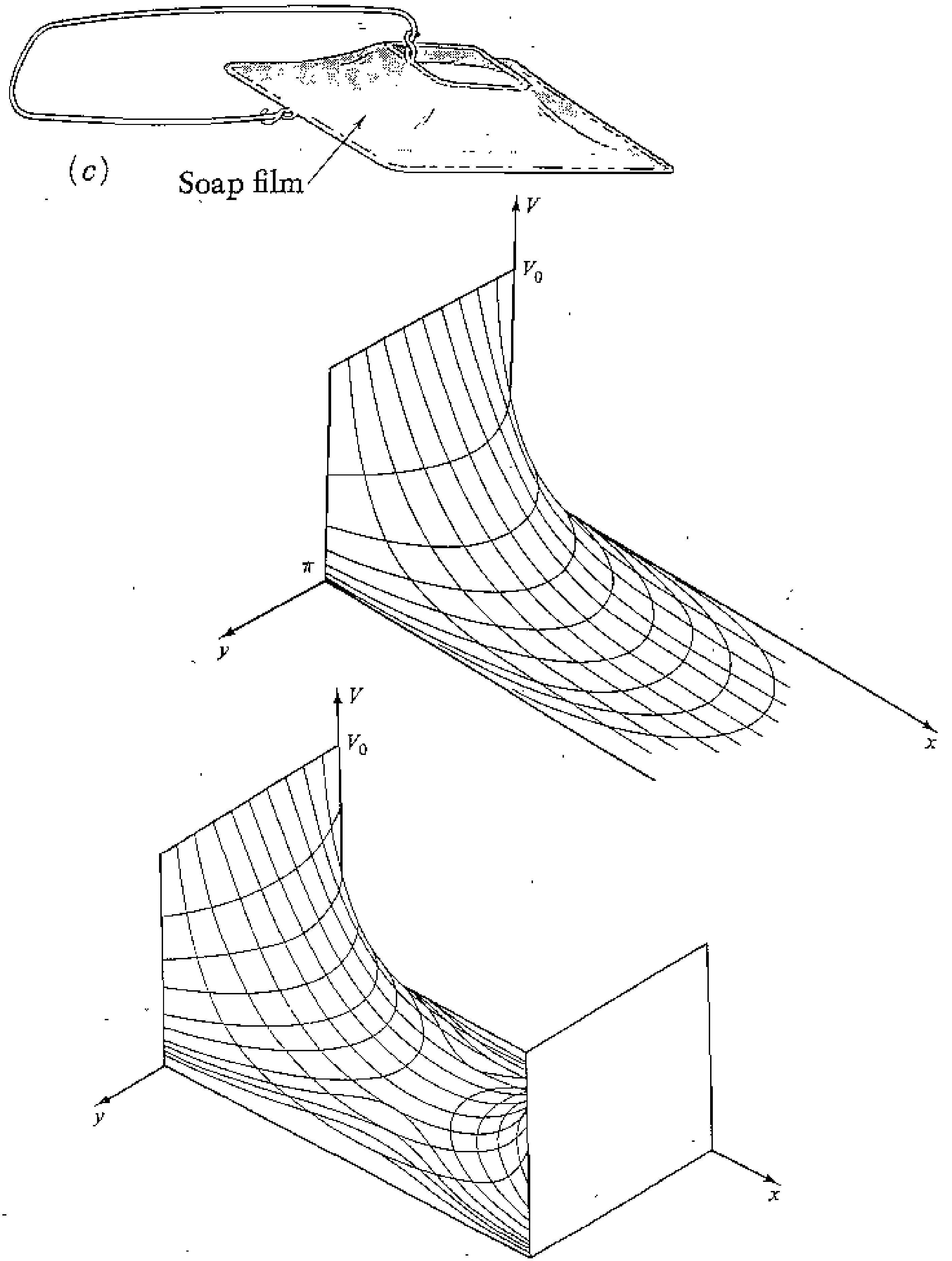


Figure 3.20 (From *Electromagnetic Fields and Waves*, Second Edition, Dale Lorrain and Paul R. Corson, W. H. Freeman and Company © 1970.)

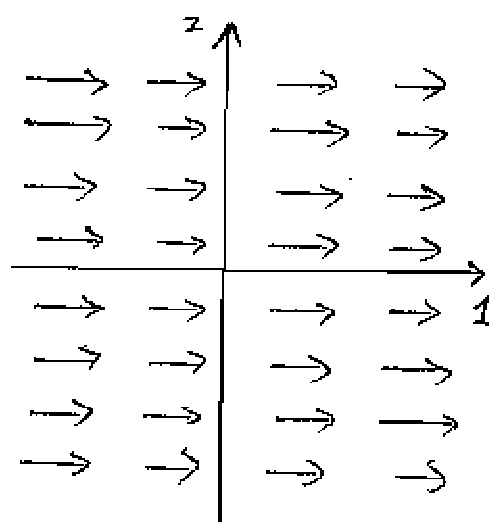
The <u>divergence</u> of a vector function $\vec{A}(x,y,z)$ is a scalar function

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

eg if $\vec{A} = \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = \begin{pmatrix} xy^2 \\ \sin(z) \\ 4 \end{pmatrix}$ then $\vec{\nabla}\cdot\vec{A} = y^2 + 0 + 0 = y^2$

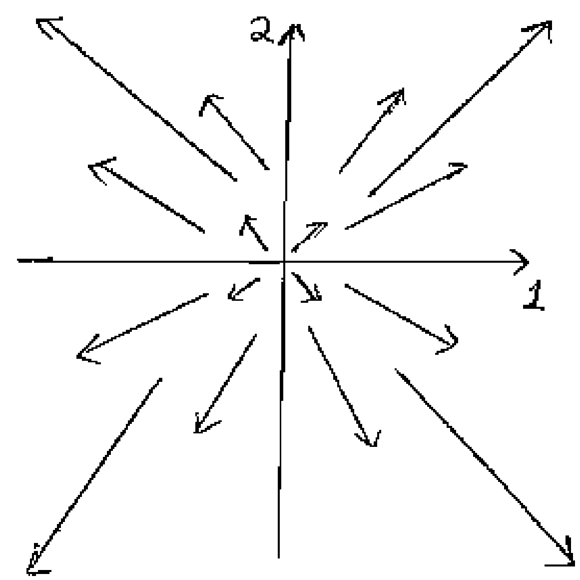
The name "divergence" comes from fluid dynamics and means a <u>net</u> flow out of (+) or into (−) a region.

$\vec{A} = 3\hat{e}_1$



$\vec{\nabla}\cdot\vec{A} = 0$

$\vec{A} = 2\vec{r} = 2x\hat{e}_1 + 2y\hat{e}_2 + 2z\hat{e}_3$



$\vec{\nabla}\cdot\vec{A} \neq 0$

(homework)

$(\vec{\nabla}\cdot\vec{A})$ is a scalar <u>function</u>, not necessarily a number, so divergence of $\vec{A}$ can vary from point to point.

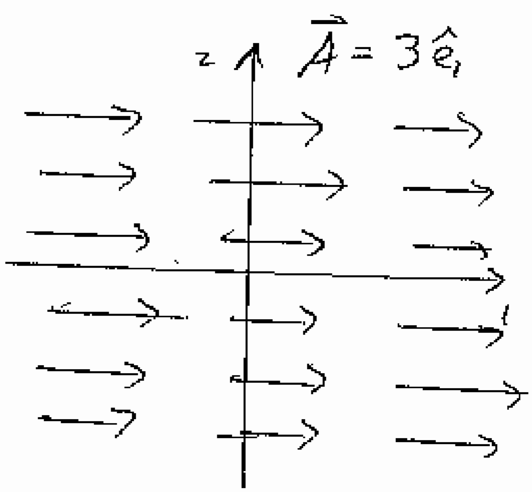
$$\vec{\nabla}\cdot\vec{A}\Big|_{(1,2,7)} \neq \vec{\nabla}\cdot\vec{A}\Big|_{(9,0,0)}$$

The curl (or rotation) of a vector function $\vec{A}$ is also a vector function.

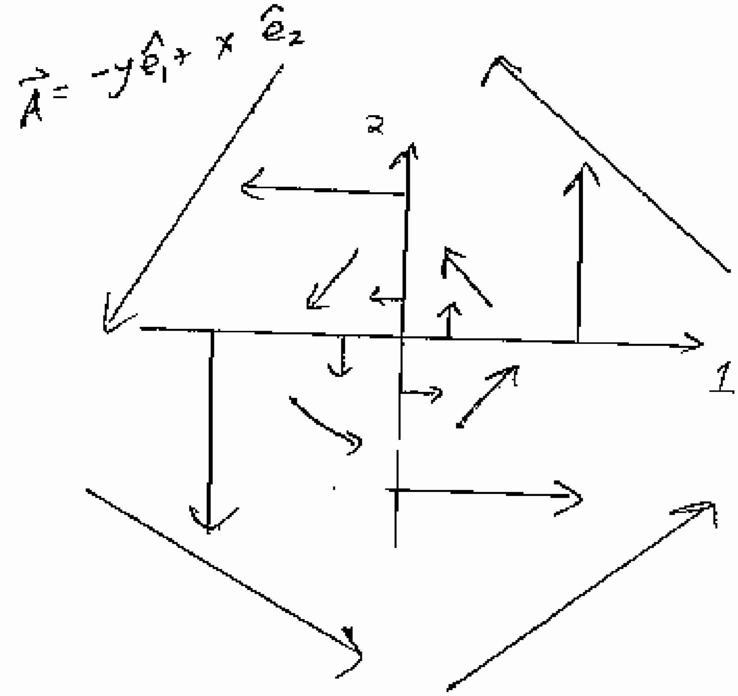
$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1(x,y,z) & A_2(x,y,z) & A_3(x,y,z) \end{vmatrix}$$

$$= \hat{e}_1\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \hat{e}_2\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) + \hat{e}_3\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

The name "curl" also comes from fluid dynamics and means the tendency for a probe immersed in the fluid to rotate.



$$\vec{\nabla} \times \vec{A} = 0$$



$$\vec{\nabla} \times \vec{A} = 0\hat{e}_1 + 0\hat{e}_2 + (1+1)\hat{e}_3$$

$$= 2\hat{e}_3$$

out of plane (right hand rule)

$(\vec{\nabla} \times \vec{A})$ is a vector function, it can vary with position.

General Coordinates in 3 dimensions

e.g.

Bipolar	EllipticCylindrical
Bispherical	OblateSpheroidal
Cartesian	ParabolicCylindrical
ConfocalEllipsoidal	Paraboloidal
ConfocalParaboloidal	ProlateSpheroidal
Conical	Spherical
Cylindrical	Toroidal

Coordinate systems.

(q_1, q_2, q_3) : Cartesian (x, y, z)

Spherical Polar (r, θ, φ)

Cylindrical Polar (ρ, φ, z)

The directions along increasing q_n are $\hat{e}_n$

They are orthonormal : $\hat{e}_n \cdot \hat{e}_m = \delta_{nm}$

Scalar functions of three coordinates : $\Phi(q_1 q_2 q_3)$

Vector functions of three coordinates:

$$\vec{A} = \begin{pmatrix} A_1(q_1 q_2 q_3) \\ A_2(q_1 q_2 q_3) \\ A_3(q_1 q_2 q_3) \end{pmatrix}$$

where A_1 is the q_1 component of $\vec{A}$ (in the direction of increasing q_1)

e.g. In Spherical Polar $\hat{e}_r$ points along increasing r, that is, radially outward. $\hat{e}_\theta$ points along increasing θ, that is, south.

The unit vectors are themselves functions of the coordinates (only constant in Cartesian system).

$$\hat{e}_i(q_1 q_2 q_3)$$

And finally, there are scale functions that also depend on the coordinates;

$$h_1(q_1 q_2 q_3) \qquad h_2(q_1 q_2 q_3) \qquad h_3(q_1 q_2 q_3)$$

(In Cartesian, $h_1 = 1 = h_2 = h_3$).

Line element: $\vec{ds}^{\,2} = ds^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$

Gradient: $\vec{\nabla}\Phi(q_1 q_2 q_3) = \frac{\hat{e}_1}{h_1}\frac{\partial\Phi}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial q_3}$

Divergence:

$$\vec{\nabla}\cdot\vec{A}(q_1 q_2 q_3) = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_3 h_1) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right]$$

Laplacian: $\nabla^2\Phi = \vec{\nabla}\cdot(\vec{\nabla}\Phi)$ divergence of gradient Φ

$$\nabla^2\Phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\Phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\Phi}{\partial q_3}\right)\right]$$

Curl: $$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{e}_1 h_1 & \hat{e}_2 h_2 & \hat{e}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

For example, the 1 component is

$$(\vec{\nabla}\times\vec{A})_1 = \frac{1}{h_1 h_2 h_3}\,\hat{e}_1 h_1 \begin{vmatrix} \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{\hat{e}_1}{h_2 h_3}\left[\frac{\partial(h_3 A_3)}{\partial q_2} - \frac{\partial(h_2 A_2)}{\partial q_3}\right]$$

And Finally, the Volume element is

$$dV = h_1 h_2 h_3 \, dq_1 \, dq_2 \, dq_3$$

Next, we will specialize these general definitions to Cylindrical Polar and Spherical Polar Coordinates.

Cylindrical: $q_1 = \rho \quad q_2 = \varphi \quad q_3 = z$

$h_1 = 1 \quad h_2 = \rho \quad h_3 = 1$

Spherical: $q_1 = r \quad q_2 = \theta \quad q_3 = \varphi$

$h_1 = 1 \quad h_2 = r \quad h_3 = r\sin\theta$

VECTOR DERIVATIVES

CARTESIAN. $d\mathbf{l} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$; $d\tau = dx\,dy\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial x}\hat{\imath} + \frac{\partial t}{\partial y}\hat{\jmath} + \frac{\partial t}{\partial z}\hat{k}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\imath} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{k}$$

Laplacian. $$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}$$

SPHERICAL. $d\mathbf{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$; $d\tau = r^2\sin\theta\,dr\,d\theta\,d\phi$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}$$

Curl. $$\nabla \times \mathbf{v} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial \phi}\right]\hat{r}$$
$$+ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r}(r v_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial \theta}\right]\hat{\phi}$$

Laplacian. $$\nabla^2 t = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial t}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial t}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 t}{\partial \phi^2}$$

CYLINDRICAL $d\mathbf{l} = dr\,\hat{r} + r\,d\phi\,\hat{\phi} + dz\,\hat{z}$; $d\tau = r\,dr\,d\phi\,dz$

Gradient. $$\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}$$

Divergence. $$\nabla \cdot \mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}$$

Curl. $$\nabla \times \mathbf{v} = \left[\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\hat{\phi}$$
$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\phi) - \frac{\partial v_r}{\partial \phi}\right]\hat{z}$$

Laplacian. $$\nabla^2 t = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 t}{\partial \phi^2} + \frac{\partial^2 t}{\partial z^2}$$

FUNDAMENTAL CONSTANTS

$\epsilon_0 = 8.85 \times 10^{-12}$ coul²/N-m²	(permittivity of free space)
$\mu_0 = 4\pi \times 10^{-7}$ N/A²	(permeability of free space)
$c = 3.00 \times 10^{8}$ m/sec	(speed of light)
$e = 1.60 \times 10^{-19}$ coul	(charge of the electron)
$m = 9.11 \times 10^{-31}$ kg	(mass of the electron)

CONVERSION FROM SPHERICAL TO CARTESIAN COORDINATES

$$\begin{cases} x = r \sin\theta \cos\phi \\ y = r \sin\theta \sin\phi \\ z = r \cos\theta \end{cases}$$

$$\begin{cases} \hat{\imath} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} \\ \hat{\jmath} = \sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} + \cos\phi\,\hat{\phi} \\ \hat{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \end{cases}$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases}$$

$$\begin{cases} \hat{r} = \sin\theta\cos\phi\,\hat{\imath} + \sin\theta\sin\phi\,\hat{\jmath} + \cos\theta\,\hat{k} \\ \hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\sin\phi\,\hat{\jmath} - \sin\theta\,\hat{k} \\ \hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} \end{cases}$$

The scale functions in Cylindrical Polar Coord's:

$$x = \rho \cos\varphi \qquad dx = d\rho \cos\varphi + \rho(-\sin\varphi)\,d\varphi$$

$$y = \rho \sin\varphi \qquad dy = d\rho \sin\varphi + \rho\cos\varphi\, d\varphi$$

$$z = z \qquad dz = dz$$

$$ds^2 = dx^2 + dy^2 + dz^2 = h_1^2 dq_1^2 + h_2^2 dq_2^2 + h_3^2 dq_3^2$$

$$= h_\rho^2 d\rho^2 + h_\varphi^2 d\varphi^2 + h_z^2 dz^2$$

$$ds^2 = (d\rho\cos\varphi - \rho\sin\varphi\, d\varphi)^2 + (d\rho\sin\varphi + \rho\cos\varphi\, d\varphi)^2 + dz^2$$

$$= (\cos^2\varphi + \sin^2\varphi)\,d\rho^2 + \rho^2(\cos^2 + \sin^2\varphi)\,d\varphi^2 + dz^2$$

$$= 1^2 d\rho^2 + \rho^2 d\varphi^2 + 1^2 dz^2$$

$$\Longrightarrow h_\rho = 1 \; ; \; h_\varphi = \rho \; ; \; h_z = 1$$

A Physical Example

Let the scalar function $\Phi(\vec{r})$ be the electric potential (voltage) due to a point charge q at the origin.

$$\Phi(\vec{r}) = \Phi(x,y,z) = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{x^2+y^2+z^2}} \qquad \text{Cartesian}$$

$$= \Phi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \qquad \text{Spherical Polar}$$

The electric field is the vector function

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi(\vec{r}) = -\left[\hat{e}_r \frac{\partial \Phi}{\partial r} + \frac{\hat{e}_\theta}{r} \cancel{\frac{\partial \Phi}{\partial \theta}}^{\,0} + \frac{\hat{e}_\varphi}{r\sin\theta} \cancel{\frac{\partial \Phi}{\partial \varphi}}^{\,0}\right]$$

$$= -\hat{e}_r\left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right) = \hat{e}_r \frac{q}{4\pi\epsilon_0 r^2} = \frac{q\vec{r}}{4\pi\epsilon_0 r^3}$$

$$\vec{E}(\vec{r}) = \begin{pmatrix} E_r(r,\theta,\varphi) \\ E_\theta(r,\theta,\varphi) \\ E_\varphi(r,\theta,\varphi) \end{pmatrix} = \begin{pmatrix} q\vec{r}/4\pi\epsilon_0 r^3 \\ 0 \\ 0 \end{pmatrix}$$

Gauss' Law

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi(\vec{r})\right) = -\nabla^2\Phi(\vec{r})$$

$$= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\, E_r\right) + 0 + 0\right]$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{q}{4\pi\epsilon_0 r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{q}{4\pi\epsilon_0}\right) = 0$$

<u>Almost</u> everywhere — not at $r=0$ (where the charge sits)