What is a vector? (3-vector)

Anything that <u>transforms</u> like $\vec{r}$ under rotations. (SO(3) rotations).

e.g. $\vec{v}, \vec{a}, \vec{P}, \vec{F}, \vec{E}, \vec{B}$ are vectors $(3\times1)$

Not $v = |\vec{v}|$ speed, $m$, $T$, $P$, Eng. $t$ = time. $(1\times1)$

Also not: $I_{ij}$ $\quad i,j \in \{1,2,3\}$

Not = $\begin{pmatrix} 1 \text{ apple} \\ 2 \text{ bananas} \\ 6 \text{ oranges} \end{pmatrix}$

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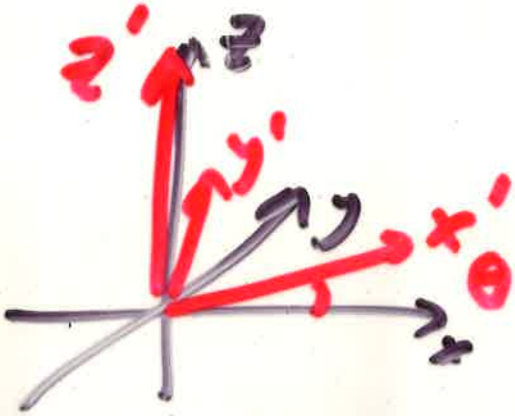
$$\underbrace{\vec{r}\,'}_{\substack{\text{vector}\\(3\times1)}} = \underbrace{\underline{\underline{R}}}_{\substack{\text{matrix}\\(3\times3)}}\ \underbrace{\vec{r}}_{\substack{\text{vector}\\(3\times1)}}$$

$\underline{\underline{R}}$ = rotation matrix

$$r_i' = \sum_{j=1}^{3} R_{ij}\, r_j$$

v1




$$\underline{\underline{R}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{r}\,' = \underline{\underline{R}}\,\vec{r} \Longrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x' = \cos\theta\, x + \sin\theta\, y + 0\, z$$
$$y' = -\sin\theta\, x + \cos\theta\, y + 0\, z$$
$$z' = 0\, x + 0\, y + 1\, z$$

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The rotation matrix $\underline{\underline{R}}$ is <u>orthogonal</u> (<u>not</u> like vectors).

$$\underline{\underline{R}}^T = \underline{\underline{R}}^{-1} \Longleftrightarrow \underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{I}}$$

transpose    inverse    identity matrix

$$\underline{\underline{R}} = \begin{pmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underline{\underline{R}}^T = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{R}}^T \underline{\underline{R}} = \underline{\underline{R}}\,\underline{\underline{R}}^T = \begin{pmatrix} c^2+s^2 & -cs+cs & 0 \\ cs-cs & c^2+s^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Rotations preserve:

1) lengths of vectors
2) angle between vectors

$$\vec{A}' = \underline{\underline{R}}\,\vec{A}, \quad \vec{B}' = \underline{\underline{R}}\,\vec{B}$$

$$\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B}$$

$$= (\underline{\underline{R}}\vec{A}) \cdot (\underline{\underline{R}}\vec{B})$$

$$= \vec{A}(\underline{\underline{R}}^T \underline{\underline{R}})\vec{B} = \vec{A}\,\underline{\underline{I}}\,\vec{B} = \vec{A} \cdot \vec{B} \quad \checkmark$$

$$\vec{A}' \cdot \vec{A}' = \vec{A} \cdot \vec{A} = |\vec{A}|^2$$

$$\vec{B}' \cdot \vec{B}' = \vec{B} \cdot \vec{B} = |\vec{B}|^2$$

$$\vec{A}' \cdot \vec{B}' = |\vec{A}|\,|\vec{B}| \cos \phi$$

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**What is a 4-vector?**

Anything that <u>transforms</u> like

$$\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = x^\mu$$

under Lorentz $= \Lambda$ Transformations

$$\Lambda = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\beta = \frac{v}{c} \quad , \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

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$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$(x^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu{}_\nu \, x^\nu$$

Define $\eta \equiv \operatorname{artanh}(\beta)$

$\uparrow$ rapidity

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{cc|cc} \cosh(\eta) & \sinh(\eta) & 0 & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

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What is preserved under Lorentz transforms?

$$\sum_{\mu=0}^{3} x'^{\mu} y'_{\mu} = \sum_{\mu=0}^{3} x^{\mu} y_{\mu}$$

$$\vec{A}' \cdot \vec{B}' = \vec{A} \cdot \vec{B} = A_{\underset{1}{x}} B_{\underset{1}{x}} + A_{\underset{2}{y}} B_{\underset{2}{y}} + A_{\underset{3}{z}} B_{\underset{3}{z}}$$

$$-x^0 y^0 + x^1 y^1 + x^2 y^2 + x^3 y^3$$

spacetime interval $= I$

"distance" between spacetime events.