E_R. Then the resonance is symmetric and $[d^2(\cot\delta(E))/dE^2]_{E=E_R} = 0$. (If the resonance is broad however, the phase-space factor for decay varies appreciably over the width, and the resonance is asymmetric). From (4.51)

$$f(E) = \frac{1}{\cot\delta - i} = \frac{\Gamma/2}{(E_R - E) - i\Gamma/2}. \tag{4.52}$$

From (4.45) and (4.50), we obtain for the elastic scattering cross-section

$$\sigma_{\text{el.}}(E) = 4\pi\lambda\!\!\!^{-2}(2l+1)\frac{\Gamma^2/4}{(E - E_R)^2 + \Gamma^2/4}. \tag{4.53}$$

This is known as the *Breit-Wigner formula*. The resonance curve of $\sigma(E)$ is shown in Fig. 4.11. The width Γ is defined so that the elastic cross section $\sigma_{\text{el.}}$ falls by a factor 2 from the peak value when $|E - E_R| = \pm\Gamma/2$.

As pointed out in Chapter 1, the width Γ and lifetime τ of the resonant state are connected by the relation $\tau = \hbar/\Gamma$. The energy dependence of the amplitude (4.53) is simply the Fourier transform of an exponential time pulse, corresponding to the radioactive decay of the resonance. The wavefunction of a nonstationary decaying state of central angular frequency $\omega_R = E_R/\hbar$ and lifetime $\tau = \hbar/\Gamma$ can be written

$$\begin{aligned} \psi(t) &= \psi(0)e^{-i\omega_R t}e^{-t/2\tau} \\ &= \psi(0)e^{-t(iE_R + \Gamma/2)} \end{aligned} \tag{4.54}$$

in units $\hbar = c = 1$. The intensity $I(t) = \psi\psi^* = I(0)e^{-t/\tau}$ obeys the normal exponential law of radioactive decay. The Fourier transform of this expression is

$$g(\omega) = \int_0^\infty \psi(t)e^{i\omega t}\,dt,$$

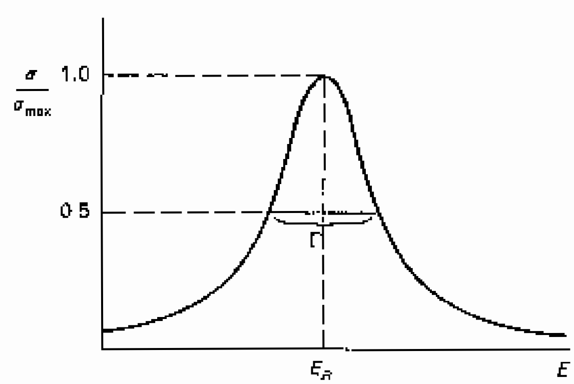


Figure 4.11 Breit-Wigner resonance curve.

incoming wave arrives—it may be proved that f traces out a circle in the *anticlockwise* direction (for an attractive potential).

4.9. AN EXAMPLE OF A BARYON RESONANCE—THE Δ(1232)

Figure 4.4 shows the π^+p and π^-p total cross-sections as a function of kinetic energy of the incident pion. There is a very obvious $I = \frac{3}{2}$ resonance at $T_\pi = 195$ MeV, corresponding to a pion-proton mass of 1232 MeV. It was discovered by Fermi and Anderson in 1949. It is designated $P_{33}(1232)$, meaning that it is a p-wave ($l = 1$) pion-nucleon resonance, of $I = \frac{3}{2}$ and $J = \frac{3}{2}$. One can also distinguish other humps and bumps in σ(total). For example, in the $I = \frac{1}{2}$ channel one can see evidence for the states $D_{13}(1520)$ and $F_{15}(1688)$ as peaks in $\sigma(\pi^-p)$, and $F_{37}(1950)$ as a peak in $\sigma(\pi^+p)$, i.e., $I = \frac{3}{2}$.

For the lowest-lying Δ(1232) πN resonance, the amplitude is almost purely elastic because of the low mass, and the "tails" of higher-lying resonances may be neglected. From (4.55) we expect $\sigma_{\text{el.}} = 2\pi\lambda\!\!\!^{-2}(2J + 1)$ at the peak. The limiting value for $J = \frac{3}{2}$, $\sigma_{\text{el.}} = 8\pi\lambda\!\!\!^{-2}$, is shown dashed in Fig. 4.12, clearly proving the Δ(1236) to be a p-wave resonance of spin-parity $J^P = \frac{3}{2}^+$.

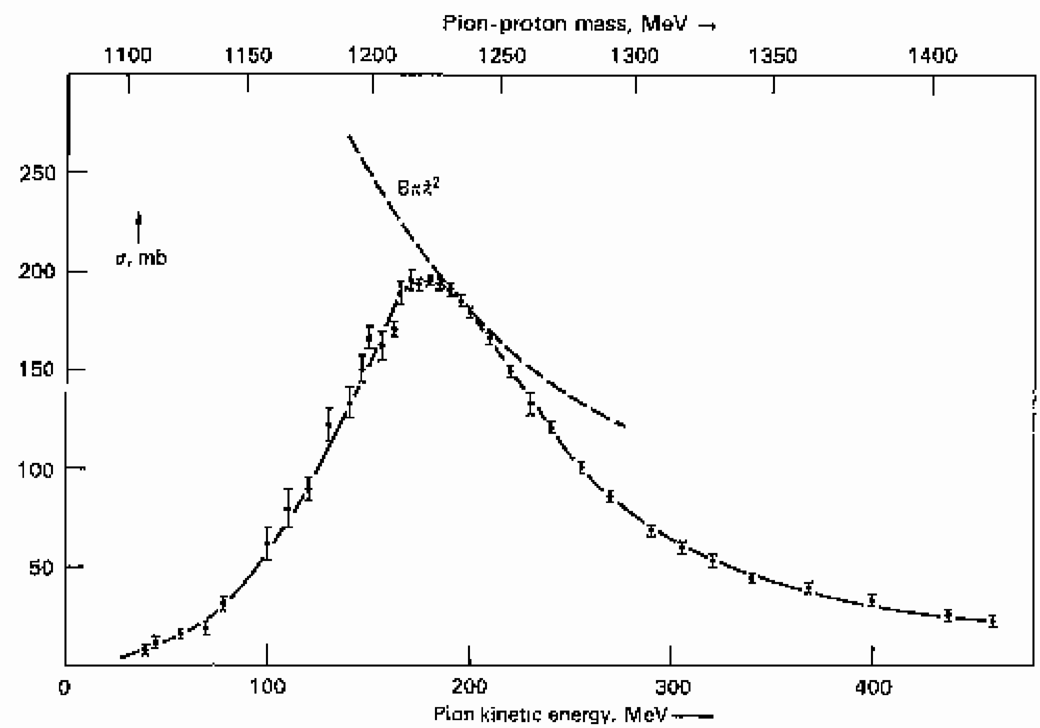


Figure 4.12 The π^+p total cross-section as a function of kinetic energy of the incident pion, or the π^+p mass, in the region of the 1232 MeV, $I = \frac{3}{2}$, $J^P = \frac{3}{2}^+$ resonance. Not all experimental points have been included. The maximum cross-section, $8\pi\lambda^2$, allowed by conservation of probability is shown dashed.

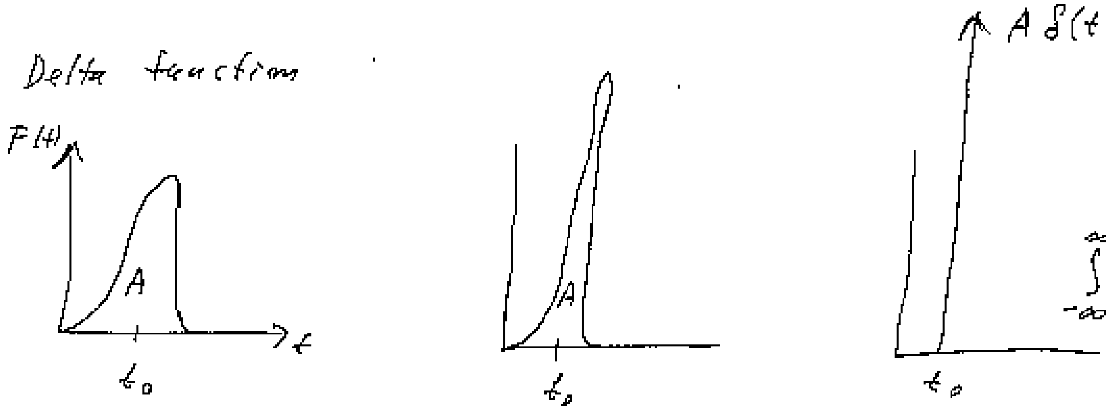
Lecture #22

13 Nov 97

Green functions

The Green function is the general solution to a differential equation with delta function (impulsive) forcing. Since general solution, initial conditions are built in. Different IC's mean a different Green function.

Delta function



Limit

zero width

infinite height

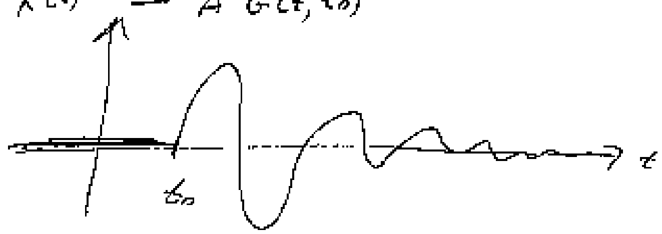
same area A

$$\int_{-\infty}^{\infty} A\,\delta(t-t_0)\,dt = A$$

e.g. underdamped $\omega_1 = \sqrt{\omega_0^2-\beta^2}$ and IC's $x(0) = 0 = v(0)$

$$G(t,t_0) = \begin{cases} 0, & t<t_0 \quad \text{(before strike)} \\ \frac{1}{m\omega_1} e^{-\beta(t-t_0)} \sin\left[\omega_1(t-t_0)\right], & t>t_0 \quad \text{(ringing after)} \end{cases}$$

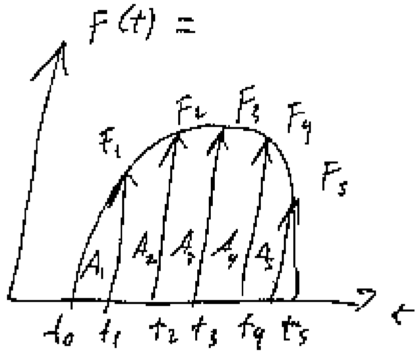
$x(t) = A\,G(t,t_0)$



An arbitrary forcing function $F(t)$ can be decomposed into:

1) sines + cosines (Fourier)
2) delta functions (Green)

$F(t) =$



$A_1 = F_1 \Delta t_1$

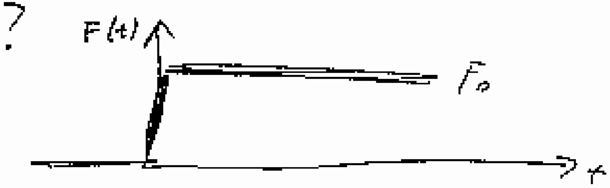
general solution:

$$x(t) = \int_{t'=-\infty}^{t} F(t')\,G(t,t')\,dt'$$

only feel effect of pings in the past — causal Green function.

How did we derive $G(t, t_0)$?

The step function is easy!



$$x_c(t) = e^{-\beta t}\left[A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right] \qquad \omega_1 = \sqrt{\omega_0^2 - \beta^2}$$

$x_p(t)$ is just a constant $\quad x_p(t) = c \quad \dot{x}_p = 0 \quad \ddot{x}_p = 0$

$$\ddot{x}_p + 2\beta \dot{x}_p + \omega_0^2 x_p = \frac{F(t)}{m} = \begin{cases} 0 & , t<0 \\ \frac{F_0}{m} & , t>0 \end{cases}$$

$$x_p(t) = \begin{cases} 0 & , t<0 \\ \frac{F_0}{m\omega_0^2} = \frac{F_0}{k} & , t>0 \end{cases}$$

general solution: $x(t) = x_c(t) + x_p(t)$

initial condition $x(0) = 0 = v(0)$

$$x(t) = \begin{cases} 0 & , t<0 \\ e^{-\beta t}\left[A_1 \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right] + \frac{F_0}{m\omega_0^2} & , t>0 \end{cases}$$

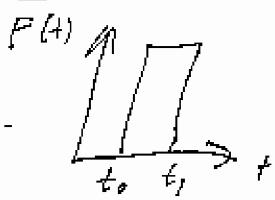
determine A_1 and A_2 from initial conditions.

$$0 = x(0) = A_1 + \frac{F_0}{m\omega_0^2} \Rightarrow A_1 = \frac{-F_0}{m\omega_0^2}$$

$$v(t) = \frac{dx}{dt} = e^{-\beta t}\left\{-\beta\left[A \cos(\omega_1 t) + A_2 \sin(\omega_1 t)\right] + \omega_1\left[-A_1 \sin(\omega_1 t) + A_2 \cos(\omega_1 t)\right]\right\}$$

$$0 = v(0) = -\beta A_1 + \omega_1 A_2 \Rightarrow A_2 = \frac{\beta A_1}{\omega_1} = -\frac{\beta F_0}{m\omega_1 \omega_0^2}$$

$$x(t) = \begin{cases} 0 & , t<0 \\ \frac{F_0}{m\omega_0^2}\left[1 - e^{-\beta t}\cos(\omega_1 t) - \frac{\beta}{\omega_1} e^{-\beta t}\sin(\omega_1 t)\right] & , t>0 \end{cases}$$



$\Rightarrow x(t - t_0) - x(t - t_1) \longrightarrow G(t - t_0)$

let $t_1 - t_0 \rightarrow 0$

$F_0 \rightarrow \infty$

with area $F_0(t_1 - t_0) = $ const. $\longrightarrow$ delta function.

Linear Superposition

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = F(t)$$

Linear (in x) inhomogeneous 2nd order ordinary differential equation

$$\ddot{x}_1 + 2\beta\dot{x}_1 + \omega_0^2 x_1 = F_1(t)$$
$$\ddot{x}_2 + 2\beta\dot{x}_2 + \omega_0^2 x_2 = F_2(t)$$

Find a general solution for forcing function $c_1 F_1(t) + c_2 F_2(t)$

$$x(t) = c_1 x_1(t) + c_2 x_2(t)$$

The response to a sum of two forces is the sum of the responses.

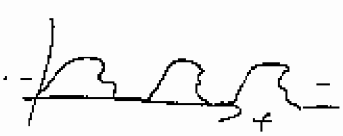
Derivative operation is linear $\frac{d}{dt}(c_1 x_1 + c_2 x_2) = c_1 \frac{dx_1}{dt} + c_2 \frac{dx_2}{dt}$

$$\mathcal{D} = \left(\frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2\right)$$

$$\mathcal{D}[c_1 x_1 + c_2 x_2] = c_1 \mathcal{D}[x_1] + c_2 \mathcal{D}[x_2]$$

We know how to find the response $x(t)$ to sinusoidal input $F_0 \cos(\omega t)$.

If we can decompose arbitrary periodic input $F(t)$ into sines and cosines (Fourier's theorem)



$$F(t) = \frac{a_0}{2}\cos(0) + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

then linear superposition gives us the output immediately

$$x(t) = \frac{x_0(t)}{2} + \sum_{n=1}^{\infty} x_n(t)$$

Next time: Fourier series.

Lecture 21

11 Nov 97

Vector Decomposition:

$\vec{F} = (3,5,7) = 3\hat{e}_1 + 5\hat{e}_2 + 7\hat{e}_3 = \sum_{i=1}^{3} a_i \hat{e}_i$

$\{a_i\}$ are components in $\{\hat{e}_i\}$ basis

Decompose in another orthonormal basis

$\hat{n}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right)$

$\hat{n}_2 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\hat{n}_3 = \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$\hat{n}_i \cdot \hat{n}_j = \delta_{ij}$

$\vec{F} = \sum_{j=1}^{3} b_j \hat{n}_j$

$\{b_i\}$ are components in $\{\hat{n}_j\}$ basis

How to find b_j — dot $\hat{n}_i$ into both sides

$\hat{n}_i \cdot \vec{F} = \sum_{j=1}^{3} b_j \, \hat{n}_i \cdot \hat{n}_j = \sum_{j=1}^{3} b_j \delta_{ij} = b_i$

$\vec{F} = \sum_{j=1}^{3} (\hat{n}_j \cdot \vec{F})\, \hat{n}_j$

$\hat{n}_1 \cdot \vec{F} = -4/\sqrt{2}$

$\hat{n}_2 \cdot \vec{F} = 15/\sqrt{3}$

$\hat{n}_3 \cdot \vec{F} = 0$

$\vec{F} = \frac{-4}{\sqrt{2}}\hat{n}_1 + \frac{15}{\sqrt{3}}\hat{n}_2 + 0.$

Functional Decomposition:

eg. Taylor series: $\vec{F}(t) = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots = \sum_{i=0}^{\infty} a_i t^i$

Use sines and cosines as a basis instead of polynomials.

Need functional dot product and orthonormal basis $\{f_i(t)\}$.

$$\langle f_i(t), f_j(t)\rangle = \frac{1}{N}\int_{t_1}^{t_2} f_i(t)\, f_j(t)\, dt = \delta_{ij}$$

Fourier basis:

$\{1 = \cos(0\omega t),\ \cos(\omega t),\ \cos(2\omega t),\ \cos(3\omega t), \ldots, \sin(\omega t), \sin(2\omega t), \ldots\}$

Check orthonormality:

$\langle 1, 1\rangle = \frac{2}{T}\int_0^T 1\cdot 1\cdot dt = 2$

$\langle 1, \cos(\omega t)\rangle = \frac{2}{T}\int_0^T 1.\cos(\omega t) = 0$

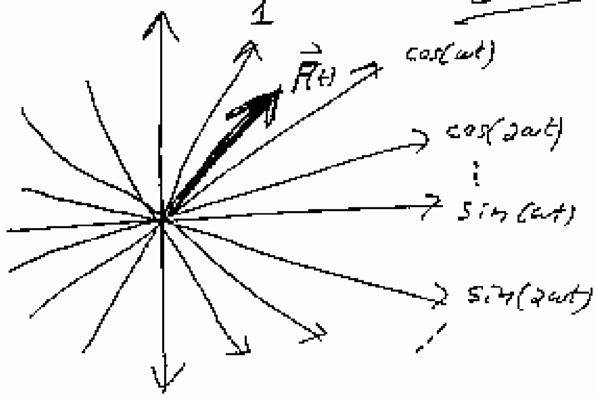
$\langle \cos(\omega t), \cos(\omega t)\rangle = \frac{2}{T}\int_0^T \cos^2(\omega t) = 1$

$\langle \cos(n\omega t), \cos(p\omega t)\rangle = \delta_{np}$

$\langle \cos(n\omega t), \sin(p\omega t)\rangle = 0$

$\langle \sin(n\omega t), \sin(p\omega t)\rangle = \delta_{np}$

Excer [illegible]



Infinite-dimensional vector space

$\vec{F}$ has components along each direction

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

Find a_0 : Multiply both sides by 1 and integrate $\frac{2}{T}\int_0^T$

$$\langle 1, F(t)\rangle = \frac{2}{T}\int_0^T 1\cdot F(t)\,dt = \frac{2}{T}\int_0^T \frac{a_0}{2}\,dt + \frac{2}{T}\int_0^T \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]dt$$

$$\boxed{\frac{2}{T}\int_0^T F(t)\,dt = a_0}$$

Find a_n $(n = 1, 2, \dots)$: Multiply both sides by $\cos(p\omega t)$ and integrate $\frac{2}{T}\int_0^T$

$$\langle \cos(p\omega t), F(t)\rangle = \underbrace{\left\langle \cos(p\omega t), \frac{a_0}{2}\right\rangle}_{0} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\langle \cos(p\omega t), \cos(n\omega t)\rangle}_{\delta_{pn}} + b_n \underbrace{\langle \cos(p\omega t), \sin(n\omega t)\rangle}_{0}\right]$$

$$\boxed{\frac{2}{T}\int_0^T \cos(p\omega t)\, F(t)\,dt = a_p}$$

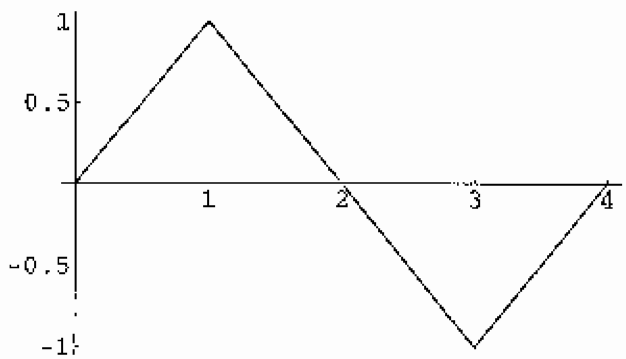
Find b_n : Multiply both sides by $\sin(p\omega t)$ and integrate $\frac{2}{T}\int_0^T$

$$\langle \sin(p\omega t), F(t)\rangle = \underbrace{\left\langle \sin(p\omega t), \frac{a_0}{2}\right\rangle}_{0} + \sum_{n=1}^{\infty}\left[a_n \underbrace{\langle \sin(p\omega t), \cos(n\omega t)\rangle}_{0} + b_n \underbrace{\langle \sin(p\omega t), \sin(n\omega t)\rangle}_{\delta_{pn}}\right]$$

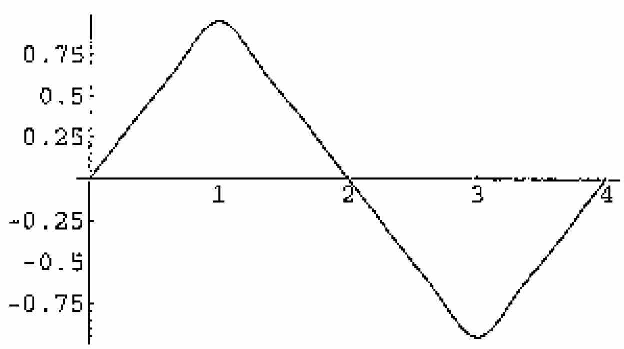
$$\boxed{\frac{2}{T}\int_0^T \sin(p\omega t)\, F(t)\,dt = b_p}$$

▪ Triangle wave

```
In[1]:= ?If

        If[condition, t, f] gives t if condition evaluates to True, and f if it evaluates
          to False. If[condition, t, f, u] gives u if condition evaluates to neither True nor False.

In[16]:= f[t_] = If[t < 1, t, If[t < 3, 2 - t, t - 4]];

In[17]:= Plot[f[t], {t, 0, 4}];
```



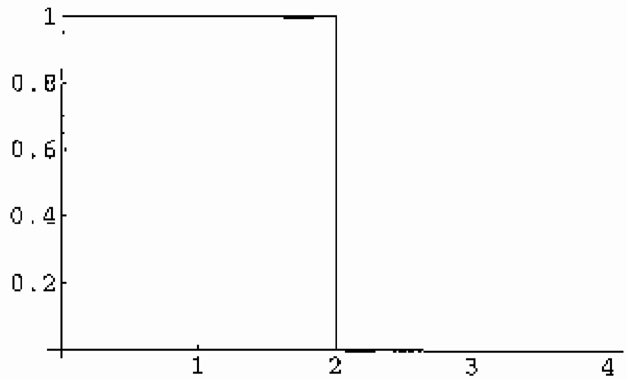

```
In[4]:= T = 4;
        w = 2 Pi / T;

In[18]:= b[n_] := 2 / T Integrate[f[t] Sin[n w t], {t, 0, 4}];

In[20]:= Plot[Sum[b[n] Sin[n w t], {n, 1, 7}], {t, 0, 4}];
```



▪ Square wave

```
In[21]:= f[t_] = If[t < 2, 1, 0];
```

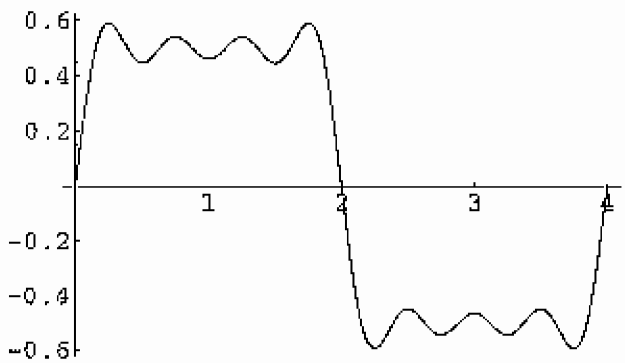
```
In[22]:= Plot[f[t], {t, 0, 4}];
```




```
In[23]:= T = 4;
         w = 2 Pi / T;

In[24]:= b[n_] := 2 / T Integrate[f[t] Sin[n w t], {t, 0, 4}];

In[25]:= b[1]
```

Out[25]= $\frac{2}{\pi}$

```
In[26]:= Plot[Sum[b[n] Sin[n w t], {n, 1, 7}], {t, 0, 4}];
```




```
In[27]:  Plot[Sum[b[n] Sin[n w t], {n, 1, 30}], {t, 0, 4}];
```