Lecture #5 27 Jan 2006

Now that we know how the displacement vector $\vec{x}$ transforms under rotations, we know how <u>any</u> vector transforms

$$\vec{x}\,' = \underline{\underline{\lambda}}\,\vec{x} \qquad\qquad x_i' = \sum_{j=1}^{3} \lambda_{ij}\, x_j$$

$$\vec{p}\,' = \underline{\underline{\lambda}}\,\vec{p} \qquad\qquad p_i' = \sum_{k=1}^{3} \lambda_{ik}\, p_k$$

Remember that scalars do not transform at all

$$s' = s$$

and rank-n tensors transform like the exterior product of n displacement vectors. So how does the exterior product $r_i r_j$ transform?

$$r_i' r_j' = \left(\sum_{k=1}^{3} \lambda_{ik}\, r_k\right)\left(\sum_{\ell=1}^{3} \lambda_{j\ell}\, r_\ell\right)$$

$$= \sum_k \sum_\ell \lambda_{ik}\,\lambda_{j\ell}\,(r_k r_\ell)$$

one factor of λ for each index

$$T_{ij}' = \sum_{k,\ell} \lambda_{ik}\,\lambda_{j\ell}\, T_{k\ell}$$

↑ 2 sums!

The generalization is straight forward

$$T'_{ijk} = \sum_a \sum_b \sum_c \lambda_{ia} \lambda_{jb} \lambda_{kc} T_{abc}$$ (3 indices $\Rightarrow$ 3 λ's)

---

the Dot Product (or Scalar Product)
between two vectors

<u>def</u> $\vec{A} \cdot \vec{B} = \sum_i A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$

The scalar product of two vectors is a <u>scalar</u>. Proof:

$$\vec{A}' \cdot \vec{B}' = \sum_i A'_i B'_i = \sum_i \left(\sum_j \lambda_{ij} A_j \right) \left(\sum_k \lambda_{ik} B_k \right)$$

$$= \sum_j \sum_k \left(\sum_i \lambda_{ij} \lambda_{ik} \right) A_j B_k = \sum_j \sum_k \delta_{jk} A_j B_k$$

$$= \sum_j A_j B_j = \vec{A} \cdot \vec{B}$$

So the scalar product in the primed system is the same as the scalar product in the unprimed system; there is no λ transformation required.

In case you are wondering about the Kronecker delta:

$$\sum_{n=1}^{3} \delta_{kn} X_n = X_k$$

Special case $- k=2$

$$\underbrace{\delta_{21}}_{0} X_1 + \delta_{22} X_2 + \underbrace{\delta_{23}}_{0} X_3 = X_2$$

The Cross Product (or Vector Product)

$$\vec{D} = \vec{A} \times \vec{B}$$

$$D_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} A_j B_k$$

where ϵ_{ijk} is the Levi-Civita symbol (permutation symbol)

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any indices are the same} \quad \epsilon_{112} = 0, \; \epsilon_{333} = 0 \\ +1, & \text{if } ijk \text{ forms an } \underline{\text{even}} \text{ permutation of } 123 \quad 123,\ 231,\ 312 \\ -1, & \text{if } ijk \text{ forms an } \underline{\text{odd}} \text{ permutation of } 123 \quad 321,\ 213,\ 132 \end{cases}$$

For example:

$$D_1 = \epsilon_{123} A_2 B_3 + \epsilon_{132} A_3 B_2 \quad \text{and the other seven terms} = 0$$

$$= A_2 B_3 - A_3 B_2$$

Triple Product

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{D} \cdot \vec{C} = \sum_i D_i C_i = \sum_i \left(\sum_j \sum_k \epsilon_{ijk} A_j B_k \right) C_i$$

All the indices are dummies – all are summed over so they can be renamed
$i \rightarrow K$
$j \rightarrow I$
$k \rightarrow J$

$$= \sum_K \sum_I \sum_J \epsilon_{KIJ} A_I B_J C_K$$

$$= \sum_K \sum_I \sum_J \epsilon_{IJK} B_J C_K A_I$$

$$= (\vec{B} \times \vec{C}) \cdot \vec{A}$$

$$\boxed{\begin{aligned} \epsilon_{KIJ} &= -\epsilon_{IKJ} \\ &= +\epsilon_{IJK} \end{aligned}}$$

every time you swap two adjacent indices you incur a minus sign.

This is $\underline{\text{not}}$ obvious! $(\vec{A} \times \vec{B})$ and $(\vec{B} \times \vec{C})$ point in completely different directions and have completely different lengths.

The triple product is often written $(\vec{A}, \vec{B}, \vec{C})$ because

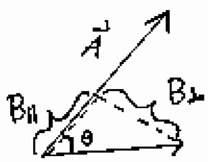
$$\vec{A}\times\vec{B}\cdot\vec{C} = \vec{B}\times\vec{C}\cdot\vec{A} = \vec{C}\times\vec{A}\cdot\vec{B}$$
$$= \vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A} = \vec{C}\cdot\vec{A}\times\vec{B}$$

$\boxed{\text{As long as } \vec{A}, \vec{B}, \vec{C} \text{ are in cyclic order, the dot and cross can be anywhere.}}$

Physical Interpretations:

Dot Product:

$$\vec{A}\cdot\vec{B} = (\text{Length of } \vec{A})(\text{the part of } \vec{B} \text{ that lies along } \vec{A}) = |\vec{A}|\, B_{\parallel}$$
$$= (\text{Length of } \vec{B})(\text{the part of } \vec{A} \text{ that lies along } \vec{B}) = |\vec{B}|\, A_{\parallel}$$



The dot product picks out <u>parallel</u> components of vectors.

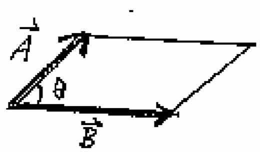
$$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}| \cos\theta$$

Cross Product:

$$|\vec{A}\times\vec{B}| = (\text{Length of } \vec{A})(\text{the part of } \vec{B} \text{ that lies perpendicular to } \vec{A})$$
$$= |\vec{A}|\, B_{\perp}$$
$$= (\text{Length of } \vec{B})(\text{the part of } \vec{A} \text{ that lies perpendicular to } \vec{B})$$
$$= |\vec{B}|\, A_{\perp}$$

The cross product picks out <u>perpendicular</u> components of vectors.

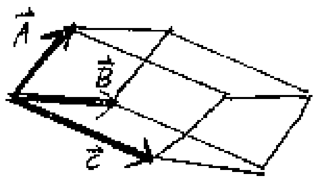
$|\vec{A}\times\vec{B}|$ is the area of the parallelogram with sides along $\vec{A}$ and $\vec{B}$



$$|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}| \sin\theta$$

Triple Products

$\vec{A} \cdot \vec{B} \times \vec{C}$ is $\pm$ Volume of the parallelopiped with edges along $\vec{A}$, $\vec{B}$, and $\vec{C}$. If $\{\vec{A}, \vec{B}, \vec{C}\}$ form a right-handed system, then $\vec{A} \cdot \vec{B} \times \vec{C}$ is positive.



A useful vector identity

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \qquad \text{[BAC-CAB rule]}$$

for the proof (not given here), you need the Levi-Civita identity:

$$\sum_{k=1}^{3} \epsilon_{ijk} \, \epsilon_{lmk} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$$

Suppose that $\hat{e}$ is a unit vector (length 1 unit) in some direction

$$\hat{e} = \frac{\vec{A}}{|\vec{A}|} \qquad \hat{e} \cdot \hat{e} = 1$$

then any vector $\vec{v}$ can be decomposed as

$$\vec{v} = \underbrace{\hat{e}(\vec{v} \cdot \hat{e})}_{\text{component of } \vec{v} \text{ along } \hat{e}} + \underbrace{\hat{e} \times (\vec{v} \times \hat{e})}_{\text{component of } \vec{v} \text{ perpendicular to } \hat{e}}$$

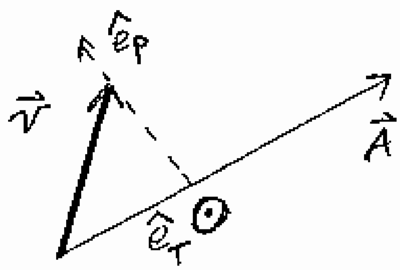
Proof:

$$\hat{e} \times (\vec{v} \times \hat{e}) = \vec{v} \underbrace{(\hat{e} \cdot \hat{e})}_{1} - \hat{e}(\hat{e} \cdot \vec{v}) \qquad \text{[BAC-CAB rule]}$$

$$\vec{v} = \hat{e}(\vec{v} \cdot \hat{e}) + \hat{e} \times (\vec{v} \times \hat{e}) \quad \blacksquare$$

1 Feb 2000

3-dimensional coordinate-free rotation



rotate $\vec{v}$ by angle φ around axis along $\hat{A}$ (unit vector)

Choose one axis along $\hat{A} = \hat{e}_A$

Choose the second axis (P) through the tip of $\vec{v}$, $\hat{e}_P$

The third axis is $\hat{e}_T = \hat{e}_A \times \hat{e}_P$

The component of $\vec{v}$ along $\hat{A}$ is $v_A = \vec{v} \cdot \hat{A}$

The component of $\vec{v}$ in the P direction is

$$v_P = |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| = |\hat{A} \times (\vec{v} \times \hat{A})|$$

The component of $\vec{v}$ in the T direction is $v_T = 0$.

Under the rotation by φ

$v_A' = v_A$ no change to component along axis

$v_P' = v_P \cos\varphi$

$v_T' = v_P \sin\varphi$

$$\vec{v}' = v_A' \hat{e}_A + v_P' \hat{e}_P + v_T' \hat{e}_T$$

$$= v_A \hat{e}_A + v_P \cos\varphi\, \hat{e}_P + v_P \sin\varphi\, \hat{e}_T$$

$$= (\vec{v} \cdot \hat{A})\hat{A} + |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| \cos\varphi\, \hat{e}_P + |\vec{v} - \hat{A}(\vec{v} \cdot \hat{A})| \sin\varphi\, \hat{e}_T$$

We are trying to express $\vec{v}'$ in terms of the givens $\hat{A}$, $\vec{v}$, and φ only.

$$\vec{v}' = (\vec{v}\cdot\hat{A})\hat{A} + \underbrace{[\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]}_{\substack{\text{already in the} \\ \rho \text{ direction}}} \cos\varphi + |\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|\, \hat{e}_\tau$$

The last change to make is to write $\hat{e}_\tau$ as

$$\hat{e}_\tau = \hat{e}_A \times \hat{e}_\rho = \hat{A} \times \frac{[\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]}{|\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|}$$

and notice that $\hat{A}\times\hat{A}=0$ in the numerator

$$\hat{e}_\tau = \frac{\hat{A}\times\vec{v}}{|\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})|}$$

$$\vec{v}' = (\vec{v}\cdot\hat{A})\hat{A} + [\vec{v}-\hat{A}(\vec{v}\cdot\hat{A})]\cos\varphi + \hat{A}\times\vec{v}\sin\varphi$$

$$= \vec{v}\cos\varphi + (\hat{A}\cdot\vec{v})\hat{A}(1-\cos\varphi) + \hat{A}\times\vec{v}\sin\varphi$$

$$v_1' = v_1\cos\varphi + (A_1v_1+A_2v_2+A_3v_3)A_1(1-\cos\varphi) + (A_2v_3-A_3v_2)\sin\varphi$$

$$v_2' = v_2\cos\varphi + (A_1v_1+A_2v_2+A_3v_3)A_2(1-\cos\varphi) + (A_3v_1-A_1v_3)\sin\varphi$$

$$v_3' = v_3\cos\varphi + (A_1v_1+A_2v_2+A_3v_3)A_3(1-\cos\varphi) + (A_1v_2-A_2v_1)\sin\varphi$$

$\vec{v}' = \underline{\lambda}\,\vec{v}$ to find λ_{21} for example look in the $2'$ line and write all occurances of v_1

$$\lambda_{21} = A_1A_2(1-\cos\varphi) + A_3\sin\varphi$$

To find λ_{12} look in the $1'$ line and write all occurances of v_2

$$\lambda_{12} = A_2A_1(1-\cos\varphi) - A_3\sin\varphi$$

Here is a complete list of the λ_{ij} direction cosines:

$$\lambda_{11} = \cos\varphi + A_1^2 (1-\cos\varphi)$$
$$\lambda_{22} = \cos\varphi + A_2^2 (1-\cos\varphi)$$
$$\lambda_{33} = \cos\varphi + A_3^2 (1-\cos\varphi)$$

$$\lambda_{12} = A_2 A_1 (1-\cos\varphi) - A_3 \sin\varphi$$
$$\lambda_{21} = A_1 A_2 (1-\cos\varphi) + A_3 \sin\varphi$$

$$\lambda_{23} = A_3 A_2 (1-\cos\varphi) - A_1 \sin\varphi$$
$$\lambda_{32} = A_2 A_3 (1-\cos\varphi) + A_1 \sin\varphi$$

$$\lambda_{31} = A_1 A_3 (1-\cos\varphi) - A_2 \sin\varphi$$
$$\lambda_{13} = A_3 A_1 (1-\cos\varphi) + A_2 \sin\varphi$$

How to use:

① Decompose the axis $\hat{A}$ into x, y, z components and make sure $\hat{A}$ is a unit vector $(\hat{A}\cdot\hat{A} = 1)$. If not, divide $\vec{A}$ by its length $|\vec{A}|$.

② Once you know A_1, A_2, A_3, and φ you can find the rotation matrix $\underline{\lambda}$

③ $\underline{\lambda}$ transforms vectors

$$\vec{X}' = \underline{\lambda}\, \vec{X}$$
$$\vec{v}' = \underline{\lambda}\, \vec{v}$$