General Theory of Relativity (not Theory of General Relativity)

(opposed to the Special Theory of Relativity.)

What's the difference?

Misconception #1

X Special Rel. can not handle accelerations (non-inertial reference frames).

Sure it can! Later: Relativistic Rocket.

The difference: In S.R the 4-dimensional spacetime is "flat". In G.R., the spacetime is "curved."

$$G_{\mu\nu} = 8\pi \, T_{\mu\nu}$$ ← Einstein Field Equation (EFE)

geometrized units $G=1$ $c=1 \frac{\text{light year}}{\text{year}}$

start unpacking $G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

G = Newton's constant = $6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2}$

c = Speed of light in vacuum = $2.9979... \times 10^{8} \frac{m}{s}$

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$ — tensor equation

Einstein tensor

Curvature of 4-dim spacetime

energy-momentum tensor = stress energy tensor

energy (including mass-energy) density, pressures, stresses.

John Archibald Wheeler:

"Matter tells space[time] how to curve." $G_{\mu\nu} = 8\pi T_{\mu\nu}$

Space[time] tells matter how to move."

Geodesic Law – later.

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tensor equations – describe form-invariant ≡ covariant transformations under:

$$\vec{F} = m\vec{a} \xrightarrow[\text{rotation}]{\text{vectors}} \vec{F}' = m\vec{a}'$$

Maxwell → $$\partial_\mu F^{\mu\nu} = 0 \xrightarrow[\text{Lorentz boosts}]{\text{4-vector}} \partial'_\mu F'^{\mu\nu} = 0$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \xrightarrow[\text{general coordinate transformations}]{} G'_{\mu\nu} = 8\pi T'_{\mu\nu}$$

Rotations

$$\begin{pmatrix} \hat{i}' \\ \hat{j}' \\ \hat{k}' \end{pmatrix} = \underbrace{\begin{pmatrix} - & - & - \\ - & - & - \\ - & - & - \end{pmatrix}}_{R} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

Boosts

$$\Lambda = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

GR

$x' = f(x, y, z, t)$

$y' = g(x, y, z, t)$

$z' =$

$t' =$

Field Equation	Equation of Motion
Charges + currents tell the field what to be.	Fields $(\vec{E}, \vec{B})$ tell charges how to move
$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \dfrac{\rho(\vec{r})}{\epsilon_0}$	$\dfrac{\cancel{m}\, d^2\vec{x}}{dt^2} = \dfrac{\vec{F}}{m} = \dfrac{q}{m}\left(\vec{E} + \vec{v}\times\vec{B}\right)$
$\vec{\nabla}\times\vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})$	
Matter tells spacetime how to curve	Spacetime tells matter how to move
$G_{\mu\nu} = \dfrac{8\pi G}{c^4} T_{\mu\nu}$	$\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\dfrac{dx^\alpha}{d\tau}\dfrac{dx^\beta}{d\tau} = 0$
$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}$	
τ = proper time	

$R_{\alpha\beta\gamma\delta}$ Riemann tensor
256 components

test for curvature.

Cartesian $g = \begin{array}{c|ccc} & x & y & z \\ \hline x & & & \\ y & & \left(\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right) & \\ z & & & \end{array}$

$R_{\alpha\beta\gamma\delta} = 0$

$$R_{\mu\nu} = \sum_{\rho=0}^{3} R^{\rho}_{\mu\rho\nu}$$

$\leftarrow$ $4 \times 4 \times 4 \times 4$ object

Ricci tensor

Riemann Curvature tensor

$R_{0000} =$

$R_{0001} =$

$R_{0002} =$

256 components $\rightarrow$ 20 independent components

$$R^{\alpha}_{\beta\gamma\delta} = \frac{\partial \Gamma_{\alpha\beta\gamma}}{\partial x^{\delta}} + \Gamma_{\rho\beta\gamma}\Gamma^{\alpha}_{\rho\delta}$$

$\Gamma_{\alpha\beta\gamma}$ = christoffel symbols

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right)$$

$g_{\mu\nu}$ = metric tensor

tells about distances between points (and other things)

$$T_{\mu\nu} = \begin{array}{cc} \begin{pmatrix} T_{00} & - & - & - \\ - & x & x & x \\ - & x & x & x \\ - & x & x & x \end{pmatrix} & \begin{array}{cc} 0 & t \\ 1 & x \\ 2 & y \\ 3 & z \end{array} \\ \begin{array}{cccc} 0 & 1 & 2 & 3 \\ t & x & y & z \end{array} & \end{array}$$

✓ Reproduces Newton's Law:

Einstein knew $\underset{tt}{G_{00}} = \underline{8\pi}\, \underset{tt}{T_{00}}$

$\Rightarrow$ must be true for all components

$G_{11} = 8\pi T_{11}$ / $G_{13} = 8\pi T_{13}$

$$\boxed{G_{\mu\nu} = 8\pi T_{\mu\nu}}$$

symmetric tensors

16 comps. $\rightarrow$ 10 independent comp.

10 2nd-order coupled non-linear inhomogeneous Partial differential Equations

Einstein

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

Ricci scalar $R = \sum_{\nu=0}^{3} R^{\nu}{}_{\nu} = R^{\nu}{}_{\nu}$

metric tensor

Ricci tensor

$$\sum_{\rho=0}^{3} g^{\mu\rho} R_{\rho\nu} = R^{\mu}{}_{\nu}$$

Contraction = trace