Free-Fall Timescale of Sun

Free-fall timescale:

The time it would take a star (or cloud) to collapse to a point if there was no outward pressure to counteract gravity.

We can calculate the free-fall timescale of the sun.

Consider a mass element *dm* at rest in the sun at a radius r_0. What is its potential energy?

$$dU = -\frac{GM(r_0)dm}{r_0}$$

Next, apply the conservation of energy:

$$E_i = E_f$$

$$U_i + K_i = U_f + K_f$$

$$-\frac{GM(r_0)\cancel{dm}}{r_0} + \frac{1}{2}\cancel{dm}(\frac{dr}{dt})^2 = -\frac{GM(r_0)\cancel{dm}}{r}$$

Remember, we are looking for free-fall time. Simplify:

$$(\frac{dr}{dt})^2 = 2[\frac{GM(r_0)}{r_0} - \frac{GM(r_0)}{r}]$$

$$(\frac{dr}{dt}) = [-2GM(r_0)(\frac{1}{r} - \frac{1}{r_0})]^{\frac{1}{2}}$$

$$\left(\frac{dr}{dt}\right) = [-2GM(r_0)(\frac{1}{r} - \frac{1}{r_0})]^{\frac{1}{2}}$$

Continuing to simplify:

$$dt = [-2GM(r_0)(\frac{1}{r} - \frac{1}{r_0})]^{-\frac{1}{2}} dr$$

To find t_{ff} we integrate:

$$\tau_{\rm ff} = \int_0^{\tau_{\rm ff}} dt = -\int_{r_0}^{0} \left[2GM(r_0)\left(\frac{1}{r} - \frac{1}{r_0}\right)\right]^{-1/2} dr$$

As 5 points extra credit (due at the beginning of next class), show that from the above equation, you get the free-fall time of

$$\boxed{\tau_{\rm ff} = \left(\frac{3\pi}{32G\bar{\rho}}\right)^{1/2}}$$

Let's examine the solution. Does it contain the correct units (use cgs)?

$$\tau_{\mathrm{ff}} = \left(\frac{3\pi}{32G\bar{\rho}}\right)^{1/2}$$

G = [erg][cm][g^{-2}]　　　ϱ = [g][cm^{-3}]　　　erg = [g][cm^2][s^{-2}]

$$units = \sqrt{[\cancel{g}\,\cancel{cm^2}\,s^{-2}][\cancel{cm}][\cancel{g^{-2}}][\cancel{g}][\cancel{cm^{-3}}]}$$

$$units = [s]$$

For the parameters of the Sun, calculate the free-fall timescale.

$$\tau_{\mathrm{ff}\odot} = \left(\frac{3\pi}{32 \times 6.7 \times 10^{-8}\ \mathrm{cgs} \times 1.4\ \mathrm{g\ cm^{-3}}} \right)^{1/2} = 1800\ \mathrm{s}$$

Observation: Without pressure support, the sun would collapse very quickly!

The sun does not collapse because it is in **hydrostatic equilibrium**. This means that the sun is in a state of balance by which the internal pressure exactly balances the gravitational pressure.

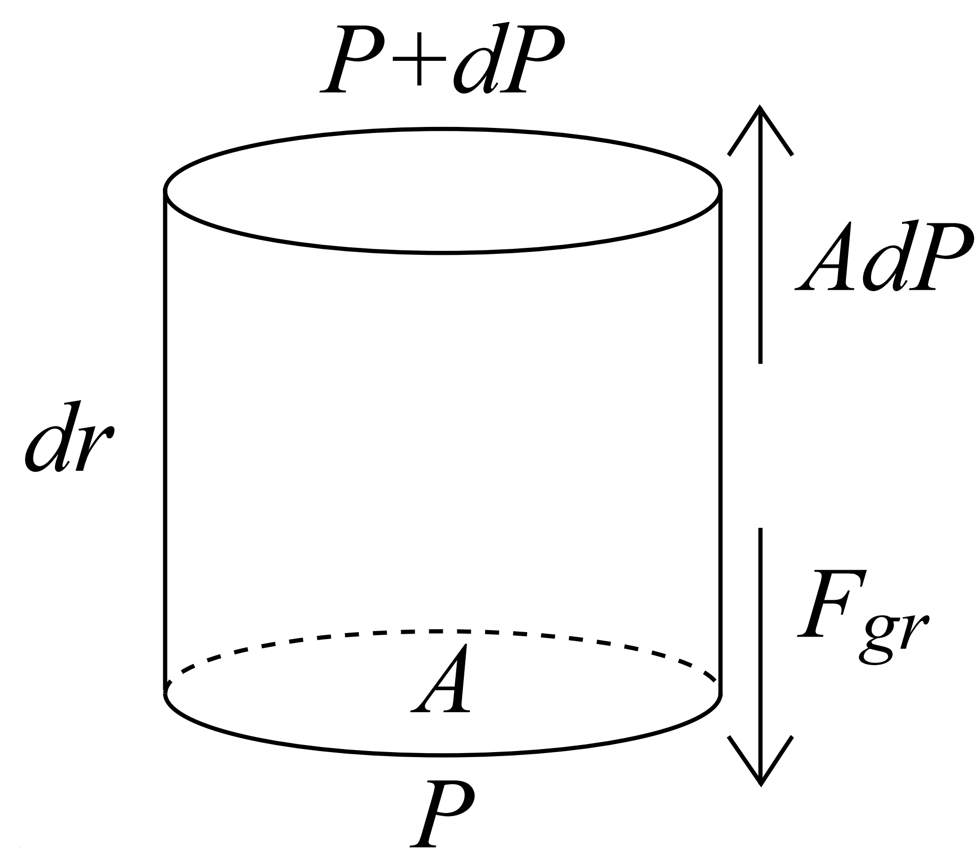
Hydrostatic Equilibrium

Back to Introductory Mechanics!

Consider a small cylinder-shaped star mass element of area A and height dr.

The pressure difference between the top and bottom of the cylinder is dP. It leads to a net force (due to pressure)

$$F_{pressure} = AdP$$



Equilibrium will exist if there is no net force.

$$\Sigma F = 0$$

$$-\frac{GM(r)dm}{r^2} - AdP = 0$$

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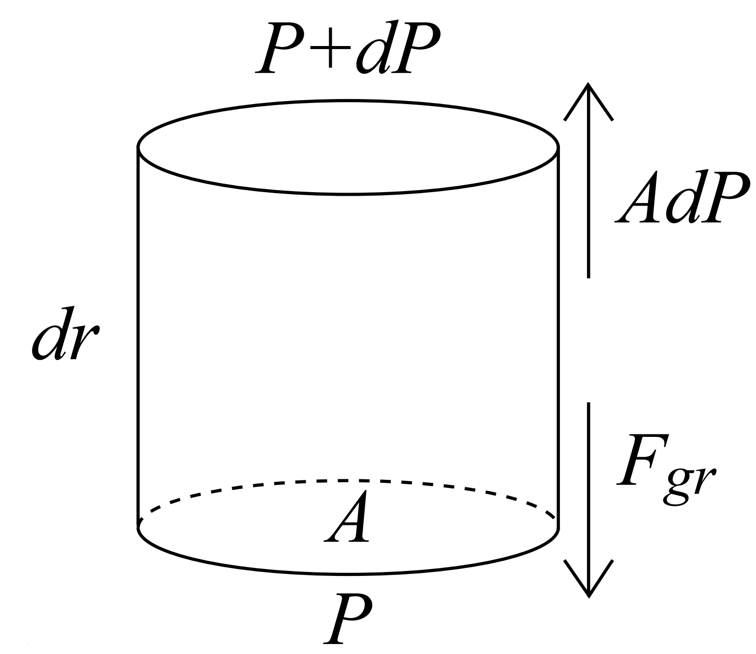
The mass element dm is given by the definition of density

$$dm = \rho(r) A dr$$



Combining with our expression for equilibrium, we find

$$AdP = -\frac{GM(r)\rho(r)}{r^2}$$

Simplifying gives us the **Equation of Hydrostatic Equilibrium**, the first equation of stellar structure that we will study.

$$\boxed{\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}}$$

Stop and Think: This pressure gradient is negative. Does that make sense?

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Now let's add some thermodynamics (for fun):

First, let's introduce a cleaver 1. Multiply both sides of our equation by $4\pi r^3 dr$.

$$4\pi r^3 \frac{dP}{dr} dr = \frac{GM(r)\rho(r)}{r^2} 4\pi r^3 dr$$

Simplify and integrate from the star's interior to it's radius:

$$\underbrace{\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr}_{} = \underbrace{-\int_0^{r_*} \frac{GM(r)\rho(r) 4\pi r^2 dr}{r}}_{}$$

This side we will have to integrate by parts to solve.

This is the gravitational self-potential energy of the star. It is equal to Egr.

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr$$

Recall integration by parts:

$$\int u dv = uv - \int v du$$

Let

$$u = 4\pi r^3 \qquad dv = \frac{dP(r)}{dr} dr$$

$$du = 3(4\pi r^2) dr \qquad v = \int \frac{dP(r)}{dr} dr = P(r)$$

Putting it together:

$$\int_0^{r_*} 4\pi r^3 \frac{dP(r)}{dr} dr = \underbrace{[4\pi r^3 P(r)]_0^{r_*}}_{0} - \int_0^{r_*} P(r)(3)(4)\pi r^2 dr$$

$$= -3 \int_0^{r_*} P(r) 4\pi r^2 dr = -3 \frac{\bar{P}}{V}$$

This is the volume-averaged pressure divided by the volume of the star

Putting it all together:

$$\int_0^{r_*} 4\pi r^3 \frac{dP}{dr} dr = -\int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$

$$-3\frac{\bar{P}}{V} = E_{gr}$$

Which gives the one form of the **viral theorem** for a gravitationally bound system.

$$\boxed{\bar{P} = -\frac{1}{3}\frac{E_{\rm gr}}{V}}$$

This tells us that the pressure inside a star is one third its gravitational energy density.

Now let's add some classical thermodynamics. If a star is composed of a classical, non relativistic, mono-atomic ideal gas of N particles, what is the gas equation of stat of the star?

$$PV = NkT \qquad \ldots(1)$$

What is its thermal energy?

$$E_{\rm th} = \frac{3}{2} NkT \qquad \ldots(2)$$

Substituting NkT from (2) into (1) we find

$$P = \frac{2}{3} \frac{E_{\rm th}}{V}$$

The local pressure is equal to 2/3 the local thermal energy density.

Multiply by $4\pi r^2$ and integrating over the volume of the star, we find

$$\bar{P}V = \frac{2}{3}E_{\rm th}^{\rm tot}$$

Recall the our first form of the viral theorem:

$$\boxed{\bar{P} = -\frac{1}{3}\frac{E_{\rm gr}}{V}}$$

Substituting gives another form of the **<u>viral theorem</u>**:

$$\boxed{E_{\rm th}^{\rm tot} = -\frac{E_{\rm gr}}{2}}$$

The second form of the viral theorem says when a star contacts and losses energy, its self gravity becomes more negative.

Pressure Inside a Star

Let's examine the following equation again.

$$E_{\rm gr} = -\int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r}$$

This is the gravitational self-potential energy of the star. It is equal to Egr.

Let's assume that the density profile is constant, then

$$E_{\rm gr} = -\int_0^{r_*} \frac{GM(r)\rho(r)4\pi r^2 dr}{r} = -\int_0^{r_*} \frac{G\frac{4\pi}{3}r^3\rho^2 4\pi r^2 dr}{r}$$

A little math and we get ….

recall:

$$\rho_{const} = \frac{M}{V} = \frac{M_*}{\frac{4}{3}\pi r_*^3}$$

$$E_{gr} = -\frac{3}{5}\frac{GM_*^2}{r_*}$$

Mean Pressure of the Sun

Take a characteristic $E_{gr} \sim -GM^2/r$ and calculate the mean pressure in the sun.

Using the first form of the viral theorem, we get

$$\bar{P} = \frac{1}{3}\frac{E_{gr}}{V} \sim \frac{1}{3}\frac{GM_\odot^2}{\frac{4}{3}\pi r_\odot^3 r_\odot} = \frac{GM_\odot^2}{4\pi r_\odot^4}$$

$$\bar{P} = \frac{(6.7\times 10^{-8} erg\ cm\ g^{-2})(2.0\times 10^{33} g)^2}{4\pi(7.0\times 10^{10} cm)^4} = 8.9\times 10^{14} erg\ cm^{-3}$$

$$\boxed{P \sim 8.9\times 10^{14} erg\ cm^{-3}}$$

Note: Your textbook uses units of dyne cm^{-2}. 1 dyne = 1 erg cm^{-1}.

Typical Temperature

Viral Temperature is the typical temperature inside a star. To find the viral temperature we start with our second form of the virial theorem.

$$\frac{3}{2}NkT_{\rm vir} = E_{\rm th}^{\rm tot} = -\frac{E_{\rm gr}}{2} \sim \frac{1}{2}\frac{GM_\odot^2}{r_\odot}$$

If the particles have a mean mass $\overline{m}$ we can write:

$$\frac{3}{2}NkT_{\rm vir} \sim \frac{1}{2}\frac{GM_\odot N\bar{m}}{r_\odot}$$

The sun is comprised of mostly ionized hydrogen gas, consisting of an equal number of protons and electrons.

$$\bar{m} = \frac{m_e + m_p}{2} = \frac{m_H}{2}$$

Note:
$m_e << m_p$, thus $m_H \sim 1/2\ m_p$.

Putting this together, we have

$$\frac{3}{2}NkT_{\rm vir} \sim \frac{1}{2}\frac{GM_\odot N\bar{m}}{r_\odot}$$

$$kT_{\rm vir} \sim \frac{GM_\odot m_H}{6r_\odot} = \frac{6.7\times10^{-8}\text{cgs}\times 2\times10^{33}\text{ g}\times 1.7\times10^{-24}\text{ g}}{6\times 7\times 10^{10}\text{ cm}}$$

$$= 5.4\times10^{-10} erg$$

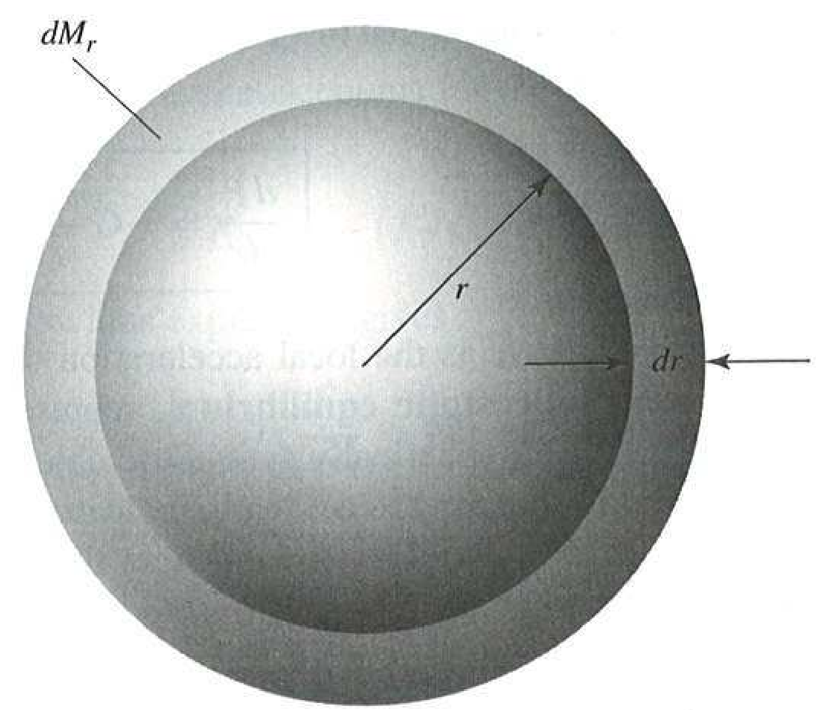
Divide by $k = 1.4\times10^{-16}$ erg K^{-1}

$$\boxed{T_{\rm vir} \sim 4 \times 10^{-6}\ \text{K}}$$

Nuclear reactions take place at temperatures of this order of magnitude. So nuclear reactions can take place and thus replenish the thermal energy that a star radiates away. This, temporarily halts the gravitational collapse.

Mass Continuity

For a spherically symmetric star, consider a shell of mass dM_r and thickness dr located a distance r from the center. If the shell is thin($dr << r$) the volume of the shell can be approximated as $dV = 4\pi r^2 dr$.



If the local density is given by $\rho(r)$, then the shell's mass is given by

$$dM(r) = \rho(r) 4\pi r^2 dr$$

Re-arranging terms yields the **Equation of Mass Continuity**.

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)}$$