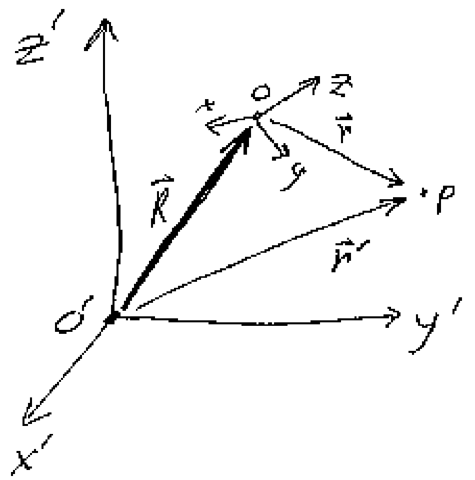
Noninertial Reference Frames

Consider two coordinate frames; one is the primed frame (x', y', z') with origin O' which is an inertial reference frame (Newton's Laws work) also called the fixed frame.



The other frame (unprimed) is possibly noninertial and is labeled the rotating frame whether or not it is actually rotating.

- O could have some velocity with respect to O'. If this velocity $\left(\frac{d\vec{R}}{dt}\right)_{\text{fixed}}$ is constant, then O is also inertial.
- O could be rotating with angular velocity $\vec{\omega}$ as seen by an observer in O'.
- O could be accelerating away from O'.

⋮

A point P in O can be at rest in O (not moving with respect to the x, y, z axes) or P can be moving in O. The movement will look different to an observer in O'.

1) Suppose P is at rest in O:

$$\vec{v}_{rot} \equiv \left(\frac{d\vec{r}}{dt}\right)_{rot} = 0 \qquad \text{What is } \vec{v}'_{fixed} \equiv \left(\frac{d\vec{r}'}{dt}\right)_{fixed} ?$$

It depends on O's motion w.r.t. O'.

1a) O has some instantaneous velocity $\left(\frac{d\vec{R}}{dt}\right)_{fixed}$ w.r.t. O':

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} \leftarrow \text{frame velocity } \vec{V}$$

1b) O is not translating, but is rotating w.r.t. O' with angular velocity $\vec{\omega}$.

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \omega\rho = \omega r \sin\theta \rightarrow \vec{\omega} \times \vec{r}$$

1c) O is translating and rotating

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega} \times \vec{r}$$

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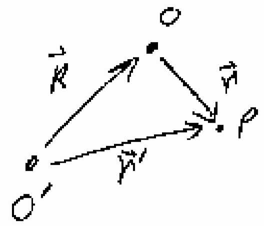
2) Suppose P is not at rest in O. $\left(\frac{d\vec{r}}{dt}\right)_{rot} \neq 0$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega} \times \vec{r}$$

$$\vec{v}'_{fixed} = \vec{v}_{rot} + \vec{V} + \vec{\omega} \times \vec{r}$$



Relation between primed and unprimed coordinates:

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\left(\frac{d\vec{r}'}{dt}\right)_{fixed} = \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \left(\frac{d\vec{r}}{dt}\right)_{fixed}$$

Combine this with the equation in (2):

$$\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r}$$

This will also hold for any vector $\vec{Q}$

$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega} \times \vec{Q}}$$

For example, the angular acceleration:

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fixed} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega}$$

this is zero

same in both frames

$$\vec{\alpha}' = \vec{\alpha} \quad \text{or} \quad \dot{\vec{\omega}}' = \dot{\vec{\omega}}$$

Newton's 2nd Law:

$$\sum \vec{F}_{real} = m\,\vec{a}\,'_{fixed} = m\left[\frac{d}{dt}\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed}\right]_{fixed}$$

We start withe the equation in (2) and take $\left(\frac{d}{dt}\right)_{fixed}$ on both sides:

$$\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \left(\frac{d\vec{R}}{dt}\right)_{fixed} + \vec{\omega}\times\vec{r}$$

$$\left[\frac{d}{dt}\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed}\right]_{fixed} = \left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{fixed} + \left[\frac{d}{dt}\left(\frac{d\vec{R}}{dt}\right)_{fixed}\right]_{fixed}$$

$$+ \left[\frac{d}{dt}(\vec{\omega}\times\vec{r})\right]_{fixed}$$

(want only <u>fixed</u> on left and only <u>rot</u> on right)

The term on the left is $\vec{a}\,'_{fixed} = \left[\frac{d}{dt}\left(\frac{d\vec{r}\,'}{dt}\right)_{fixed}\right]_{fixed}$

The first term on the right is evaluated with the boxed equation with $\vec{Q} = \left(\frac{d\vec{r}}{dt}\right)_{rot}$

$$\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega}\times\vec{Q}$$

$$\left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{fixed} = \left[\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{rot}\right]_{rot} + \vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot}$$

$$= \vec{a}_{rot} + \vec{\omega}\times\vec{v}_{rot}$$

The second term on the right is $\ddot{\vec{R}}_{fixed}$ — the frame acceleration

The third term on the right is

$$\left[\frac{d}{dt}(\vec{\omega}\times\vec{r})\right]_{fixed} = \underbrace{\dot{\vec{\omega}}}_{\text{same in both frames}}\times\vec{r} + \vec{\omega}\times\underbrace{\left(\frac{d\vec{r}}{dt}\right)_{fixed}}_{\text{evaluate using boxed equation}}$$

$$= \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}\right]$$

$$= \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\left[\vec{v}_{rot} + \vec{\omega}\times\vec{r}\right]$$

All together

$$\vec{a}\,'_{fixed} = \vec{a}_{rot} + \vec{\omega}\times\vec{v}_{rot} + \ddot{\vec{R}}_{fixed} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times\vec{v}_{rot} + \vec{\omega}\times(\vec{\omega}\times\vec{r})$$

same

$$\boxed{\vec{a}\,'_{fixed} = \vec{a}_{rot} + \ddot{\vec{R}}_{fixed} + 2\vec{\omega}\times\vec{v}_{rot} + \dot{\vec{\omega}}\times\vec{r} + \vec{\omega}\times(\vec{\omega}\times\vec{r})}$$

$$\left.\begin{array}{l}\vec{r}\\ \vec{v}_{rot}\\ \vec{a}_{rot}\end{array}\right\}\ \text{measured by a noninertial observer in } O.$$

What happens when you try to apply Newton's Laws in a noninertial reference frame (where you have no right to do so)?

$$m\vec{a}_{rot} = \,? \;\equiv \sum \vec{F}_{effective}$$

$$= \underbrace{m\vec{a}\,'_{fixed}}_{\sum \vec{F}_{real}} \underbrace{- m\ddot{\vec{R}}_{fixed} - 2m\vec{\omega}\times\vec{v}_{rot} - m\dot{\vec{\omega}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{\text{fictitious forces!}}$$

$-m\ddot{\vec{R}}_{fixed}$: translational force ; eg. thrown back in your seat when you step on the gas peddle.

$-m\dot{\vec{\omega}}\times\vec{r}$: azimuthal force ; results from angular acceleration $\vec{\alpha} = \dot{\vec{\omega}}$

$-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$: centrifugal force : e.g. thrown to <u>outside</u> of curve when you turn a corner, e.g. salad spinner, rinse cycle.

$-2m\vec{\omega}\times\vec{v}_{rot}$: Coriolis force : depends on velocity w.r.t. unprimed axes.
e.g. try to walk on a Merry-go-round
e.g. hurricanes.