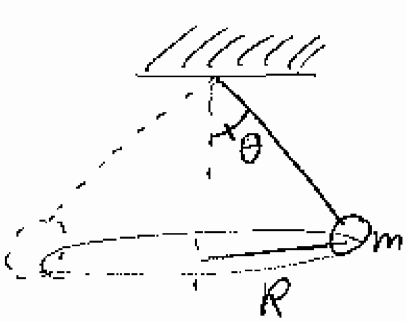
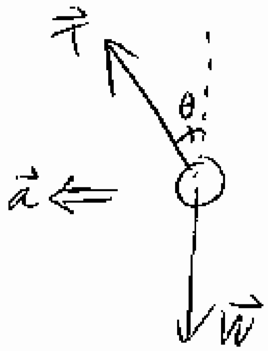
## E.g. The Conical Pendulum




**Inertial Frame Analysis:**

The point mass $m$ moving in a circle must accelerate toward the center of the circle.




Two real forces act on the mass, weight and tension in the string.

Use cylindrical polar coordinates: $\{\rho, \varphi, z\}$. Newton's Second Law must hold in each direction.

$$\sum F_z = m \cancel{a_{z_{fixed}}}^{\to 0}$$

$$T\cos\theta - W = 0 \implies T = \frac{mg}{\cos\theta}$$

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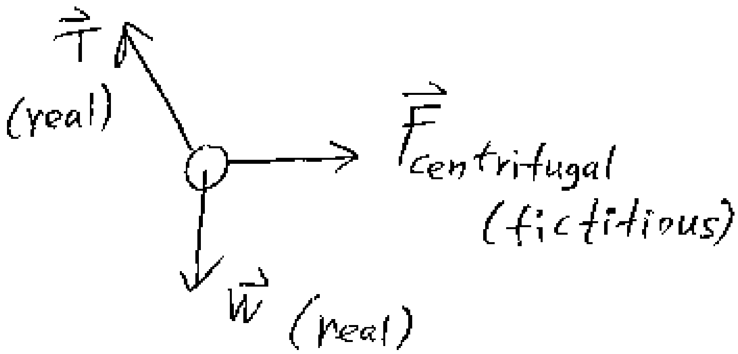
$$\sum F_\rho = m a_{\rho_{fixed}}$$

$$T\sin\theta = \frac{mv^2}{R}$$

$T\sin\theta$ provides the centripetal acceleration. (toward center)

$$\left(\frac{mg}{\cos\theta}\right)\sin\theta = \frac{mv^2}{R} \implies \theta = \arctan\left(\frac{v^2}{Rg}\right)$$

# Non-inertial (rotating) Frame Analysis:




An observer riding on the mass will see the mass at rest with the world rotating around the mass,

$$\sum F_z = m \cancel{a_{z_{rot}}}^{\,0}$$

$$T\cos\theta - W = 0 \implies T = \frac{mg}{\cos\theta}$$

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$$\sum F_x = m \cancel{a_{x_{rot}}}^{\,0}$$

$$-T\sin\theta + F_{centrifugal} = 0$$

$$-T\sin\theta + \frac{mv^2}{R} = 0$$

$$-\left(\frac{mg}{\cos\theta}\right)\sin\theta + \frac{mv^2}{R} = 0 \implies \theta = \arctan\left(\frac{v^2}{Rg}\right)$$

same answer, of course

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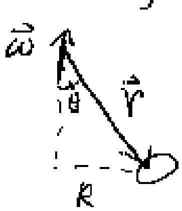
$$\vec{F}_{centrifugal} = -m\vec{\omega}\times(\vec{\omega}\times\vec{r})$$

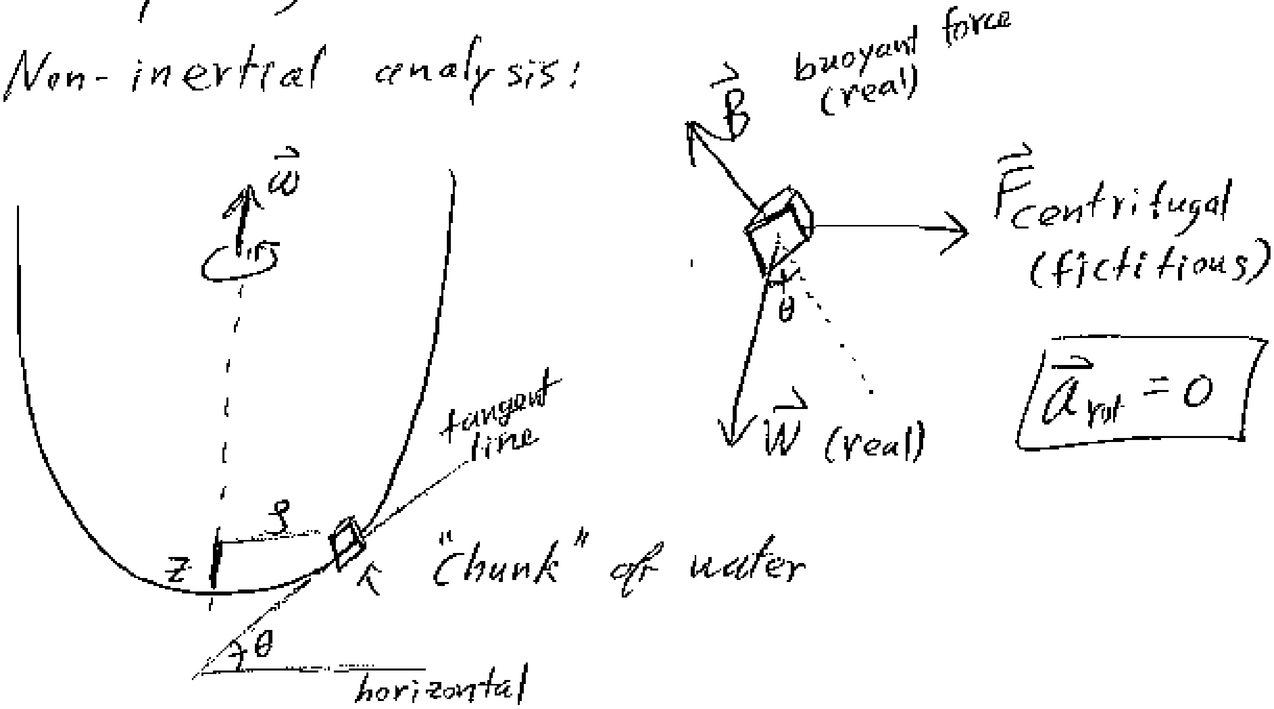



$$|\vec{\omega}\times\vec{r}| = \omega r\sin\theta = \omega R = v$$

$$|\vec{\omega}\times(\vec{\omega}\times\vec{r})| = \omega^2 R = \frac{v^2}{R}$$

E.g. Shape of water-air interface in a spinning container.

Non-inertial analysis:




$$\tan\theta = \frac{opp}{adj} = \frac{F_{centrif}}{W} = \frac{\cancel{m}\,\omega^2\varrho}{\cancel{m}g} = \frac{dz}{d\varrho} = \text{slope of tangent line}$$

$$dz = \frac{\omega^2\varrho\,d\varrho}{g} \implies z = \frac{\omega^2\varrho^2}{2g} + \text{constant}$$

(parabola!)

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Effective Potential Energy:

$$U = mgz - \tfrac{1}{2}m\omega^2\varrho^2$$

Check: $-\vec{\nabla}U = \underset{\text{weight}}{-mg\hat{e}_z} + \underset{\text{Centrifugal force}}{m\omega^2\varrho\,\hat{e}_\varrho} \equiv m\,\vec{g}_{eff}$

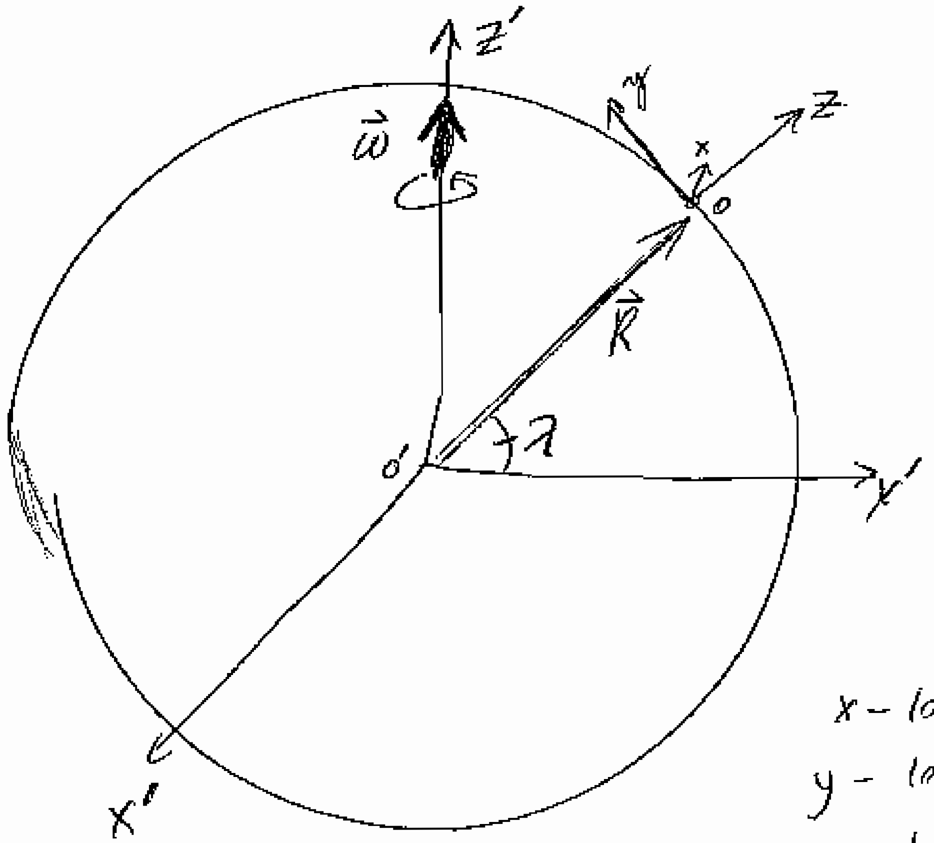
Effective gravity: $\vec{g}_{eff} = \vec{g}_0 - \vec{\omega}\times(\vec{\omega}\times\vec{r})$   $\vec{g}_0$ is true gravity.

Water surface is a surface of constant $U_{eff}$.

$mgz - \frac{1}{2}\omega^2\varrho^2 = \text{constant}$ is a parabola.

# Motion close to the Earth's surface

Assume: Earth is spherical and uniform, and acceleration due to gravity does not change appreciably with altitude




$\{x', y', z'\}$ are inertial (fixed) coordinates.

$\{x, y, z\}$ are non-inertial (rotating) coordinates.

x - locally east
y - locally north
z - locally up

Latitude: $\lambda = \begin{cases} +90^\circ & \text{north pole} \\ 0^\circ & \text{equator} \\ -90^\circ & \text{south pole} \end{cases}$

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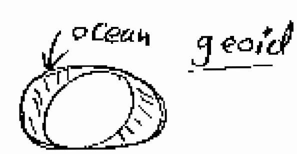
$$\vec{g}_{eff} = \vec{g}_0 - \vec{\omega} \times [\vec{\omega} \times (\vec{r} + \vec{R})]$$

measure magnitude of $g_{eff}$ with a pendulum: $T = 2\pi\sqrt{\frac{\ell}{g}}$

get direction of $\vec{g}_{eff}$ with a plumb line.

(ocean surface is perpendicular to $\vec{g}_{eff}$)

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$$\vec{F}_{eff} = \underset{\substack{\text{than}\\ \text{gravity}}}{\vec{S}_{other}} + m\vec{g}_0 - m\ddot{\vec{R}}_{fixed} - m\cancel{\dot{\vec{\omega}}}\times\vec{r} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\vec{v}_{rot}$$

$\dot{\vec{\omega}} \to 0$: ignore Earth spin-down (leap second) $\Rightarrow$ no angular acceleration

Concentrate on the inertial force term: $-m\ddot{\vec{R}}_{fixed}$

Remember $\left(\frac{d\vec{Q}}{dt}\right)_{fixed} = \left(\frac{d\vec{Q}}{dt}\right)_{rot} + \vec{\omega}\times\vec{Q}$

$$\ddot{\vec{R}}_{fixed} = \left(\frac{d\dot{\vec{R}}_{fix}}{dt}\right)_{fixed} = \cancel{\left(\frac{d\dot{\vec{R}}_{fix}}{dt}\right)_{rot}} + \vec{\omega}\times\dot{\vec{R}}_{fixed}$$

$\to 0$: $\vec{R}$ does not change according to a rotating observer.

$$= \vec{\omega}\times\dot{\vec{R}} = \vec{\omega}\times\left(\frac{d\vec{R}}{dt}\right)_{fixed}$$

$$= \vec{\omega}\times\left[\cancel{\left(\frac{d\vec{R}}{dt}\right)_{rot}} + \vec{\omega}\times\vec{R}\right] = \vec{\omega}\times(\vec{\omega}\times\vec{R})$$

$\to 0$

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$$\vec{F}_{eff} = \vec{S} + \underbrace{m\vec{g}_0 - m\vec{\omega}\times[\vec{\omega}\times(\vec{r}+\vec{R})]}_{\equiv m\vec{g}_{eff}} - 2m\vec{\omega}\times\vec{v}_{rot}$$

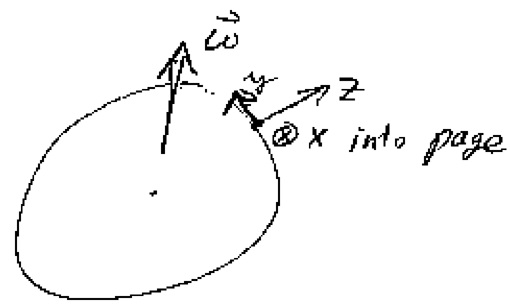
$$\vec{F}_{eff} = \vec{S} + m\vec{g}_{eff} - 2m\vec{\omega}\times\vec{v}_{rot}$$

Effective weight

Coriolis force

Deflection of a particle dropped from rest from height $h$.

rotating frame: $x$ - east, $y$ - north, $z$ - up




$$\begin{cases} \omega_x = 0 \\ \omega_y = \omega \cos\lambda \\ \omega_z = \omega \sin\lambda \end{cases} \qquad \begin{cases} v_{rot_x} = \mathcal{O}(\omega) \\ v_{rot_y} = \mathcal{O}(\omega) \\ v_{rot_z} = \underbrace{-gt}_{\text{freefall}} + \mathcal{O}(\omega) \end{cases}$$

We only need $\vec{v}_{rot}$ to zeroeth order in $\omega$.

$$\vec{\omega} \times \vec{v}_{rot} = \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 0 & -gt \end{vmatrix} = -\omega g t \cos\lambda \; \hat{e}_x$$

Coriolis acceleration: $-2\vec{\omega} \times \vec{v}_{rot} = 2\omega g t \cos\lambda \; \hat{e}_x$

$$\begin{cases} a_{rot_x} = 2\omega g t \cos\lambda + \mathcal{O}(\omega^2) = \ddot{x} \\ a_{rot_y} = \mathcal{O}(\omega^2) \\ a_{rot_z} = -g + \mathcal{O}(\omega^2) \end{cases}$$

$\leftarrow$ integrate twice to get deflection $\Delta x$.

$$v_{\text{rot},x} = \int_{t'=0}^{t} a_{\text{rot},x}(t')\,dt' = \omega g t^2 \cos\lambda + \cancel{v_0}^{\,0} \quad \text{(starts from rest)}$$

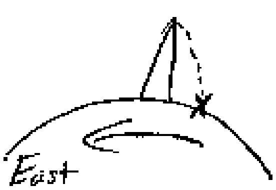
$$\Delta x = \int_{t'=0}^{t} v_{\text{rot},x}(t')\,dt' = \frac{1}{3}\omega g t^3 \cos\lambda$$

Now we need the freefall time to zeroeth order in $\omega$

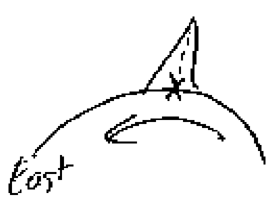
$$h = \tfrac{1}{2} g t^2 \implies t = \sqrt{\frac{2h}{g}} + \mathcal{O}(\omega)$$

Final deflection to lowest order in $\omega$:

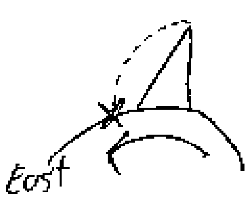
$$\Delta x = \frac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2} \cos\lambda \qquad \text{East!}$$




Pre-Galileo




Galileo




Coriolis

On a merry-go-round in the night,
Coriolis was shaken with fright.
Despite how he walked,
Thus like he was stalked
By some fiend always pushing him right.

Northern Hemisphere only!

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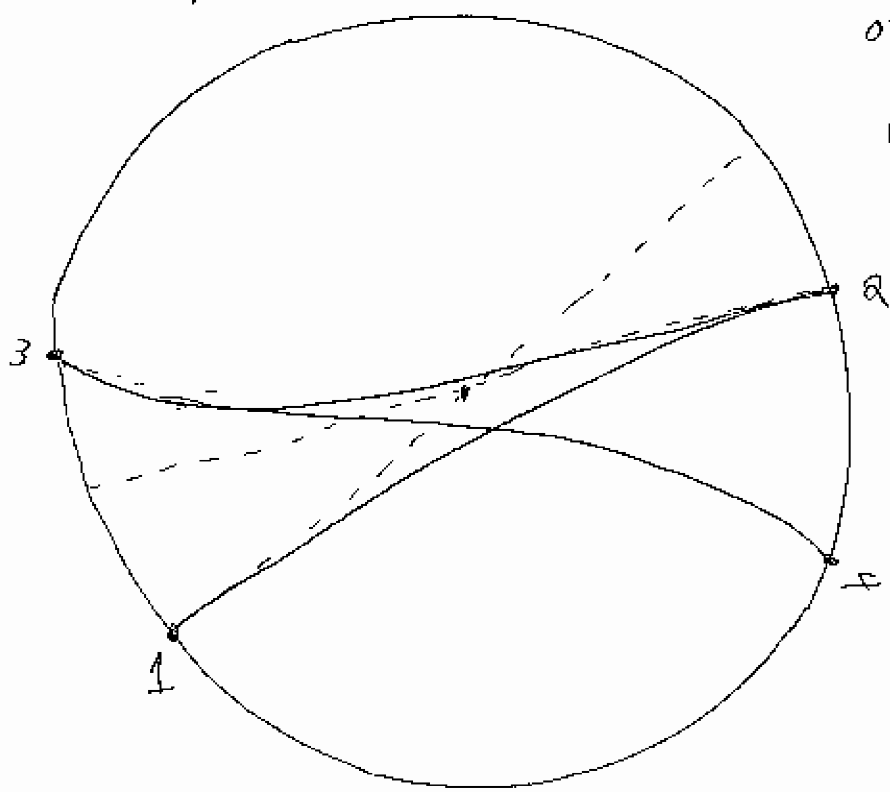
In the Northern Hemisphere, $\vec{\omega}$ has a locally <u>up</u> component. $\vec{F}_{\text{Coriolis}}$ always has a component to the right of the velocity $\vec{v}_{\text{rot}}$ (can also have an up or down component).

## Foucault's Pendulum

top view




Plane of oscillation of pendulum precesses <u>clockwise</u>. (Northern Hemisphere)

Proof that the Earth <u>rotates</u>!


At the North Pole, a Foucault pendulum oscillates in a plane in inertial space while the Earth turns under it. A non-inertial observer on the Earth (rotating frame) would see the pendulum precess clockwise in 1 day.

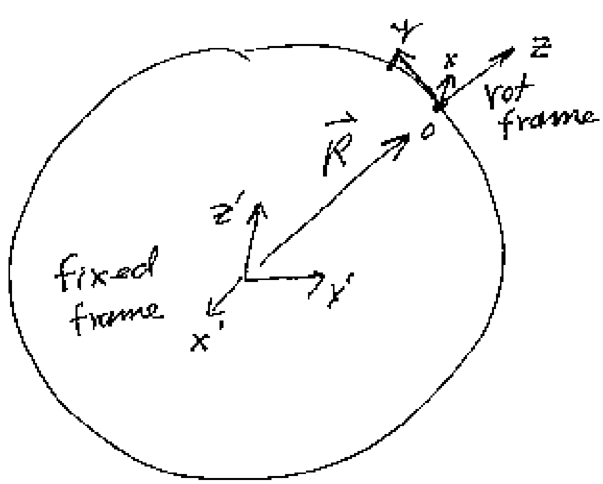
A Foucault pendulum on the Earth's equator does not precess. How about latitudes in between?

This is a different derivation from the one in Marion.

$$\vec{F}_{\substack{\text{eff}\\ \text{rot}}} = \underset{\substack{\uparrow\\ \text{tension}\\ \text{in string}}}{\vec{T}} + m\vec{g}_{\text{eff}} - \underset{\substack{\uparrow\\ \text{Coriolis}\\ \text{force}\\ \text{(fictitious)}}}{2m\vec{\omega}\times\vec{v}_{\text{rot}}}$$

(rot) is the Earth Frame.

Consider a new frame precessing along with the plane of the Foucault pendulum, (prec) frame.




The (rot) and (prec) frames have the same origin.

Call the vector that connects (rot) origin to (prec) origin $\vec{P}$. Then $\vec{P}=0$, $\dot{\vec{P}}=0$, $\ddot{\vec{P}}=0$.

The angular velocity of the precessing frame is $\vec{\Omega}$ (locally down in Northern Hemisphere).

$\dot{\vec{\Omega}} = 0$ no angular acceleration.

Remember $\left(\frac{d\vec{r}}{dt}\right)_{fixed} = \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega}\times\vec{r}$ holds for any vector $\vec{Q}$.

So by analogy

$$\left(\frac{d\vec{r}}{dt}\right)_{rot} = \left(\frac{d\vec{r}}{dt}\right)_{prec} + \vec{\Omega}\times\vec{r}$$

also holds for any vector $\vec{Q}$.

$$\vec{v}_{rot} = \vec{v}_{prec} + \vec{\Omega}\times\vec{r}$$

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Take one more rotating time derivative of both sides

$$\left(\frac{d\vec{v}_{rot}}{dt}\right)_{rot} \equiv \vec{a}_{rot} = \frac{d}{dt}\left[\vec{v}_{prec} + \vec{\Omega}\times\vec{r}\right]_{rot}$$

$$= \frac{d}{dt}\left[\vec{v}_{prec} + \vec{\Omega}\times\vec{r}\right]_{prec} + \vec{\Omega}\times\left[\vec{v}_{prec} + \vec{\Omega}\times\vec{r}\right]$$

$$= \left(\frac{d\vec{v}_{prec}}{dt}\right)_{prec} + \cancelto{0}{\dot{\vec{\Omega}}\times\vec{r}} + 2\vec{\Omega}\times\vec{v}_{prec} + \vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$= \vec{a}_{prec} + \underbrace{2\vec{\Omega}\times\vec{v}_{prec}}_{\text{new Coriolis}} + \underbrace{\vec{\Omega}\times(\vec{\Omega}\times\vec{r})}_{\text{new Centrifugal}}$$

$$\vec{F}_{\substack{eff\\prec}} = \vec{F}_{\substack{eff\\rot}} - 2m\vec{\Omega}\times\vec{v}_{prec} - m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

$$\hookrightarrow \vec{v}_{prec} = \vec{v}_{rot} - \vec{\Omega}\times\vec{r}$$

$$\hookrightarrow \vec{T} + m\vec{g}_{eff} - 2m\vec{\omega}\times\vec{v}_{rot}$$

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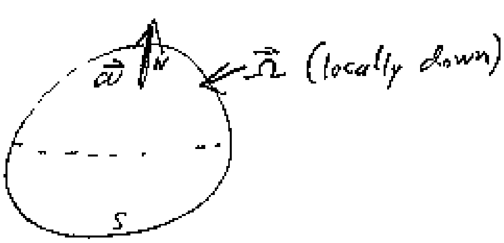
$$\vec{F}_{\substack{eff\\prec}} = \vec{T} + m\vec{g}_{eff} - 2m(\vec{\omega}+\vec{\Omega})\times\vec{v}_{rot} + m\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$$

OK, here's the punchline – an observer in this (prec) frame precessing with the plane of the Foucault pendulum will see <u>no precession</u>, that is <u>no Coriolis force.</u> The pendulum appears to oscillate in one plane and does not change. Therefore, the term $-2m(\vec{\omega}+\vec{\Omega})\times\vec{v}_{rot}$ must vanish. (Ignore the $\mathcal{O}(\Omega^2)$ term $\vec{\Omega}\times(\vec{\Omega}\times\vec{r})$).

But $\vec{\omega}$ and $\vec{\Omega}$ are not in the same direction.




The $\vec{v}_{rot}$ has only x and y components (locally east-west and north-south), then $(\vec{\omega}+\vec{\Omega})$ can have zero cross product with $\vec{v}_{rot}$ if

$$|\vec{\Omega}| = \omega \sin\lambda$$

check N. pole $\lambda = +90^\circ$ $\Rightarrow \Omega = \omega$ ✓

Equator $\lambda = 0^\circ$ $\Omega = 0$ ✓