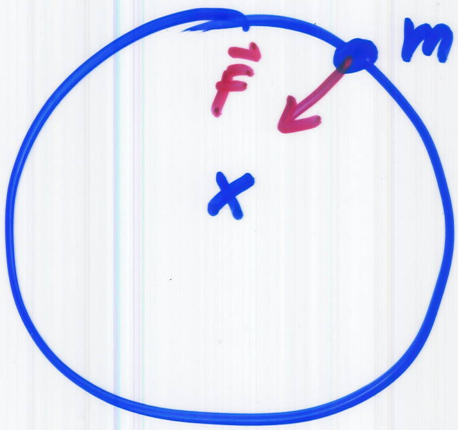
Virial theorem

Example 1 $F \propto \frac{1}{r^2}$

Newton's Gravity or Coulomb

special case: circular motion



$$U = -\frac{GMm}{r}$$

$$\vec{F} = -\vec{\nabla} U = \frac{-GMm}{r^2}\hat{r}$$

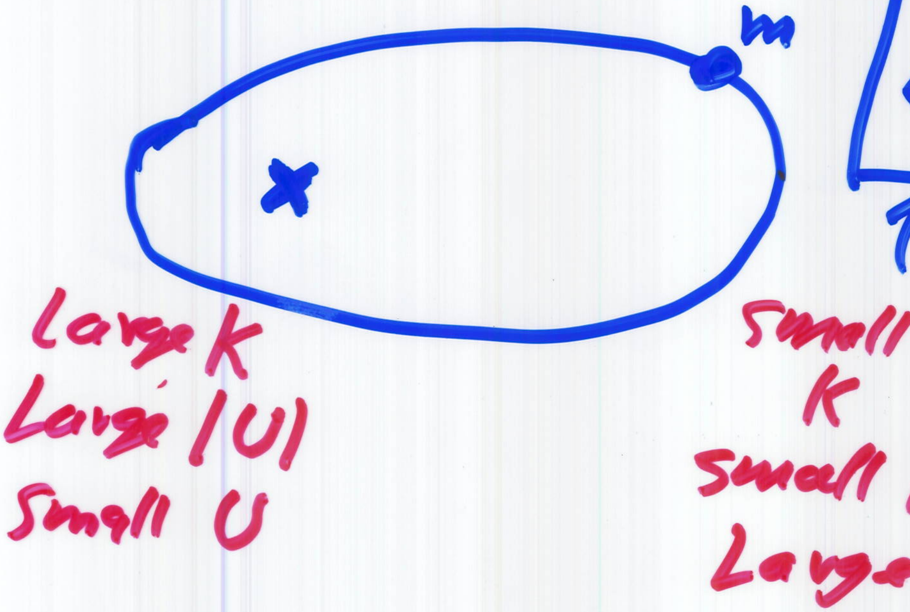
$$K = \frac{1}{2}mv^2, \quad |F_{centripetal}| = \frac{mv^2}{r}$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow K = \frac{GMm}{2r}$$

$$= -\frac{1}{2}U$$

$$\boxed{K = -\frac{1}{2}U}$$

Virial Theorem holds for elliptic orbits too



$$\boxed{\langle K \rangle = -\frac{1}{2}\langle U \rangle}$$

↑ time averages.

Conservation of Energy

$$E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U$$

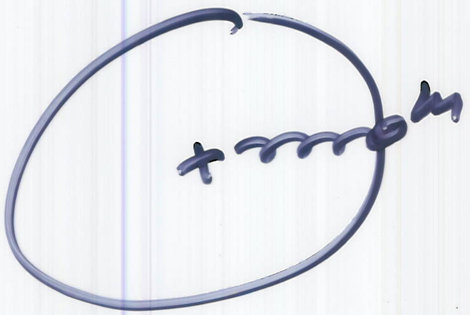
$$E < 0 \Longrightarrow \text{bound}$$

Example #2

Hooke's Law $F \propto r$

$$U = \frac{1}{2}kr^2 \qquad \vec{F} = -\vec{\nabla}U = -kr\hat{r}$$

special case: circular motion



$$K = \frac{1}{2} m v^2$$

$$|F| = kr = \frac{mv^2}{r}$$

$$\Rightarrow K = \frac{1}{2} k r^2 = U \Rightarrow \boxed{K = U}$$

In general: $\langle K \rangle = \langle U \rangle$

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another special case: linear motion

$$F = ma$$

$$-kx(t) = m\ddot{x}(t)$$

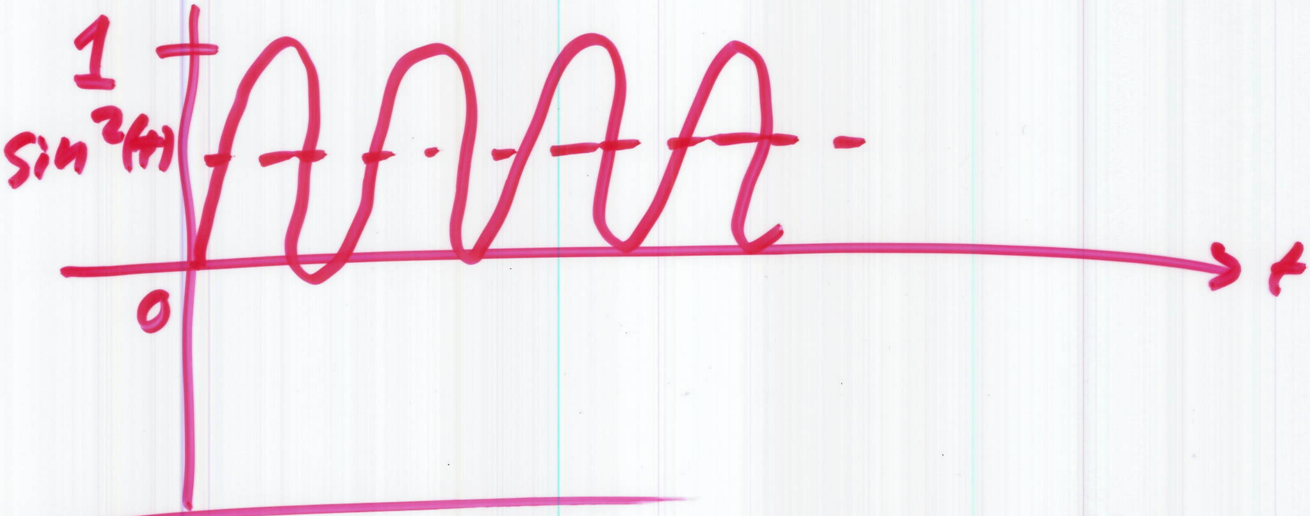
$$\Rightarrow \ddot{x}(t) + \frac{k}{m} x(t) = 0$$

2nd order

$$x(t) = A \sin(\omega t + \varphi), \quad \omega = \sqrt{\frac{k}{m}}$$

$$v(t) = \frac{dx}{dt} = \omega A \cos(\omega t + \varphi)$$

$$\langle U\rangle = \frac{1}{2}k\langle x(t)\rangle^2 = \frac{k}{2}A^2\cdot\frac{1}{2} = \frac{kA^2}{4}$$



$$\langle K\rangle = \frac{1}{2}m\langle v(t)^2\rangle = \frac{m}{2}\omega^2A^2\cdot\frac{1}{2}$$

$$= \frac{kA^2}{4} = \langle U\rangle \checkmark$$

In general, if $F \propto r^n$

then $\langle K\rangle = \frac{n+1}{2}\langle U\rangle$

$n=-2 \rightarrow$ Gravity

$n=+1 \rightarrow$ Hooke's Law