Black body Spectrum

photons – massless, spin-1 particles

quantum number : s (angular momentum)

z-projection $m_s = \pm 1 \hbar$ ($0\hbar$ missing)

length of spin vector $|\vec{s}| = \sqrt{s(s+1)}\,\hbar = \sqrt{2}\,\hbar$

$\omega = 2\pi\nu$

QM Stat. Mech.

Energy states $E_j = (j+\frac{1}{2})\hbar\omega = (j+\frac{1}{2})h\nu$

Ground state energy $E_0 = \frac{1}{2}\hbar\omega$

j = # of photons, mode number, occupation #

Partition Function

$$Z = \sum_{j=0}^{\infty} e^{-\beta E_j} = \sum_{j=0}^{\infty} e^{-\beta (j+\frac{1}{2})\hbar\omega}$$

$$= e^{-\beta\frac{\hbar\omega}{2}} + e^{-\beta\frac{\hbar\omega}{2} - \beta\hbar\omega} + e^{-\beta\frac{\hbar\omega}{2} - 2\beta\hbar\omega} + \dots$$

$$= e^{-\beta\frac{\hbar\omega}{2}}\Big(\underbrace{1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \dots}_{1 + r + r^2 + \dots}\Big)$$

$$\sum_j r^j = \frac{1}{1-r}$$

$$Z = e^{-\frac{\beta\hbar\omega}{2}} \sum_{j=0}^{\infty} e^{-\beta\hbar\omega j} = e^{-\frac{\beta\hbar\omega}{2}} \left(\frac{1}{1-e^{-\beta\hbar\omega}} \right)$$

$$\ln(Z) = \ln\left(e^{-\frac{\beta\hbar\omega}{2}}\right) - \ln\left(1-e^{-\beta\hbar\omega}\right)$$

$$= -\frac{1}{2}\beta\hbar\omega - \ln\left(1-e^{-\beta\hbar\omega}\right)$$

$$U_1 = \langle E_1 \rangle = -\frac{\partial \ln(Z)}{\partial \beta} = \frac{\hbar\omega}{2} + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

multiply the last term by

$$\left(\frac{e^{+\beta\hbar\omega}}{e^{+\beta\hbar\omega}} \right) = 1$$

$$U_1 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{+\beta\hbar\omega}-1}$$ single mode

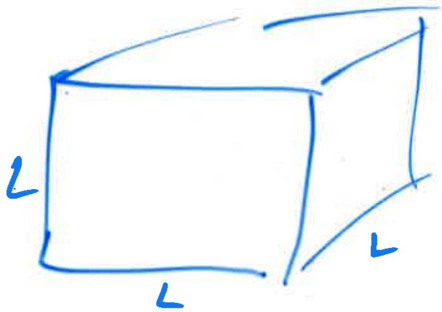
Wavenumber $k = \frac{2\pi}{\lambda}$

Dispersion Relation

$\omega(|\vec{k}|) = c|\vec{k}|$

even though no dispersion here.

Now put the photon gas in a box



$$k_x = \frac{2\pi}{\lambda} = \frac{n_x \pi}{L}$$

$$k_y = \frac{n_y \pi}{L}$$

$$k_z = \frac{n_z \pi}{L} \rightarrow n_z = \frac{k_z L}{\pi}$$

Sum over all modes in cavity:

$$U(T) = \sum_{k_x} \sum_{k_y} \sum_{k_z} \sum_{\epsilon=1}^{2} \left[\hbar\omega(\vec{k}) \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega(\vec{k})} - 1} \right) \right]$$

polarizations

$$\boxed{\frac{1}{e^{\beta\hbar\omega} - 1}} \quad \text{Planck Distribution}$$

polarizations

$$U(T) = \underbrace{V}_{L^3} \left[\underbrace{u_0}_{\infty} + 2 \iiint \left(\frac{\hbar c |\vec{k}| \cancel{\beta}}{e^{\beta \hbar c |\vec{k}|} - 1} \right)^{k_B T} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{dk_z}{2\pi} \right]$$

Cartesian $k_x, k_y, k_z \rightarrow$ Spherical Polar k_r, θ_k, φ_k

Introduce dimensionless $x = \beta\hbar c|\vec{k}| = \beta\hbar c k_r$, $dx = \beta\hbar c\, dk_r$

$$U = V\Big[u_0 + 2\underbrace{\int_{\theta_k=0}^{\pi} \sin\theta_k\, d\theta_k \int_{\varphi_k=0}^{2\pi} d\varphi_k}_{4\pi} \int_{k_r=0}^{\infty} \frac{k_r^2}{(8\pi^3)} \frac{\beta\hbar c k_r (k_B T)}{e^{\beta\hbar c k_r}-1}\, dk_r$$

$$U_{(T)} = V\Big[u_0 + \frac{k_B T}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \underbrace{\int_{x=0}^{\infty} \frac{x^3}{e^x-1}\, dx}_{\frac{\pi^4}{15}}$$

$$\frac{U_{(T)}}{V} = u_0 + \underbrace{\frac{8\pi^5 k_B^4}{15 h^3 c^3}}_{\frac{4}{c}\sigma \leftarrow \text{Stefan-Boltzmann}} T^4$$