Random Walk (Drunkard's Walk)

1-dimension

N total steps. Probability p of taking one step to the right. Prob $q = (1-p)$ of stepping to the left. Start at origin.

Prob. that we end up N steps to the right?

$$P(N) = ? = p^N = \underbrace{R \text{ and } R \text{ and } R \ldots R}_{N \text{ times}}.$$

Prob that we end up N steps to the left?
= Prob " " " 0 steps to the right.

$$P(0) = q^N = (1-p)^N$$

Prob that we take exactly one step to the right.

$$P(1) = N p^1 q^{N-1}$$

$$\binom{N}{1} \rightarrow = N p (1-p)^{N-1}$$

$$\left.\begin{matrix} R\,L\,L\,L \cdots L \\ L\,R\,L\,L \cdots L \\ L\,L\,R\,L \cdots L \\ \vdots \\ L\,L\,L \cdots L\,R \end{matrix}\right\} \underline{N} \text{ ways} = N \text{ microstates}$$

Prob. that we take n steps to the right.

$$P(n) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}$$

Binomial Distribution

Remember $(p+q)^N = 1p^N + Np^{N-1}q + \frac{N(N-1)}{2!}p^{N-2}q^2 + \dots$

$\dots + 1q^N$

$\binom{N}{N} \quad \binom{N}{N-1} \quad \binom{N}{N-2} \quad \binom{N}{1}$

Normalized?

$$\sum_{n=0}^{N} P(n) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n \underbrace{(1-p)}_{q}^{N-n}$$

$$= (p+q)^N = (1)^N = 1$$

Mean number of steps to the right

$$\langle n_R \rangle = \langle n \rangle = \sum_{n=0}^{N} P(n)\, n$$

$$= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n p^n q^{N-n}$$

p, q independent

$$n p^n = p\left(\frac{\partial}{\partial p} p^n\right)_q$$

$$\langle n \rangle = p \frac{\partial}{\partial p} \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n} = p\frac{\partial}{\partial p}\left[(p+q)^N\right]_q$$

$\langle n \rangle = p N (p+q)^{N-1}$ use $p+q=1$

$$\boxed{\langle n \rangle = pN = \langle n_R \rangle}$$

$\langle n_L \rangle = qN = (1-p)N$

$\langle n_R + n_L \rangle = \langle n_R \rangle + \langle n_L \rangle = N$

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displacement $d = n_R - n_L = 2n_R - N = 2n - N$

$\langle d \rangle = \langle n_R \rangle - \langle n_L \rangle = pN - (1-p)N = (2p-1)N$

$$\langle n^2 \rangle = \sum_{n=0}^{N} n^2 P(n) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n^2 p^n q^{N-n}$$

$$n^2 p^n = \left(p\frac{\partial}{\partial p}\right)^2 p^n \Big|_q$$

$$\langle n^2 \rangle = \left(p\frac{\partial}{\partial p}\right)^2 \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n} = \left(p\frac{\partial}{\partial p}\right)^2 (p+q)^N$$

$$= p\frac{\partial}{\partial p}\left[pN(p+q)^{N-1}\right]$$

$$= p\left[N(p+q)^{(N-1)} + pN(N-1)(p+q)^{N-2}\right]$$

$$= p\left[N + pN(N-1)\right] = pN + \underline{p^2N^2} - p^2N$$

$$\langle n^2\rangle = \underline{\langle n\rangle}^2 + pN(1-p) = \langle n\rangle^2 + Npq$$

Standard Deviation of steps to the right

$$\boxed{\sigma_n = \sqrt{\langle n^2\rangle - \langle n\rangle^2} = \sqrt{Npq}}$$

Displacement $d = n_R - n_L$

$$\langle d\rangle = 2\langle n\rangle - N = (2p-1)N$$

$$\sigma_d = \sqrt{\langle d^2\rangle - \langle d\rangle^2} = 2\sqrt{Npq}$$

If $p = \frac{1}{2} = q \Longrightarrow \boxed{\sigma_d = \sqrt{N}}$ root mean square

Large N, Binomial distribution $\rightarrow$ Gaussian distribution

discrete continuous

Mean $\mu = Np$

$$\sigma_n = \sigma = \sqrt{Npq} = \sqrt{Np(1-p)}$$

$$P(n) \rightarrow \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(n-\mu)^2}{2\sigma^2}\right]$$

Central Limit Theorem.