From Last Time:

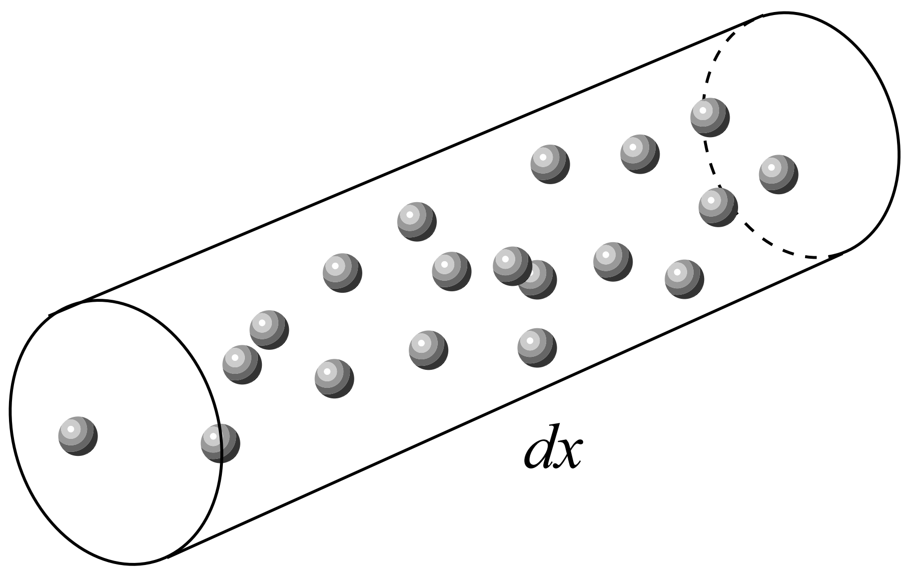
Mean mass $\bar{m}$

$$\bar{m} = \frac{m_e + m_p}{2} = \frac{m_H}{2}$$

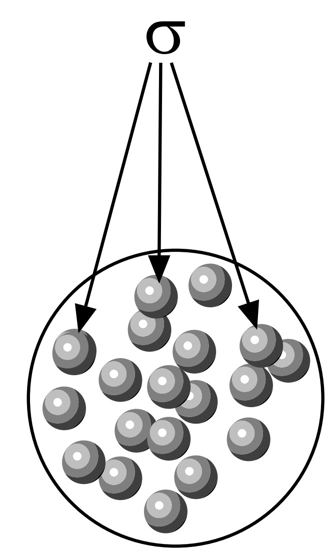
The mean mass is 1/2 the mass of the hydrogen atom. Since, the $m_e \ll m_p$, the mean mass of hydrogen is close to 1/2 the mass of a proton.

Radiative Energy Transport

The radial gradient in pressure that supports a star is produced by a gradient in density and temperature. Pressure, density and temperature are functions of radius.



A photon transverses a path dx, filled with "targets" with a number density.



σ = **effective cross section** for absorption or scattering.

Projected number of targets per unit area lying in the path of the photon is given as ndx.

For a straight line along the length of the path

$$\boxed{\# \text{ of interactions} = n\sigma dx}$$

Mean free path:

The typical distance a particle will travel between interactions.

$$l = \frac{1}{n\sigma}$$

For stellar matter, we need to modify this equation. Stellar matter consists of a variety of absorbers and scatters, each with its own density and cross section.

$$l = \frac{1}{\sum n_i \sigma_i} \equiv \frac{1}{\rho\kappa}$$

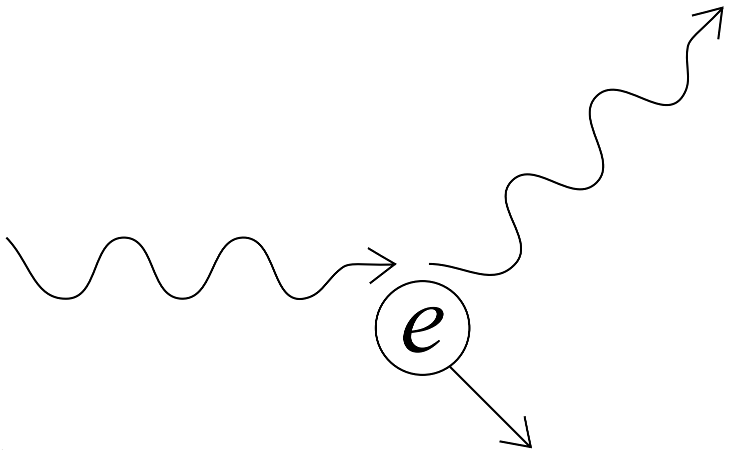
mean free path of stellar matter

Here we have defined a new term, **opacity** = κ. It is found by adding together the cross-sections of all the absorbers and scatterers in the shell, and dividing by the total mass of the shell.

$$\kappa = \frac{\Sigma n_i \sigma_i}{\rho}$$

Thomson Scattering

In Thomson Scattering a photon is scattered off a free electron.



The cross section for this process is given by:

$$\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$$

Note: Thomson cross section is independent of temperature and photon energy.

Calculate the Thomson cross section (using cgs units):

$$\sigma_T = \frac{8\pi}{3}\left(\frac{(4.8\times 10^{-10} esu)^2}{9.1\times 10^{-28} g \times (3\times 10^{10} cm\ s^{-1})^2}\right)^2 = 6.7\times 10^{-25}\mathrm{cm}^2$$

Note: 1 esu = 1cm/sqrt(dyne) = $g^{1/2}$ $cm^{3/2}$ s^{-1}
1 dyne = erg cm^{-1} ; 1 erg = g cm^2 s^{-2}

Let's simplify. Assume all the gas is hydrogen and that there is one electron per atom of mass m_H.

$$n_e \approx \frac{\rho}{m_H}$$

Calculate the mean free path for electron scattering in this star. Assume the mean density to be that of the sun, 1.4 g cm^{-3}.

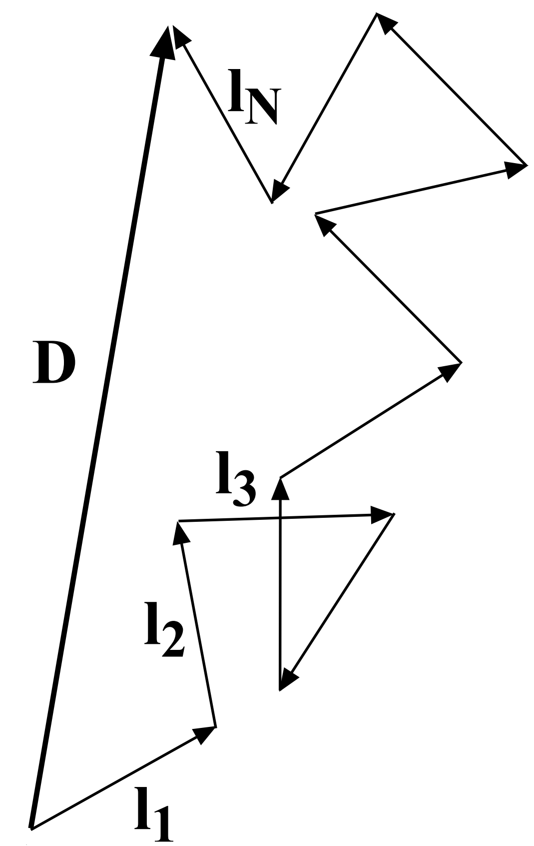
$$l_{\mathrm{es}} = \frac{1}{n_e \sigma_T} \approx \frac{m_H}{\rho\, \sigma_T} \approx \frac{1.7 \times 10^{-24}\ \mathrm{g}}{1.4\ \mathrm{g\ cm^{-3}} \times 6.7 \times 10^{-25}\ \mathrm{cm^2}}$$

$$l_{\mathrm{es}} \approx 2\ \mathrm{cm}$$

In reality, there are regions where electron scattering is more prominent and regions where it is less important. As a result, the typical photon mean free path is 1 mm.

In which areas of the star do you suppose electron scattering is more dominant?

So, we can see that the photons only travel a tiny distance before being scattered or absorbed and remitted in a new direction.



Thus, photons in the sun follow a **random walk** process.

$$\mathbf{D} = \mathbf{l_1} + \mathbf{l_2} + \mathbf{l_3} + ... + \mathbf{l_N}$$

The square of the linear distance covered is

$$\mathbf{D}^2 = |\mathbf{l_1}|^2 + |\mathbf{l_2}|^2 + ... + |\mathbf{l_N}|^2 + 2(\mathbf{l_1}\cdot\mathbf{l_2} + \mathbf{l_1}\cdot\mathbf{l_3} + ...)$$

This term approaches zero for a large number of steps.

Thus, we can write

$$\langle\mathbf{D}^2\rangle = Nl^2 \quad \text{and} \quad \langle\mathbf{D}^2\rangle^{1/2} = D = \sqrt{N}\, l$$

expectation value

linear distance covered in random walk

Photon Travel Time

Calculate how long it takes for a photon to travel from the center of the sun to the surface.

The time it takes for a photon to reach the surface is the time it takes in each step, times the number of steps in the random walk.

$$\tau_{rw} = t_{step} N$$

The time for each step is given by

$$t_{step} = \frac{\ell}{c}$$

And N is given by our equation for linear distance covered by random walk.

$$D = \sqrt{N}\ell \longrightarrow N = \frac{D^2}{\ell^2} = \frac{r_{sun}^2}{\ell^2}$$

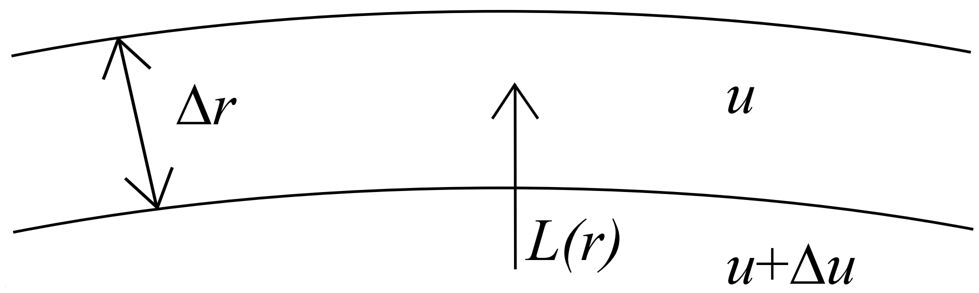
Put it all together and simplify.

$$\tau_{rw} = \frac{\ell}{c}\frac{r_{sun}^2}{\ell^2} = \frac{r_{sun}^2}{c\ell} = \frac{(7 \times 10^{10} cm)^2}{(3 \times 10^{10} cm\ s^{-1})(10^{10} cm)} = 1.6 \times 10^{12} s = 52,000\ years$$

Equation of Radiative Transport

Inside the sun, it is a good approximation that every volume element radiates as a blackbody. However, there is still a net flow of radiation outward. This implies that there is a higher density at smaller radii.

L(r) = net flow of radiation through a mass shell at radius r.



$$L(r) = \frac{excess\ energy}{transit\ time}$$

Δu = excess energy density

$\tau_{rw} = \frac{r^2}{c\ell}$ = transit time for photons to cross shell

Thus, we can write:

$$L(r) \approx -\frac{4\pi r^2 \Delta r \Delta u}{(\Delta r)^2/lc} = -4\pi r^2 lc \frac{\Delta u}{\Delta r}$$

WARNING: this is an approximation

A rigorous derivation yields:

diffusion coefficient

gradient in energy

$$\frac{L(r)}{4\pi r^2} = \underbrace{-\frac{cl}{3}}_{\text{diffusion coefficient}}\underbrace{\frac{du}{dr}}_{\text{gradient in energy}}$$

Notes: This **diffusion equation** describes the outward flow of energy.

The opacity is reflected in the mean free path (l).

For low opacity (large l), flow is unobstructed and luminosity high.

$$\frac{L(r)}{4\pi r^2} = -\frac{cl}{3}\frac{du}{dr}$$

Let's revisit blackbody radiation.

$$u = aT^4$$

At every radius, we approximate a blackbody. Thus, we can write:

$$\frac{du}{dr} = \frac{du}{dT}\frac{dT}{dr} = 4aT^3\frac{dT}{dr}$$

Substituting yields (and letting $l = (\kappa\rho)^{-1}$:

$$\boxed{\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT^3(r)}}$$

This is the **Equation of Radiative Energy Transport**.

The Sun's Luminosity

Using the diffusion equation together with the mean free path estimate we can estimate the luminosity of the sun.

$$\frac{L(r)}{4\pi r^2} = -\frac{cl}{3}\frac{du}{dr}$$

Approximate

$$-\frac{du}{dr} \sim \frac{u}{r_{sun}} = \frac{aT^4}{r_{sun}}$$

Recall:

$$u = aT^4 \longrightarrow \frac{u}{r} = \frac{aT^4}{r}$$

Thus, we can write

$$L_\odot \sim 4\pi r_\odot^2 \frac{cl}{3}\frac{aT^4}{r_\odot} = \frac{4\pi}{3} 7\times 10^{10}\ \mathrm{cm} \times 3\times 10^{10}\ \mathrm{cm\ s^{-1}} \times 10^{-1} \times 7.6\times 10^{-15}\ \mathrm{cgs} \times (4\times 10^6\ \mathrm{K})^4$$

$$L_\odot \sim 2\times 10^{33}\ \mathrm{ergs^{-1}}$$ (compared to 3.8×10^{33} $\mathrm{ergs^{-1}}$)

Energy Conservation

The luminosity of a star is produced by nuclear reactions. Let ε(r) be the power per unit mass of stellar material. The star's luminosity due to energy production in a thin shell at radius *r* is given by

$$dL = \epsilon dm = \epsilon \rho 4\pi r^2 dr$$

rearranging terms

$$\boxed{\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \epsilon(r)}$$

This is the **Equation of Energy Conservation**.

Equations of Stellar Structure

We now have all 4 of the equations for stellar structure!

$$\boxed{\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2},}$$

$$\boxed{\frac{dM(r)}{dr} = 4\pi r^2 \rho(r),}$$

$$\boxed{\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3},}$$

$$\boxed{\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r).}$$

And it hardly hurt at all ....

To solve these equations, we need to define boundary conditions.

$$M(r = 0) = 0$$

$$L(r = 0) = 0$$

$$P(r = r_*) = 0$$

$$M(r = r_*) = M_*$$

Vogt-Russell Conjecture:

Properties and evolution of a star are fully determined by its initial mass and its chemical composition.

From this we can determine a stars observable parameters: surface temperature, radius and luminosity.

Ancillary Equations

The Equations of Stellar Structure do not give us enough information to understand how the gas in the sun behaves. So, we add to the following equations to connect pressure, opacity, energy generations, density and temperature.

$$P = P(\rho, T, \text{composition});$$ "equation of state"

$$\kappa = \kappa(\rho, T, \text{composition});$$ "opacity"

$$\epsilon = \epsilon(\rho, T, \text{composition}).$$ "energy generation"

The mass abundances of He, H and heavier elements are parameterized by

$$X \equiv \frac{\rho_{\rm H}}{\rho}, \quad Y \equiv \frac{\rho_{\rm He}}{\rho}, \quad Z \equiv \frac{\rho_{\rm metals}}{\rho}$$

Equation of State

In most normal stars, the classical, nonrelativistic ideal gas law is a good approximation for the equation of state:

$$P_g = nkT$$

The number density n is related to the mean mass. Consider gas of three kinds of particles.

$$\bar{m} = \frac{n_1 m_1 + n_2 m_2 + n_3 m_3}{n_1 + n_2 + n_3} = \frac{\rho}{n}$$

Which means we can write

This is the kinetic gas pressure

$$P_g = \frac{\rho}{\bar{m}} kT$$

$$X \equiv \frac{\rho_{\rm H}}{\rho}, \quad Y \equiv \frac{\rho_{\rm He}}{\rho}, \quad Z \equiv \frac{\rho_{\rm metals}}{\rho}$$

We can more generally write the number densities of H, He and metals.

$$n_H = \frac{X\rho}{m_H}, \quad n_{\rm He} = \frac{Y\rho}{4m_H}, \quad n_A = \frac{Z_A\rho}{Am_H}$$

How many particles results from the complete ionization of hydrogen? Helium?

Thus, for an ionized gas:

$$n = 2n_H + 3n_{\rm He} + \sum \frac{A}{2} n_A$$

Here we used the fact X + Y + Z = 1

$$n = \frac{\rho}{m_H}(2X + \frac{3}{4}Y + \frac{1}{2}Z) = \frac{\rho}{2m_H}\left(3X + \frac{Y}{2} + 1\right)$$

Recall, $\bar{m} = \rho / n$

$$\frac{\bar{m}}{m_H} = \frac{\rho}{nm_H} = \frac{2}{1 + 3X + 0.5Y}$$

$$\frac{\bar{m}}{m_H} = \frac{2}{1 + 3X + 0.5Y}$$

The mean mass to hydrogen mass ratio for pure hydrogen = 1/2. (X = 1, Y = 0, Z = 0).

For solar abundances, X = 0.71, Y = 0.27 and Z = 0.02. Calculate the mean to hydrogen mass ratio.

$$\bar{m}/m_H = 0.61$$

In the core of the sun, X = 0.34, Y = 0.64 and Z = 0.02. Calculate the mean to hydrogen mass ratio.

$$\bar{m}/m_H = 0.85$$

Notice: X + Y + Z = 1.

Radiation Pressure

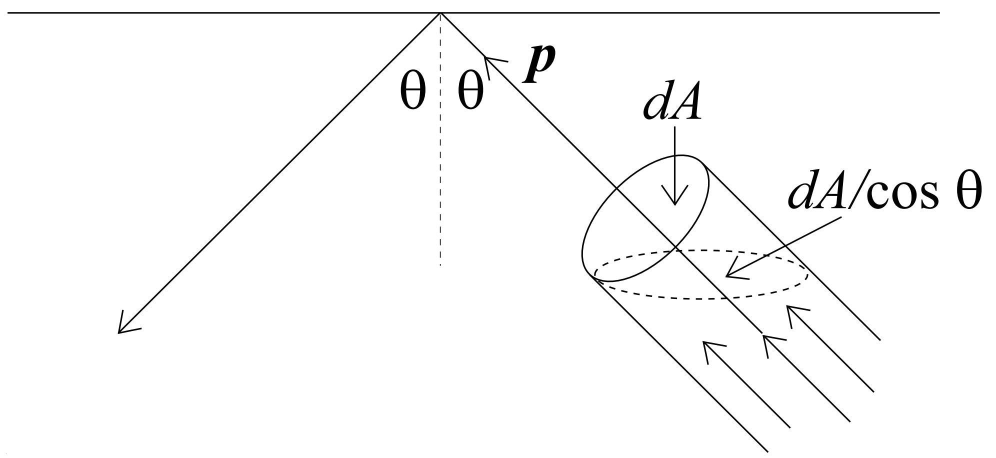
Since the gas in a star locally emits as a blackbody, the blackbody photons will produce radiation pressure.

The intensity of this radiation (B) is integrated over all frequencies and directions.

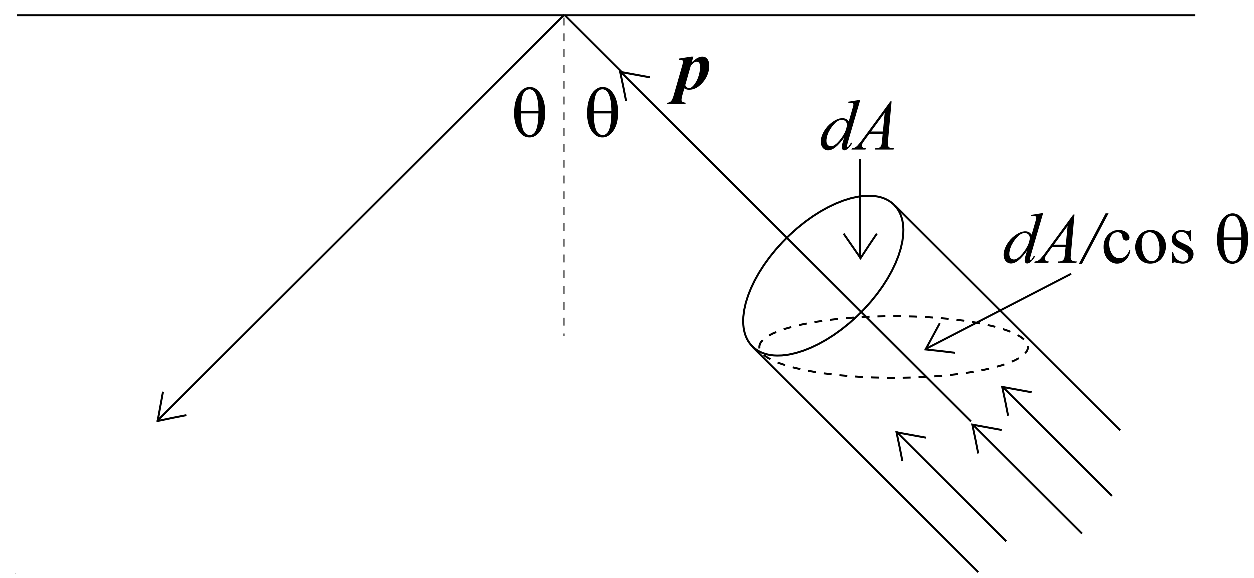
$$B = \int I_\lambda d\lambda$$

Reminder:
Intensity = power / area
Power = energy / time

The projected area of the beam is increased by *1/cosθ*, thus the energy per unit time (power) arriving at the surface is *Bcosθ*.



For a photon, p=E/c, incident at angle θ, the photon surface density is reduced by a factor cosθ. The momentum transfer is then



$$\Delta p = [\frac{E\cos\theta}{c} - (-\frac{E\cos\theta}{c})] = \frac{2E}{c}\cos\theta$$

So we can write

$$P = \frac{F}{A} = \frac{dp/dt}{A} = \int_{\pi=0}^{2\pi}\int_{\frac{\pi}{2}}^{\pi} \frac{2}{c} B\cos^2\theta\sin\theta d\theta d\phi = \frac{4\pi}{c} B \int_{\frac{\pi}{2}}^{\pi} \cos^2\theta\sin\theta d\theta$$

Recall: The energy per unit time (power) arriving at the surface is *Bcosθ*.

Hint: $\int \cos ax^n \sin ax dx = -\frac{\cos^{n+1} ax}{(n+1)a}$

$$\boxed{P = \frac{4\pi}{3c} B = \frac{1}{3} u}$$

Where we have used the relationship:

$$B = \frac{c}{4\pi} u$$

$$P = \frac{4\pi}{3c}B = \frac{1}{3}u$$

We can relate this to the energy density and the temperature using relations for blackbody radiation.

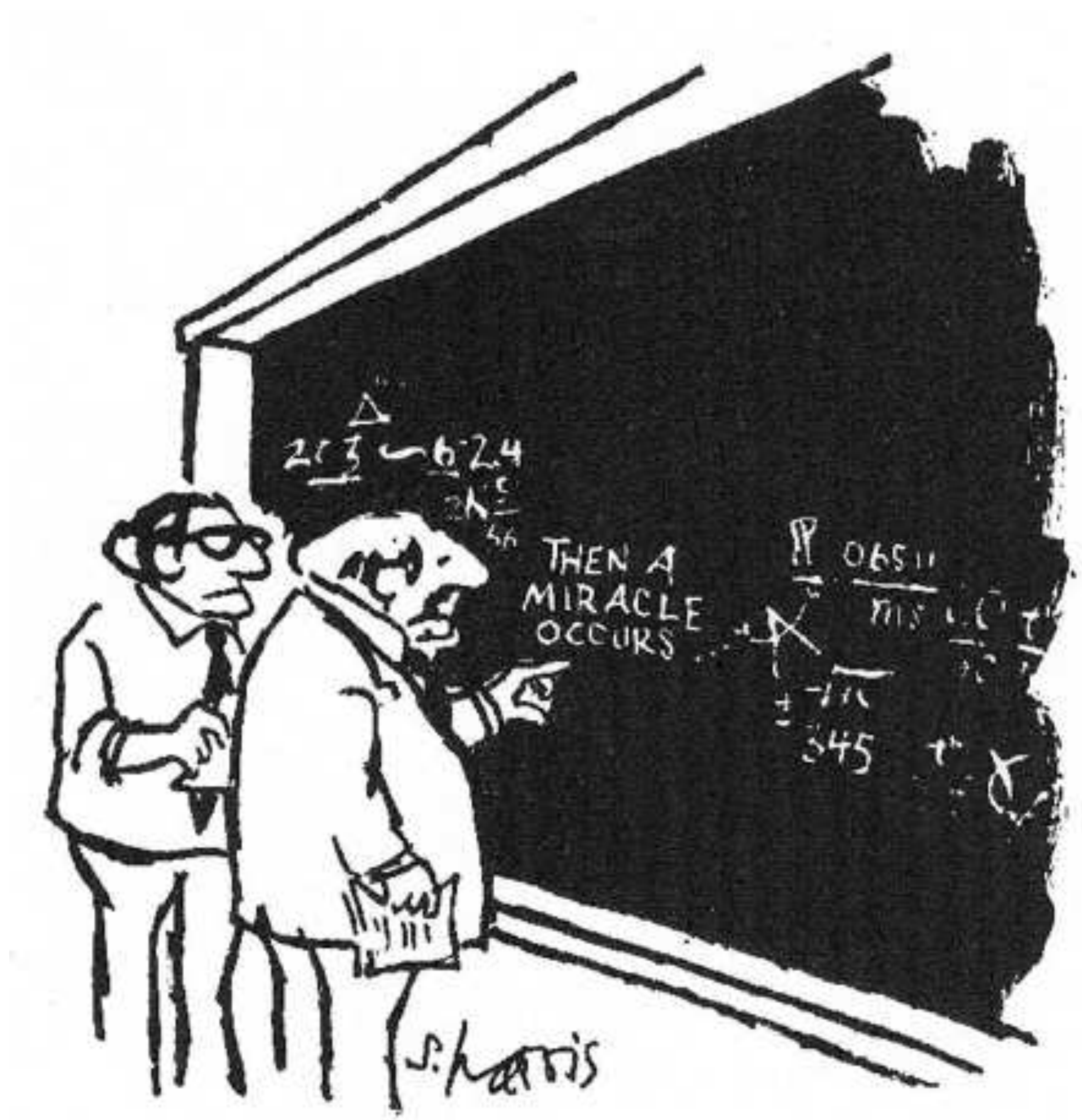
$$P_{\rm rad} = \frac{1}{3}u = \frac{1}{3}aT^4$$

The total pressure is then the sum of the gas pressure and the radiation pressure.

$$\boxed{P = P_g + P_{\rm rad} = \frac{\rho k T}{\bar{m}} + \frac{1}{3}aT^4}$$

In normal starts, the gas ("kinetic") pressure usually dominates. Can you think of when this might not be the case?

Stay Tuned!



"I think you should be more explicit here in step two."