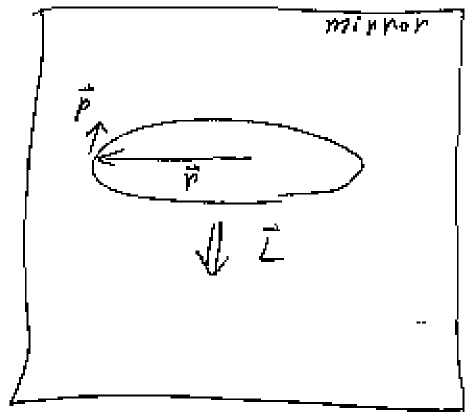
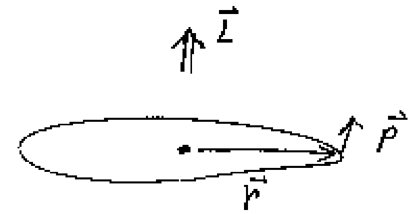
Lecture #2 20 Jan 2000

## Reflections




$\vec{r}$, $\vec{p}$ are vectors (polar vectors)

$\vec{L} = \vec{r} \times \vec{p}$ is a <u>pseudo</u> vector (axial vector)

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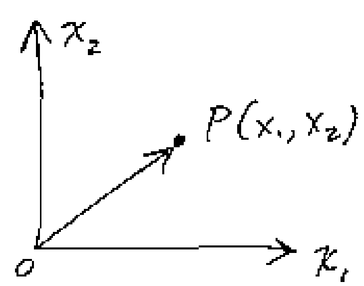
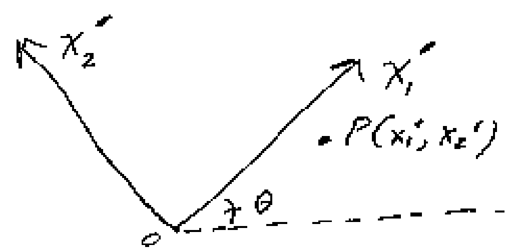
How does $\vec{r}$ change under rotations?

Special case: rotations in 2 dimensions (can't rotate in 1-d)

Later we will generalize to higher dimensions.

$R_\theta$ is the rotation transformation by angle $\theta$ (one parameter)




Passive transformation: $P$ stays fixed, axes are rotated by angle $\theta$ counterclockwise about origin.

Active transformation: axes stay fixed, $P$ is rotated by angle $\theta$ clockwise about origin.

$R_\theta$ preserves: the lengths of vectors
the angle between vectors

such a transformation is called "orthogonal."

Two-dimensional rotations form a mathematical "group" $U(1)$

Properties: 1) $R_{\theta_2} R_{\theta_1} (\vec{x}) = R_{\theta_3} (\vec{x})$ $\qquad \theta_3 = \theta_1 + \theta_2$

two successive rotations are equivalent to a third rotation.

2) $R_0 = \mathbb{I}$ identity. $\quad R_0 \vec{x} = \vec{x} \qquad R_{2\pi n} \vec{x} = \vec{x}$

$n$ integer

3) $R_{-\theta} R_\theta (\vec{x}) = \mathbb{I} \vec{x} = \vec{x}$

every element of the group has an inverse.

The order of 2-d rotations is irrelevant. This kind of group is called "abelian".

$$R_{\theta_2} R_{\theta_1} (\vec{x}) = R_{\theta_1} R_{\theta_2} (\vec{x})$$

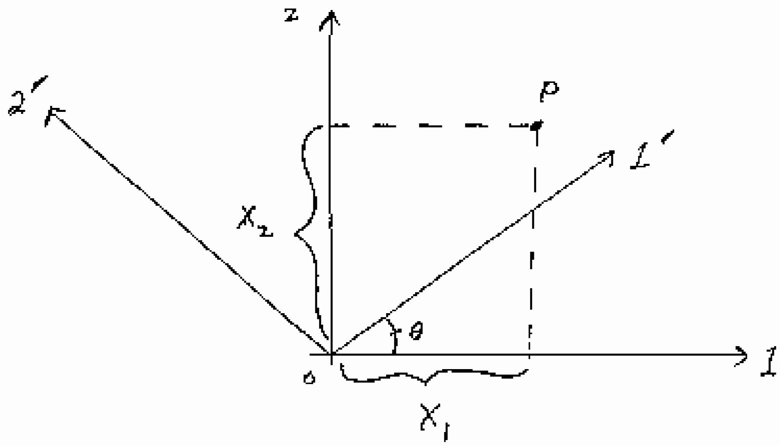
or

$$R_{-\theta_2} R_{-\theta_1} R_{\theta_2} R_{\theta_1} (\vec{x}) = \vec{x}$$

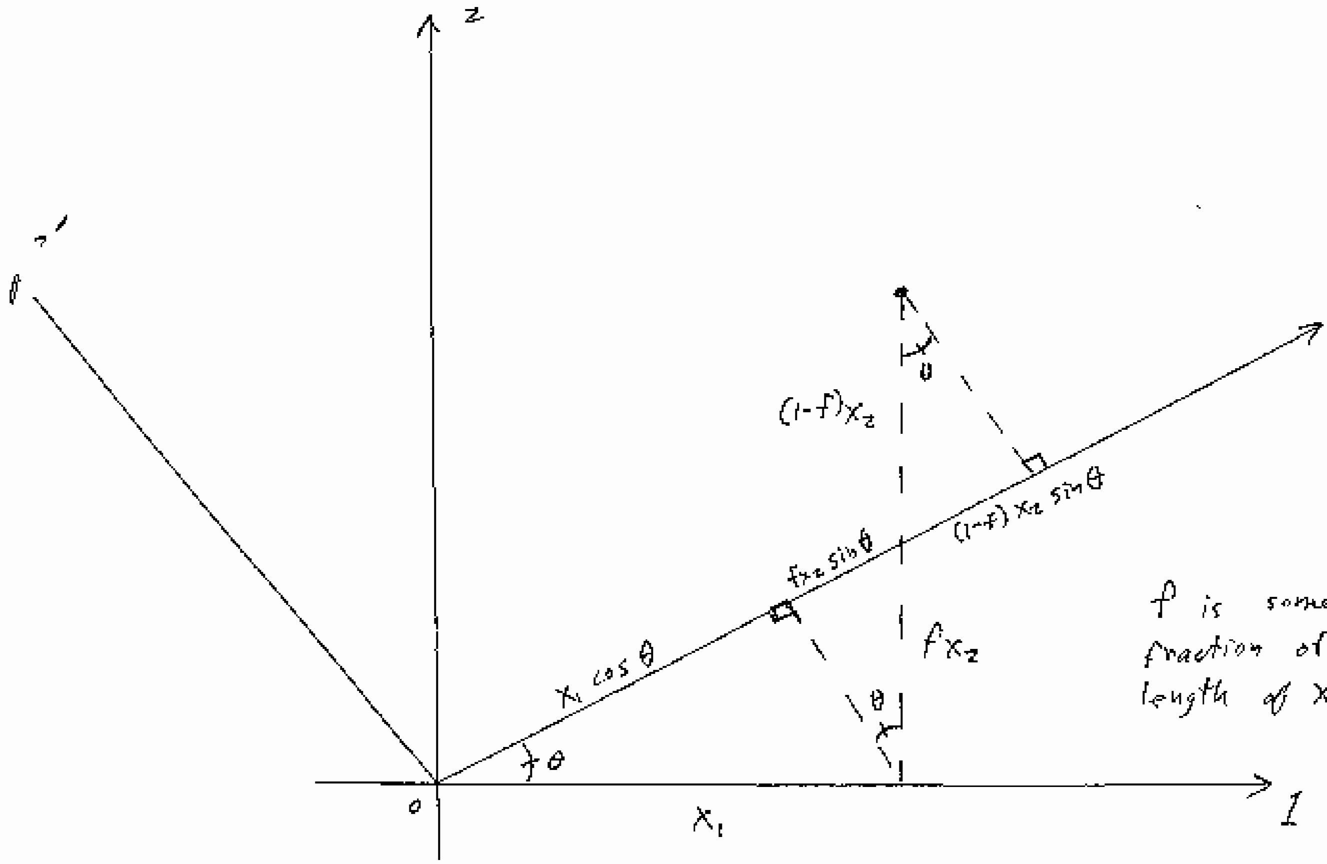
Not true for 3-d rotations (e.g. rotations of book).

The world is non-abelian: You cook your food, then you eat it!

2-dimensional rotations




Passive Transformation: leave P fixed and rotate axes by angle θ counterclockwise.




f is some fraction of the length of $X_2$.

$$X_1' = X_1 \cos\theta + f\,X_2 \sin\theta + (1-f)\,X_2 \sin\theta$$

now you see the fraction f is irrelevant

$$X_1' = X_1 \cos\theta + X_2 \sin\theta$$

By a similar geometrical construction, you can find $X_2'$:

$$X_1' = X_1 \cos\theta + X_2 \sin\theta$$
$$X_2' = -X_1 \sin\theta + X_2 \cos\theta$$

Rewrite these equations in matrix form

$$\begin{pmatrix} X_1' \\ X_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$ check this!

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And remember how to multiply matrices

$$\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} = \underline{\underline{C}}$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

$$c_{11} = a_{11} b_{11} + a_{12} b_{21} \qquad c_{12} = a_{11} b_{12} + a_{12} b_{22}$$
$$c_{21} = a_{21} b_{11} + a_{22} b_{21} \qquad c_{22} = a_{21} b_{12} + a_{22} b_{22}$$

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One more time in condensed form

$\vec{X}' = \underline{\underline{\lambda}} \, \vec{X}$ where $\underline{\underline{\lambda}}$ is the 2-dim transformation matrix with components

$$\underline{\underline{\lambda}} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \qquad \lambda_{11} = \cos\theta \quad \lambda_{12} = \sin\theta$$
$$\lambda_{21} = -\sin\theta \quad \lambda_{22} = \cos\theta$$

($\underline{\underline{\lambda}}$ is what we called $R_\theta$ last time.)

It is often useful to write the transformation in

Index Notation

$$\vec{x}\,' = \underset{\approx}{\lambda}\,\vec{x} \quad \Longleftrightarrow \quad x_i' = \sum_{j=1}^{2} \lambda_{ij}\, x_j$$

ith row jth column entry of $\lambda$ matrix

For example, when $i=1$ this gives

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 = \cos\theta\, x_1 + \sin\theta\, x_2$$

which is the same equation we had previously.

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Two things to notice:

① $j$ is a "dummy" index — because $j$ is summed over, one can call it by <u>any</u> name, $k$ for example

$$x_i' = \sum_{k=1}^{2} \lambda_{ik}\, x_k$$

② In matrix multiplication the sum occurs over <u>adjacent</u> indices. In the example above, the $k$'s touch each other.

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One can solve the transformation equations

$$x_1' = x_1 \cos\theta + x_2 \sin\theta$$
$$x_2' = -x_1 \sin\theta + x_2 \cos\theta$$

(2 equations in 2 unknowns)

for the unprimed variables using algebra to get

$$x_1 = x_1' \cos\theta - x_2' \sin\theta$$
$$x_2 = x_1' \sin\theta + x_2' \cos\theta$$

But this is the hard way. The easy way is to notice that primed and unprimed are arbitrary labels for the axes and it doesn't matter which is which. If you change

$$\text{primed} \longleftrightarrow \text{unprimed}$$

then you must also change $\theta \leftrightarrow -\theta$.

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Write the inverse transformation in matrix form

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} X_1' \\ X_2' \end{pmatrix} \quad \text{or} \quad \vec{X} = \underline{\underline{\mu}}\, \vec{X}'$$

where the matrix $\underline{\underline{\mu}}$ is the <u>transpose</u> of $\underline{\underline{\lambda}}$

$$\underline{\underline{\mu}} = \underline{\underline{\lambda}}^T \qquad \text{(switch rows and columns)}$$

$$\mu_{ij} = \lambda_{ji}$$

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Now we have $\vec{X} = \underline{\underline{\mu}}\, \vec{X}'$ from above and $\vec{X}' = \underline{\underline{\lambda}}\, \vec{X}$ from before. Combining these

$$\vec{X} = \underline{\underline{\mu}}(\vec{X}') = \underline{\underline{\mu}}(\underline{\underline{\lambda}}\, \vec{X}) = (\underline{\underline{\mu}}\, \underline{\underline{\lambda}})\, \vec{X}$$

this can only be true if

$$\underline{\underline{\mu}}\, \underline{\underline{\lambda}} = \underline{\underline{I}} \quad (2\times 2 \text{ identity matrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

or

$$\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{I}} \qquad \text{also} \qquad \underline{\underline{\lambda}}\, \underline{\underline{\lambda}}^T = \underline{\underline{I}}$$

or

$$\lambda^{-1} = \lambda^T \qquad (\text{inverse} = \text{transpose})$$

A matrix $\underline{\underline{\lambda}}$ that obeys $\underline{\underline{\lambda}}^T\underline{\underline{\lambda}} = \underline{\underline{\mathbb{I}}}$ is called "orthogonal." That means the transformation preserves:

1) lengths of vectors
2) Angles between vectors

Let's write the orthogonality relation in index notation for practice:

$$\underline{\underline{\lambda}}^T\underline{\underline{\lambda}} = \underline{\underline{\mathbb{I}}} \iff \sum_{j=1}^{2} (\lambda^T)_{ij}\, \lambda_{jk} = \mathbb{I}_{ik}$$

adjacent index
is summed over
in matrix multiplication

Now two alterations:

$$(\lambda^T)_{ij} = \lambda_{ji}$$ transpose switches $i \leftrightarrow j$

$\mathbb{I}_{ik}$ is the element in the ith row, jth column of the identity matrix.

$1$ if $i = k$ and $0$ if $i \neq k$

There is a symbol for this

$$\delta_{ik} = \begin{cases} 1 & \text{if } i=k \\ 0 & \text{if } i \neq k \end{cases}$$ The Kronecker delta

$$\sum_{j=1}^{2} \lambda_{ji}\, \lambda_{jk} = \delta_{ik}$$ orthogonality

No longer matrix multiplication since the summed indices don't touch.

## Direction Cosines

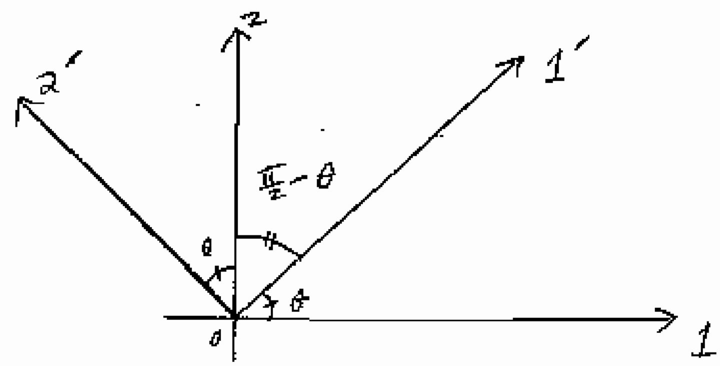
$$\underset{=}{\lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos(\frac{\pi}{2}-\theta) \\ \cos(\frac{\pi}{2}+\theta) & \cos\theta \end{pmatrix}$$

$\lambda_{11}$ is the cosine of the angle between $1'$ and $1$ axes

$\lambda_{12}$ " $1'$ $2$

$\lambda_{21}$ " $2'$ $1$

$\lambda_{22}$ " $2'$ $2$

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the orthogonality condition imposes 3 constraints on the 4 elements of the $\underset{=}{\lambda}$ matrix, leaving one "degree of freedom."

$$\sum_{j=1}^{2} \lambda_{ji}\lambda_{jk} = \delta_{ik}$$

① $\lambda_{11}\lambda_{11} + \lambda_{21}\lambda_{21} = 1$ $(i=1,\ k=1)$

② $\lambda_{12}\lambda_{12} + \lambda_{22}\lambda_{22} = 1$ $(i=2,\ k=2)$

③ $\lambda_{11}\lambda_{12} + \lambda_{21}\lambda_{22} = 0$ $(i=1,\ k=2$ or $i=2,\ k=1)$

same constraint

## Generalize to 3 dimensions

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3$$
$$x_2' = \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3$$
$$x_3' = \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3$$

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In matrix form: $\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

In condensed form: $\vec{x}' = \underline{\underline{\lambda}} \, \vec{x}$

In index notation: $x_p' = \sum_{a=1}^{3} \lambda_{pa} x_a$

The 3-dimensional rotation matrix is also orthogonal:

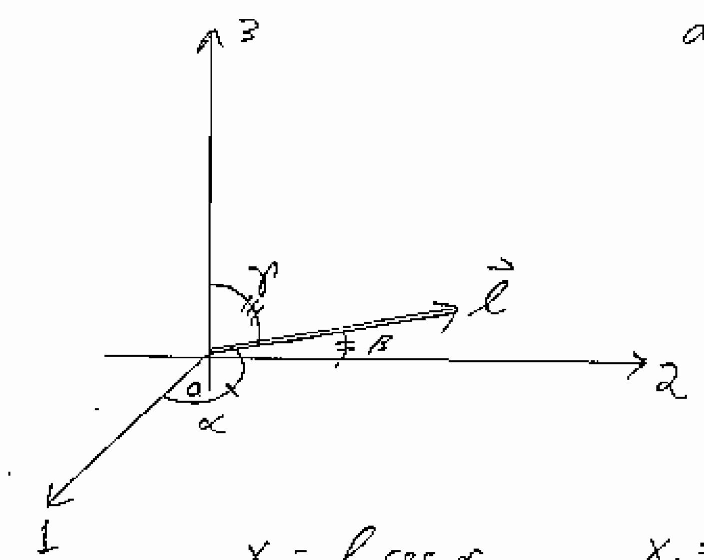
$$\underline{\underline{\lambda}}\,\underline{\underline{\lambda}}^T = \underline{\underline{I}} \iff \sum_{j=1}^{3} \lambda_{ij} (\lambda^T)_{jk} = \delta_{ik}$$

The elements of $\underline{\underline{\lambda}}$ are still direction cosines, but now in 3 dimensions:

$\lambda_{11}$ is the cosine of the angle between 1′ and 1 axes

$\lambda_{23}$ 〃 2′ and 3

$\lambda_{ij}$ 〃 i′ and j




$\alpha$ is the angle between the vector $\vec{\ell}$ and the 1 axis

$\beta$ between $\vec{\ell}$ and 2 axis

$\gamma$ between $\vec{\ell}$ and 3 axis

$$X_1 = \ell \cos\alpha \qquad X_2 = \ell\cos\beta \qquad X_3 = \ell\cos\gamma$$

The three direction cosines are not independent:

$$(\text{Length of } \vec{\ell})^2 = \ell^2 = X_1^2 + X_2^2 + X_3^2 \quad \begin{pmatrix}\text{3-dim} \\ \text{Pythagorean} \\ \text{Theorem}\end{pmatrix}$$

$$\ell^2 = \ell^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)$$

$$\boxed{1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma}$$

What does this have to do with rotations in 3-dim?

Take $\vec{\ell}$ in the direction of the $1'$ axis:

$$\lambda_{11} = \cos\alpha \qquad \lambda_{12} = \cos\beta \qquad \lambda_{13} = \cos\gamma$$

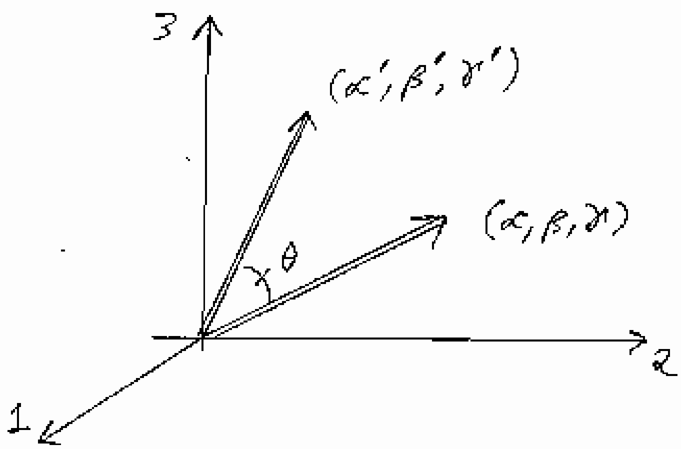
$$\boxed{\lambda_{11}^2 + \lambda_{12}^2 + \lambda_{13}^2 = 1}$$

Take $\vec{\ell}$ in the direction of the $2'$ axis:

$$\boxed{\lambda_{21}^2 + \lambda_{22}^2 + \lambda_{23}^2 = 1}$$

$\vec{\ell}$ along $3'$ $\Longrightarrow$ $\boxed{\lambda_{31}^2 + \lambda_{32}^2 + \lambda_{33}^2 = 1}$

So far, we have 3 constraints on the 9 elements of $\underline{\underline{\lambda}}$. We will get 3 more after a digression into math.




Here are 2 lines with different direction cosines. The angle between the lines is $\theta$.

Part of your homework is to prove that:

$$\cos\theta = \cos\alpha \cos\alpha' + \cos\beta \cos\beta' + \cos\gamma \cos\gamma'$$

The angle $\theta$ between the $1'$ and $2'$ axes is $\frac{\pi}{2}$, so

$$\boxed{\cos\left(\frac{\pi}{2}\right) = 0 = \lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23}}$$

The $2'$ and $3'$ axes are also $\frac{\pi}{2}$ apart:

$$\boxed{0 = \lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{23}\lambda_{33}}$$

And finally, the $3'$ and $1'$ axes are $\frac{\pi}{2}$ apart:

$$\boxed{0 = \lambda_{31}\lambda_{11} + \lambda_{32}\lambda_{12} + \lambda_{33}\lambda_{13}}$$

These equations are 3 more constraints on $\lambda_{ij}$.
There are a total of 6 constraints on the 9 elements, leaving 3 degrees of freedom for 3-dimensional rotations.

The 6 constraints can be summarised in index notation:

$$\left.\begin{array}{l}
\lambda_{11}\lambda_{11} + \lambda_{12}\lambda_{12} + \lambda_{13}\lambda_{13} = 1 \\
\lambda_{21}\lambda_{21} + \lambda_{22}\lambda_{22} + \lambda_{23}\lambda_{23} = 1 \\
\lambda_{31}\lambda_{31} + \lambda_{32}\lambda_{32} + \lambda_{33}\lambda_{33} = 1 \\
\lambda_{11}\lambda_{21} + \lambda_{12}\lambda_{22} + \lambda_{13}\lambda_{23} = 0 \\
\lambda_{21}\lambda_{31} + \lambda_{22}\lambda_{32} + \lambda_{23}\lambda_{33} = 0 \\
\lambda_{31}\lambda_{11} + \lambda_{32}\lambda_{12} + \lambda_{33}\lambda_{13} = 0
\end{array}\right\} \quad \sum_{j=1}^{3} \lambda_{ij}\lambda_{kj} = \delta_{ik}$$

check this!

And notice that it is exactly the orthogonality relation $\underline{\underline{\lambda}}^T \underline{\underline{\lambda}} = \underline{\underline{I}}$

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The 3 degrees of freedom can be seen as follows.

You must specify a direction for the axis of rotation. This requires 2 numbers (latitude and longitude) or ($\theta$ and $\varphi$ in spherical polar coordinates). Then you must specify the angle of twist about that axis — one number.