Why does this work?

$$I = \int_{x_1}^{x_2} f[y(x), y'(x); x]\, dx$$

functional

generalized "velocities"

$$S = \int_{t_1}^{t_2} L[q(t), \dot{q}(t); t]\, dt$$

action

generalized coordinates

Euler-Lagrange

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

The $y(x)$ that solves this D.E. guarantees $\delta I = 0$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

.... $q(t)$ ....

$\delta S = 0$

action is stationary

e.g. mass m in one dimension subject to potential energy $V(x)$

Lagrangian: $L = T - V$

Kinetic energy: $T = \frac{1}{2} m \dot{x}^2$

$$S = \int_{t_1}^{t_2} \underbrace{\left[\frac{1}{2} m \dot{x}^2 - V(x)\right]}_{L} dt$$

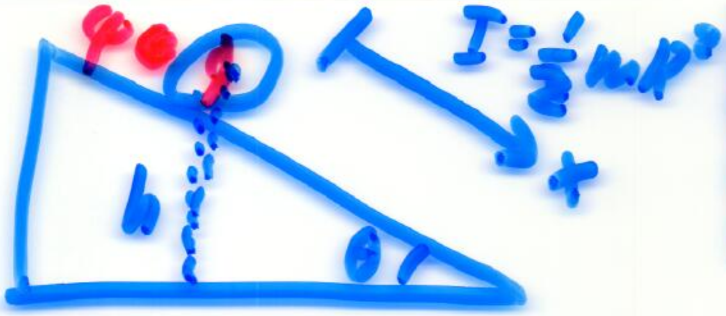
Euler-Lagrange $\Rightarrow$ equation of motion of the mass

differential equation with solution $x(t)$ that makes S stationary

$$\frac{\partial L}{\partial q} = \frac{\partial L}{\partial x} = \frac{-\partial V}{\partial x} = F = \text{force}$$

$$\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \Bigg| \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\Longrightarrow m\ddot{x} - F = 0 \longrightarrow F = ma$$



1 degree of freedom.

Constraint
$v = R\omega$
$\dot{x} = R\dot{\phi}$
$\Rightarrow x = R\phi$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{4} m \dot{x}^2$$

$$= \frac{3}{4} m \dot{x}^2$$ kinetic energy

$$V = - m g x \sin\phi + C$$ potential energy

Lagrangian $\mathcal{L} = T - V$

$$\mathcal{L} = \frac{3}{4} m \dot{x}^2 + m g x \sin\theta$$

$$\frac{\partial \mathcal{L}}{\partial x} = m g \sin\theta \quad \Bigg| \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{3}{2} m \dot{x}$$

Euler-Lagrange Equation $\Rightarrow$

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = 0$$

makes $\underbrace{S}_{\text{action}} = \int \mathcal{L}\, dt$ stationary

$$mg\sin\theta - \frac{d}{dt}\left(\frac{3}{2}m\dot{x}\right) = 0$$

$$mg\sin\theta = \frac{3}{2}m\ddot{x} \qquad \text{if } m \neq 0$$

$$\ddot{x} = \frac{2}{3}g\sin\theta \quad \checkmark$$

Two degrees of freedom with one constraint.

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\varphi}^2$$

$$V = -mgx\sin\theta$$

Constraint (holonomic) $v = R\omega \Rightarrow \dot{x} = R\dot{\varphi} \Rightarrow x = R\varphi$

$$f = x - R\varphi = 0$$

new Lagrangian

$$L' = T - V - \lambda f$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{4}mR^2\dot{\varphi}^2 + mgx\sin\theta - \lambda(x - R\varphi)$$

$\dfrac{\partial L'}{\partial x} = mg\sin\theta - \lambda$	$\dfrac{\partial L'}{\partial \dot{x}} = m\dot{x}$
$\dfrac{\partial L'}{\partial \varphi} = \lambda R$	$\dfrac{\partial L'}{\partial \dot{\varphi}} = \frac{1}{2} m R^2 \dot{\varphi}$

Euler - Lagrange (Equations of motion)

① $\dfrac{\partial L'}{\partial x} - \dfrac{d}{dt}\left(\dfrac{\partial L'}{\partial \dot{x}}\right) = 0 \Longrightarrow mg\sin\theta - \lambda - m\ddot{x} = 0$

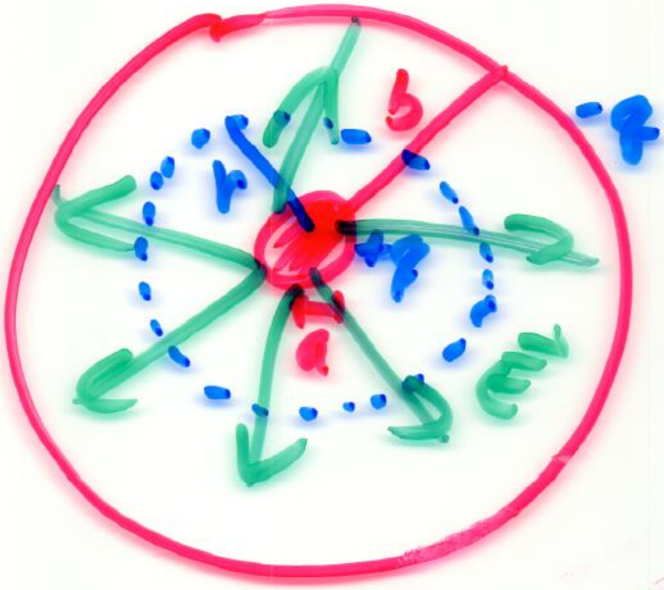
② $\dfrac{\partial L'}{\partial \varphi} - \dfrac{d}{dt}\left(\dfrac{\partial L'}{\partial \dot{\varphi}}\right) = 0 \longrightarrow \lambda R - \frac{1}{2} m R^2 \ddot{\varphi} = 0$

③ $f = 0 \Longrightarrow x = R\varphi, \ \dot{x} = R\dot{\varphi}, \ \ddot{x} = R\ddot{\varphi}$

② $\Longrightarrow \lambda = \frac{1}{2} m R \ddot{\varphi}$ (use ③) $\longrightarrow \lambda = \frac{1}{2} m \ddot{x}$

Lagrange multiplier is the "force" that imposes the constraint. force of static friction f_s

① $\longrightarrow \ddot{x} = \frac{2}{3} g \sin\theta$



$E=0 \quad r<a$

$E=0 \quad b<r$

Gauss' law

$$\oiint \vec{E}\cdot d\vec{A} = \frac{Q_{encl}}{\epsilon_0}$$

$$E\cdot A = \frac{+q}{\epsilon_0}$$

$$E\, 2\pi r l = \frac{q}{\epsilon_0}$$

unit length $l=1$

$$E = \frac{q}{2\pi\epsilon_0 r}$$

$$\Delta V_{AB} = \int_a^b \vec{E}\cdot d\vec{l} = \frac{q}{2\pi\epsilon_0}\int_a^b \frac{1}{r}\,dr$$

$$V = \frac{q}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$

capacitor : $Q = CV$

$$\frac{C}{2\pi\epsilon_0} = \frac{1}{\ln(b/a)}$$