Numerical Integration $\quad F(x,v,t) = ma$

For simplicity of presentation; one dimension, constant mass, boundary conditions at the same time v_0, x_0.

$$\text{given} \quad \left.\begin{array}{l} t_0 \\ x_0 \\ v_0 \end{array}\right\} \Rightarrow F_0 = F(x_0, v_0, t_0) \Rightarrow a_0 = \frac{F_0}{m}$$

$$\text{then} \quad \left.\begin{array}{l} t_1 = t_0 + \Delta t \\ x_1 = x_0 + v_0 \Delta t \\ v_1 = v_0 + a_0 \Delta t \end{array}\right\} \Rightarrow F_1 = F(x_1, v_1, t_1) \Rightarrow a_1 = \frac{F_1}{m}$$

and continue until the desired final time is reached.

Free-body Diagrams

Action-Reaction Pairs $\{\vec{W}, \vec{W}'\}, \{\vec{N}, \vec{N}'\}$

satisfy
1. Equal Magnitude
2. Opposite Direction
3. same cause (both gravitational, or both contact)
4. Never act on the <u>same</u> object
5. Switch "acts on" and "caused by"

$\vec{W}$ acts on <u>m</u> caused by <u>Earth</u>, $\vec{W}'$ acts on Earth caused by <u>m</u>.

Conservation of Linear Momentum

Consider a system of just two particles

$$\vec{F}_{\substack{total \\ on\ 1}} = \underset{\substack{on\ 1 \\ due\ to\ 2}}{\vec{F}_{1,2}} + \vec{F}_{1,ext} = \frac{d\vec{P}_1}{dt}$$

$$\vec{F}_{\substack{total \\ on\ 2}} = \underset{\substack{on\ 2 \\ due\ to\ 1}}{\vec{F}_{2,1}} + \vec{F}_{2,ext} = \frac{d\vec{P}_2}{dt}$$

$$\vec{F}_{\substack{total \\ on\ system}} = 0 + \vec{F}_{\substack{system \\ external}} = \frac{d}{dt}(\vec{P}_1 + \vec{P}_2)$$

Action-Reaction partners cancel in pairs if Newton's Third Law holds true (forces must be central, not $\vec{v}$-dependent)

Generalize to N particles

$$\frac{d}{dt}\left(\sum_{a=1}^{N} \vec{P}_a\right) = \vec{F}_{\substack{system \\ external}}$$

Internal forces cancel in pairs

If the net external force on a system of particles vanishes, then the linear momentum of the system is conserved (does not change in time.

$$\sum \vec{F}_{ext} = 0 = \frac{d}{dt}\left(\sum_{a=1}^{N} \vec{P}_a\right) \Longrightarrow \sum_{a=1}^{N} \vec{P}_a = \text{constant}$$

Two equivalent ways of doing physics:

1) Newton's Laws (vectors)

2) The Work-Energy Theorem (scalar)

$$W_{net} = \Delta T = T_f - T_i = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

total work done on a particle — change in kinetic energy of the particle

$$\underset{\substack{1\to 2 \\ \text{any path}}}{W} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}_{net}\cdot d\vec{r} = \int \left(m\frac{d\vec{v}}{dt}\right)\cdot d\vec{r} = m\int \frac{d\vec{v}}{dt}\cdot\frac{d\vec{r}}{dt}\,dt$$

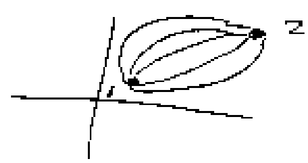
$$= m\int\left(\frac{d\vec{v}}{dt}\cdot\vec{v}\right)dt = \frac{m}{2}\int\frac{d}{dt}(\vec{v}\cdot\vec{v})\,dt = \frac{m}{2}\int d(v^2)$$

$$= \frac{m}{2}v^2\Big|_1^2 = \frac{m}{2}(v_2^2 - v_1^2) = T_2 - T_1$$

The net work is composed of two pieces

$$W_{net} = W_{cons} + W_{n.c.}$$

The conservative work is independent of the path between $\vec{r}_1$ and $\vec{r}_2$



e.g. friction, viscous drag, and aerodynamic drag are <u>non</u>-conservative

The conservative work depends only on the end points $\vec{r}_1$ and $\vec{r}_2$

$$\underset{1\to 2}{W_{cons}} \equiv U(\vec{r}_1) - U(\vec{r}_2) = -\Delta U \text{ (looks wrong, but it's right!)}$$

where $U(\vec{r})$ is the potential energy function (scalar, like T)

One can always add an arbitrary constant to $U(\vec{r})$.
Only changes ΔU are physically measurable.

$$\Delta U = U_f - U_i \equiv -W_{\substack{cons\\ i\to f}} = -\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{cons} \cdot d\vec{r}$$

$$\vec{F}_{cons}(\vec{r}) = -\vec{\nabla} U(\vec{r})$$

Consider a scalar function $\Phi(x,y,z)$

$$d\Phi = \frac{\partial \Phi}{\partial x} dx + \frac{\partial \Phi}{\partial y} dy + \frac{\partial \Phi}{\partial z} dz$$

$$= (\vec{\nabla}\Phi) \cdot d\vec{r}$$

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla}\Phi \cdot d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} d\Phi = \Phi(\vec{r}_2) - \Phi(\vec{r}_1) = \Delta\Phi$$

$$W_{net} = \Delta T$$

$$= W_{cons} + W_{nc} = \Delta T$$

$$= -\Delta U + W_{n.c} \quad \Delta T \quad \Rightarrow \quad W_{n.c.} = \Delta U + \Delta T$$

Define total mechanical energy $E = U + T$

$$\Delta U + \Delta T = (U_f - U_i) + (T_f - T_i) = (U_f + T_f) - (U_i + T_i)$$

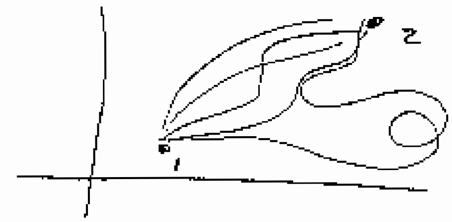
$$= E_f - E_i = \Delta E = W_{n.c}$$

If only <u>conservative</u> forces act on a system, then the <u>total</u> energy $E \equiv T+U$ is conserved.

These 3 conservation laws $\vec{p}, \vec{L}, E$ survive post Newtonian Mechanics
symmetries: translation, orientation, time

How can one tell if a force field is conservative?

Plan A Integrate over all possible paths between 1 and 2 and get the same answer.



Plan B

$$\vec{F}_{cons}(\vec{r}) = -\vec{\nabla} U(\vec{r})$$

$$\vec{\nabla} \times \vec{F}_{cons} = -\vec{\nabla} \times \left[\vec{\nabla} U\right] \equiv 0 \quad \text{for all } U(\vec{r})$$

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eg gravity: $\vec{F} = -mg\hat{y}$ (constant)

$\vec{\nabla} \times \vec{F} = 0 \implies$ force is conservative

$\implies$ there is a potential energy function $U(\vec{r})$

$$U(\vec{r}) = mgy + c$$

$$\vec{F} = -\vec{\nabla} U = -mg\hat{y} \quad \checkmark$$

$$= -\left(\hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}\right)(mgy + c)$$

Real gravity is also, $\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{e}_r$

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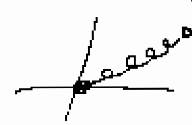
e.g. Hooke's law spring force: $\vec{F} = -k\vec{r}$



$\vec{\nabla} \times \vec{F} = 0$ conservative

$$U(\vec{r}) = \frac{kr^2}{2}$$

$$\vec{F} = -\vec{\nabla} U = -\left[\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right]\frac{kr^2}{2}$$

$$= -k\vec{r} \quad \checkmark$$

If total mechanical energy E is conserved, then in 1-dimension:

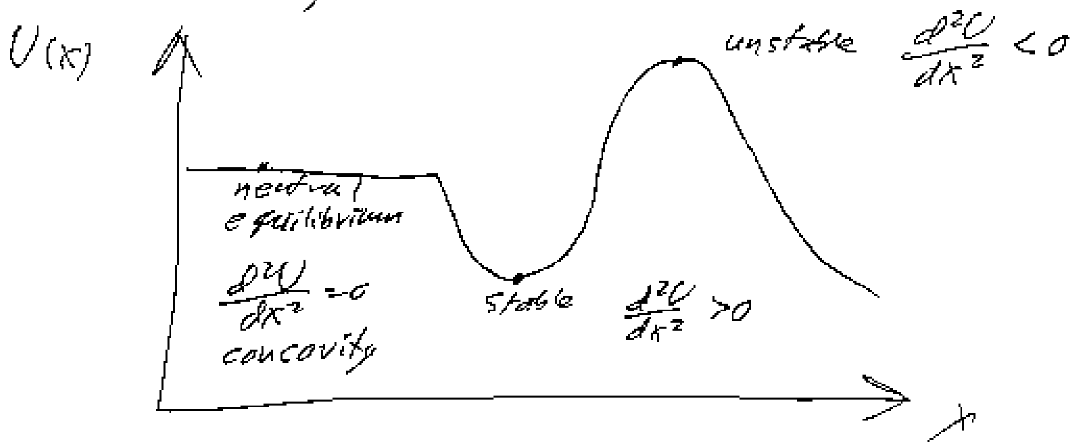
$$E = T + U = \frac{1}{2} m v^2 + U(x)$$

$$v(x) = \sqrt{\frac{2}{m}\left[E - U(x)\right]} = \frac{dx}{dt}$$

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\left[E - U(x)\right]}}$$

$$\int_{t'=t_0}^{t} dt' = \int_{x'=x_0}^{x(t)} \frac{dx'}{\sqrt{\frac{2}{m}\left[E - U(x')\right]}} \qquad \text{solve for } x(t)$$

~~Stability~~ Equilibrium — if $\vec{F} = 0 \Rightarrow \frac{dU}{dx} = 0$ in 1-dim



neutral → small push, particle experiences no force

stable → restoring force (toward equilib.)

unstable → force away from equilibrium