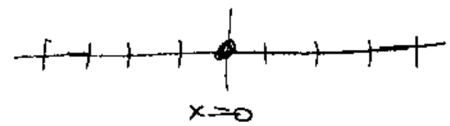
- a chance to put to use basic prob. in an interesting physics problem
- build some tools useful for rest of course

RANDOM WALK

- MODEL FOR PARTICLE(S) MOVING RANDOMLY THRU FLUID (DIFFUSION) (REIF gives other applications; tools are general)

1 DIM:

PARTICLE STARTS AT $X=0$



ASSUMPTIONS:

① GOES DISTANCE l BEFORE HITTING NEXT PARTICLE ($\approx$ MEAN FREE PATH); REPEATS

② AFTER COLLISION, DIR. OF NEXT STEP INDEP. OF PREVIOUS STEP

PROB NEXT STEP TO RT: p $(0 < p < 1)$ } particle is at the mercy of the fluid particles

" " " TO LT: $q = 1 - p$

(IF ISOTROPIC: $p = q = \frac{1}{2}$; IF $p > q$ ~ FLUID FLOWS TO RT)

(Treating w/ probability since don't want to get into details of each collision)

PROBLEM:

AFTER N STEPS, FIND PROB. $P(m)$ FOUND AT $X = ml$ (m AN INTEGER)

{CLASSIC FORMULATION: DRUNK AT LAMP POST ON SIDEWALK }

PROBABILITY: IN LARGE ENSEMBLE OF IDENT SYSTEMS, AFTER N STEPS

$P(m) = (\#\text{ AT } ml)/N_{ENS}$.

REALIZATIONS:

(a) 1 PART. EACH IN N_{ENS} FLUIDS

(b) " " N_{ENS} TIMES IN 1 "

* (c) MANY PARTICLES IN 1 " ⇒ { ex DIFFUSION OF DYE, $P \propto$ DENSITY, CAN SEE DIRECTLY

{REIF: PROB. DEPENDS ON ENSEMBLE

ex prob seed yields red flower – diff if seed $\in$ [tulips] vs seed $\in$ [all plants, incl trees]

⇒ "prob given the info or constraints on ensemble }

skip

EXPECT:

IF $p = q = \frac{1}{2}$:

- USUALLY: END CLOSE TO $x = 0$
- $\bar{x} = 0$ BY SYMM · BUT AVE DIST $\neq 0$
- RARE: $m\ell = \pm N\ell$ (ESP FOR N LARGE)
    - CAN ONLY BE DONE 1 WAY (ALL R OR L STEPS)
    - LOTS OF WAYS TO GET TO $x \sim 0$

IF $p > q$:

- MORE R THAN L STEPS ON AVE $\Rightarrow$ MIGRATES R
- GUESS: $\bar{x} = N(p-q)\ell$ (WILL CONFIRM)

START SIMPLE:

$N = 3$

m	ROUTE	PROB
$+3$	$\bullet \xrightarrow{p} \xrightarrow{p} \xrightarrow{p}$ $x=0$	p^3 $\Rightarrow$ $\boxed{P(3) = p^3}$ R and R and R in 3 steps $\Rightarrow$ mult. probs
$+2$	CAN'T DO IT	0
$+1$	$\bullet \xrightarrow{p} \xrightarrow{p}$, $\xleftarrow{q}$	$p^2 q$
	OR $\xrightarrow{p}$, $\xleftarrow{q}$, $\xrightarrow{p}$	$p q p$
	OR $\xleftarrow{q}\bullet$, $\xrightarrow{p} \xrightarrow{p}$	$q p^2$

path 1 or 2 or 3

$\Rightarrow$ add

TOTAL: $\boxed{P(1) = 3p^2 q}$

~~K.3~~

1,3

$\underline{m}$	ROUTE	PROB
0	CAN'T	0

-1 ETC $\rightarrow$ JUST REPLACE $p \leftrightarrow q$ $\Rightarrow \boxed{P(-1) = 3pq^2}$

-3 $\qquad q^3 \qquad \boxed{P(-3) = q^3}$

$\left\{ \text{ex } p = q = \tfrac{1}{2}: \quad \begin{matrix} P(3) = P(-3) = \tfrac{1}{8} \\ P(1) = P(-1) = \tfrac{3}{8} \end{matrix} \right\}$

HAVE ALL CASES.

PROB THAT PARTICLE GOES ANYWHERE? 1

CHECK: $\sum_{m=-3}^{3} P(m) = 1$

$$\Rightarrow \quad p^3 + 3p^2q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1 \checkmark$$

(any p, q)

IN GENERAL:

$n_R, n_L \equiv$ # RT, LT STEPS (reif uses n_1, n_2)

$$\boxed{\begin{matrix} N = n_R + n_L \\ m = n_R - n_L = 2n_R - N \end{matrix}} \quad \text{OR} \quad \boxed{\begin{matrix} n_R = \tfrac{1}{2}(N+m) \\ n_L = \tfrac{1}{2}(N-m) \end{matrix}}$$

} same on board

$n_{R,L} = 0, 1, 2 \cdots N$

N, m BOTH EVEN OR BOTH ODD

TYPICAL ROUTE TO $m\ell$:

STEP	1	2	3					N
	R	R	L	R	L	L	--	R

$\left.\begin{matrix} n_R & R\text{'s} \\ n_L & L\text{'s} \end{matrix}\right\}$

PROB FOR THIS SPECIFIC ROUTE $\quad p^{n_R} q^{n_L} = p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)}$

(note: if N, m fixed, so are n_R, n_L)

$(\rightarrow)$

HOW MANY WITH THIS n_R & n_L ?

NEED # WAYS TO ARRANGE n_R R'S AND n_L L'S IN N STEPS / SPACES:

STEP: $\underline{\ 1\ }$ $\underline{\ 2\ }$ $\underline{\ 3\ }$ ... $\underline{\ N\ }$

(1) ASSUME HAVE N DISTINCT OBJECTS

1ST : N CHOICES

2ND : N − 1 " ETC } for each one of the N choices for 1ST

⇒ N! WAYS TO ARRANGE

(2) WE OVER COUNTED!
NOW RECOGNIZE THAT R'S SAME, L'S SAME

R R L R L L L L R R

SAME IF REARRANGE R'S AMONG SELVES; ALSO L'S

$n_R!$ WAYS TO ARRANGE R'S (counting same way)

$n_L!$ WAYS TO " L'S

⇒ NUMBER OF DISTINCT ARRANGEMENTS WITH n_R R'S & n_L L'S:

$$\frac{N!}{n_L!\, n_R!} = \frac{N!}{(N-n_R)!\, n_R!} \equiv \binom{N}{n_R}$$

"BINOMIAL COEFFICIENT"

$$= \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\left[\frac{1}{2}(N-m)\right]!}$$

cf

BINOMIAL EXPANSION:

$$(x+y)^N = \binom{N}{0}x^N + \binom{N}{1}x^{N-1}y^1 + \binom{N}{2}x^{N-2}y^2 + \cdots + \binom{N}{N-1}xy^{N-1} + \underbrace{\binom{N}{N}}_{1}y^N$$

$$= \sum_{n=0}^{N} \binom{N}{n} x^{N-n} y^n \qquad (\text{NOTE } 0! \equiv 1)$$

WHY?

ex $(x+y)^3 = (x+y)(x+y)(x+y)$

$$= x^3 + (\underbrace{x^2y + xyx + yx^2}_{3x^2y}) + \cdots$$

⇒ ALL POSSIBLE TERMS w/ 2x's & 1y

⇒ FOR EACH OF 3 FACTORS, PICK EITHER AN x OR y SUCH THAT HAVE 2 x's & 1y

SAME COUNTING AS PICK. L & R's FROM 3 STEPS

FINALLY

$$\boxed{P(m) = \frac{N!}{[\frac{1}{2}(N+m)]!\,[\frac{1}{2}(N-m)]!}\, p^{\frac{1}{2}(N+m)}\, q^{\frac{1}{2}(N+m)}}$$

"BINOMIAL DISTRIBUTION"

$q \equiv 1-p$

m, N BOTH ODD OR BOTH EVEN (ELSE $P=0$)

EQUIVALENTLY IN TERMS OF n_R:

$$\boxed{W(n_R) \equiv P(m) = \frac{N!}{n_R!\,(N-n_R)!}\, p^{n_R}\, q^{(N-n_R)}}$$

SAME DISTR. w/ CHG OF VARS

CHECK:

(1) SYMM IF $p \leftrightarrow q$ AND $m \to -m$

(2) q SMALL: $P(m)$ SMALL UNLESS $N \approx +m$ (IN POWER)

(3) BIN. COEFF LARGE WHEN $m \sim 0$ $\quad n_R \sim n_L \sim N/2$

⇒ PICTURES

COMMENTS ON PLOTS: (say)

$p = q = \frac{1}{2}$

- GIVES BOTH $P(m)$ & $W(n_R)$ } since m & n_R tied together, corresp. probs same
- PEAKED AT $m = 0$
- UNLIKELY FOR $m = \pm 20 = \pm N$
- SHAPE DUE ENTIRELY TO BIN. COEFF (ie # ROUTES)
  → PROB. OF EACH ROUTE SAME

$p = 0.6 \quad q = 0.4$ (note - only good for m even)

- DRIFTS RT.
- SEE INTERPLAY BETW. PREF. TO MOVE RT VS MANY MORE WAYS TO END AT ORIGIN
  ex STILL UNLIKELY TO END AT $m = +20$

IN BOTH - WILL SEE THAT (AS EXPECT): } save
MEAN NEAR PEAK
$\Delta^* x \sim$ WIDTH

1.5.1

$N = 20$
$p = q = \frac{1}{2}$

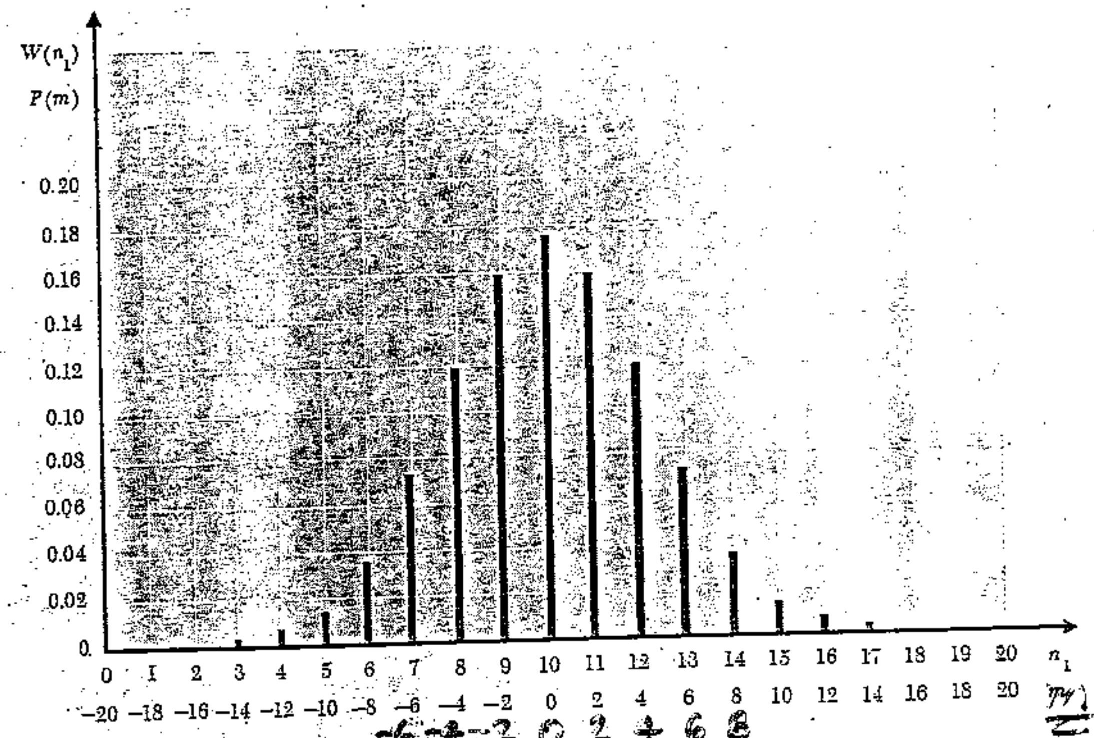
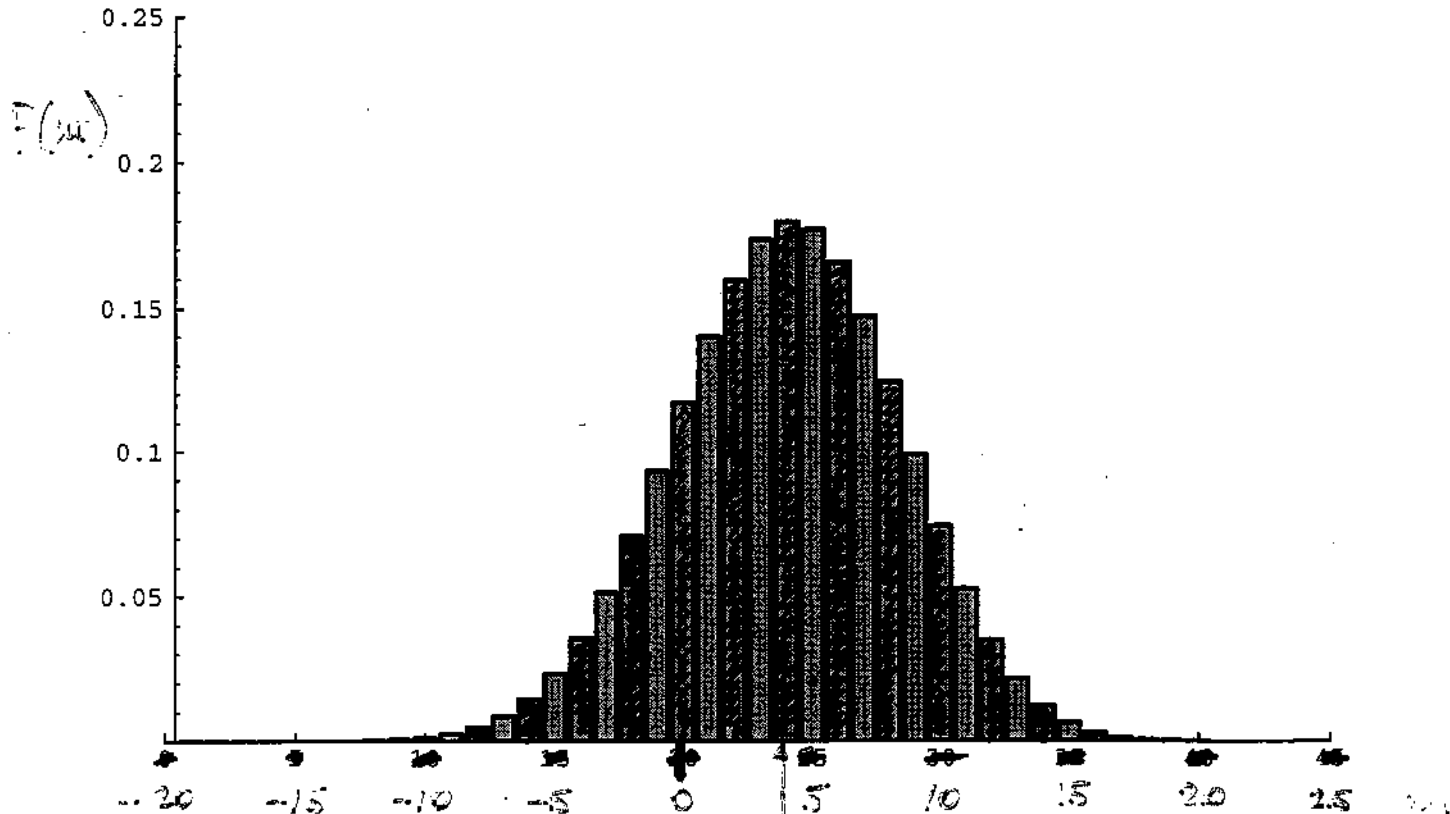


Fig. 1·2·3 Binomial probability distribution for $p = q = \frac{1}{2}$ when $N = 20$ steps. The graph shows the probability $W_N(n_1)$ of n_1 right steps, or equivalently the probability $P_N(m)$ of a net displacement of m units to the right.

$N = 20$

$p = .6 \quad q = .4$

(ONLY GOOD FOR m EVEN)



$\bar{m} = N(p-q) = 4$

$\sqrt{\overline{(\Delta m)^2}} = \sqrt{4pqN}$

$$\sum_{m=-N}^{N} P(m) \stackrel{?}{=} 1$$

EASIEST IF USE FORM

$$W(n_R) = \frac{N!}{(N-n_R)!\, n_R!}\, p^{n_R} q^{N-n_R}$$

don't re-write if don't have to

skip (WITH $q = 1-p$ $\quad n_R + n_L = N$
$n_R - n_L = m$)

$$\sum_{n_R=0}^{N} W(n_R) = \sum_{n_R=0}^{N} \binom{N}{n_R} p^{n_R} q^{N-n_R}$$

$$= (p+q)^N = 1 \quad \checkmark$$

skip {WILL SEE SEVERAL TRICKS FOR SUMMING SERIES.
cf GRADSHTEYN & RYZHIK; SYMBOLIC PROGRAMS ~ MATHEMATICA
MAPLE}

skip *ex* USEFUL FOR PROB 1.5.C (cf A.1)

$$f(x) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \cdots x^N$$

$$x f(x) = x + x^2 + \cdots + x^N + x^{N+1}$$

SUBTR

$$(1-x) f(x) = 1 - x^{N+1}$$

$$f(x) = \frac{1-x^{N+1}}{1-x}$$

(converges if $|x|<1$)

$N = \infty \Rightarrow 1 + x + x^2 + \cdots = \frac{1}{1-x}$

also $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

SPECIAL CASE:

$$\sum_{n=0}^{N} e^{ny} = \sum_{n=0}^{N} \underbrace{(e^y)}_{\equiv x}{}^n \Rightarrow \text{SAME AS ABOVE}$$

skip

$$= \frac{1-e^{(N+1)y}}{1-e^y}$$

COMPUTE SOME AVES w/ RAN WALK:

(EASIEST TO WORK w/ $W(n_R)$ → DON'T HAVE TO WORRY ABOUT EVEN/ODD)

(1) HAVE $\displaystyle\sum_{n_R=0}^{N} W(n_R) = 1$

(2) $$\overline{n}_R = \sum_{n_R} W(n_R)\, n_R = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} \cdot n_R$$

TRICK:

(a) TREAT p, q AS INDEP. ; ie $\overline{n}_R(p,q)$ } ie $q \neq 1-p$ yet

(b) NOTE $p \frac{\partial}{\partial p} p^{n_R} = n_R\, p^{n_R}$

(c) $$\overline{n}_R(p,q) = \sum_{n_R} \binom{N}{n_R} \left(p \frac{\partial}{\partial p} p^{n_R}\right) q^{N-n_R}$$

$$= p \frac{\partial}{\partial p} \Big\{ \underbrace{\sum_{n_R} \binom{N}{n_R} p^{n_R} q^{N-n_R}}_{(p+q)^N} \Big\}$$

CAN NOW SUM :

DERIV: $p\, N (p+q)^{N-1} = \overline{n}_R(p,q)$ FOR ANY p, q

$$\Rightarrow \cancel{\overline{n}_R(p,q) =} \sum \binom{N}{n_R} p^{n_R} q^{N-n_R}\, n_R = p N (p+q)^{N-1}$$

~~true for any~~ p, q

(d) NOW TAKE $q = 1-p$

$$\overline{n}_R = \overline{n}_R(p,q)\Big|_{q=1-p} = Np$$

(YOU ~~WOULD~~ DID GUESS THIS)

$$\boxed{\overline{n}_R = Np}$$

IN SAME WAY $\boxed{\overline{n}_L = Nq}$ ⇒ $\boxed{\overline{m} \overset{= \overline{n_R - n_L}}{=} \overline{n}_R - \overline{n}_L = N(p-q)}$

(w/ $q = 1-p$)

(3) VARIANCE

$$\overline{(\Delta m)^2} \equiv (\Delta^* m)^2 = \overline{(m-\bar{m})^2}$$

USE n_R, $W(n_R)$:

$$\Delta m = m - \bar{m} = (2n_R - N) - (2\bar{n}_R - N) = 2\Delta n_R$$

$$\overline{(\Delta m)^2} = 4\,\overline{(\Delta n_R)^2} = 4(\overline{n_R^2} - \bar{n}_R^2)$$

$$\uparrow N^2 p^2$$

SAME TRICK:

$$\overline{n_R^2} = \sum_{n_R} W(n_R)\, n_R^2 = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} n_R^2$$

$$= \left(p\frac{\partial}{\partial p}\right)\left(p\frac{\partial}{\partial p}\right)(p+q)^N \Big|_{q=1-p}$$

SOME ALGEBRA, AFTER $q = 1-p$ $\Rightarrow$

$$\overline{n_R^2} = N^2 p^2 + Npq$$

$$\overline{(\Delta m)^2} = 4(\overline{n_R^2} - \bar{n}_R^2) = 4Npq$$

$$\boxed{\Delta^* m = \sqrt{4Npq}}$$ note $\sim N^{\frac{1}{2}}$

cf previous plot

(WILL WANT N LARGE (AND INCR w/ t) FOR DIFFUSION MODEL :)

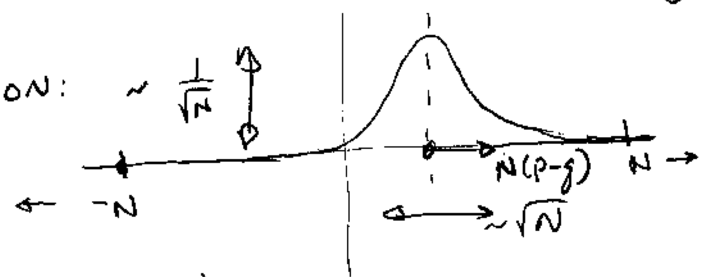
GEN'L FEATURES VS N:

AVE: $\bar{m} = N(p-q)$ ≈ LOC. OF PEAK

WIDTH: $\Delta^* m = \sqrt{4Npq}$ WIDTH REL TO EXTENT: $\frac{\Delta^* m}{2N} = \left(\frac{pq}{N}\right)^{\frac{1}{2}}$

HEIGHT: (will see): $P_{max} \approx \left(\frac{1}{2\pi Npq}\right)^{\frac{1}{2}}$

DISTRIBUTION:



(picture - moving plot)

SHRINKS & SPREADS, BUT AREA CONST (why? prob dist)

(WILL SEE IN HW, CAN THINK OF AS DESCRIBING t DEVELOPMENT OF DIFFUSION PROCESS)

(THERE $N\ell = vt$
ℓ ave v)

IN TERMS OF n_R & $W(n_R)$:

$\bar{n}_R = Np$

$\Delta^* n_R = \frac{1}{2}\Delta^* m = \sqrt{Npq}$

① also; cp width to max extent = 2N
② mention ⇒ sharply peaked

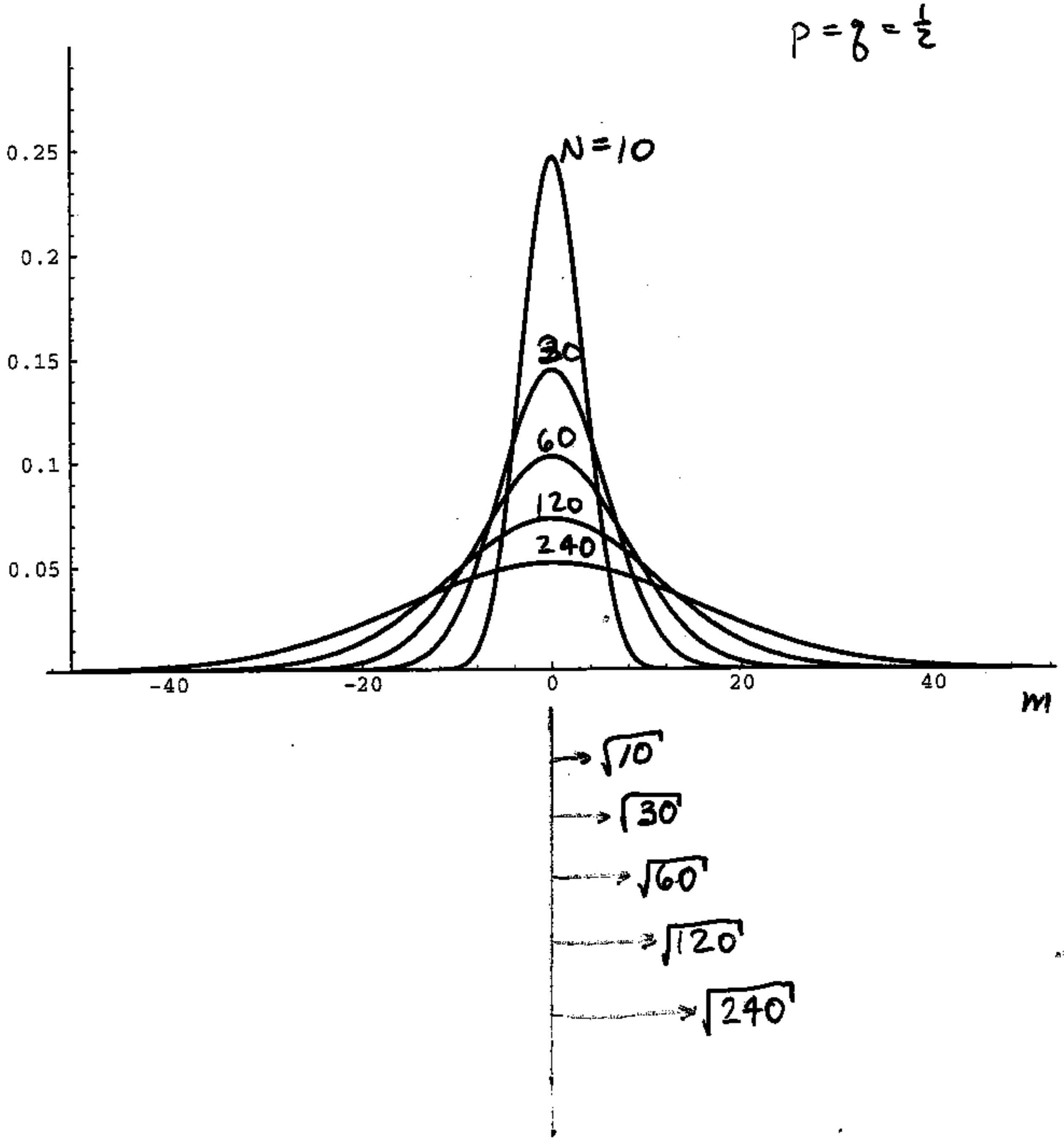
RELATIVE UNCERTAINTY: $\frac{\Delta^* n_R}{\bar{n}_R} = \frac{1}{\sqrt{N}}\left(\frac{q}{p}\right)^{\frac{1}{2}}$

SHRINKS w/ N ⇒ % ACCURACY INCREASES

(ex $p=q=\frac{1}{2}$ $\bar{n}_R$ MORE FOCUSED AT $N/2$)

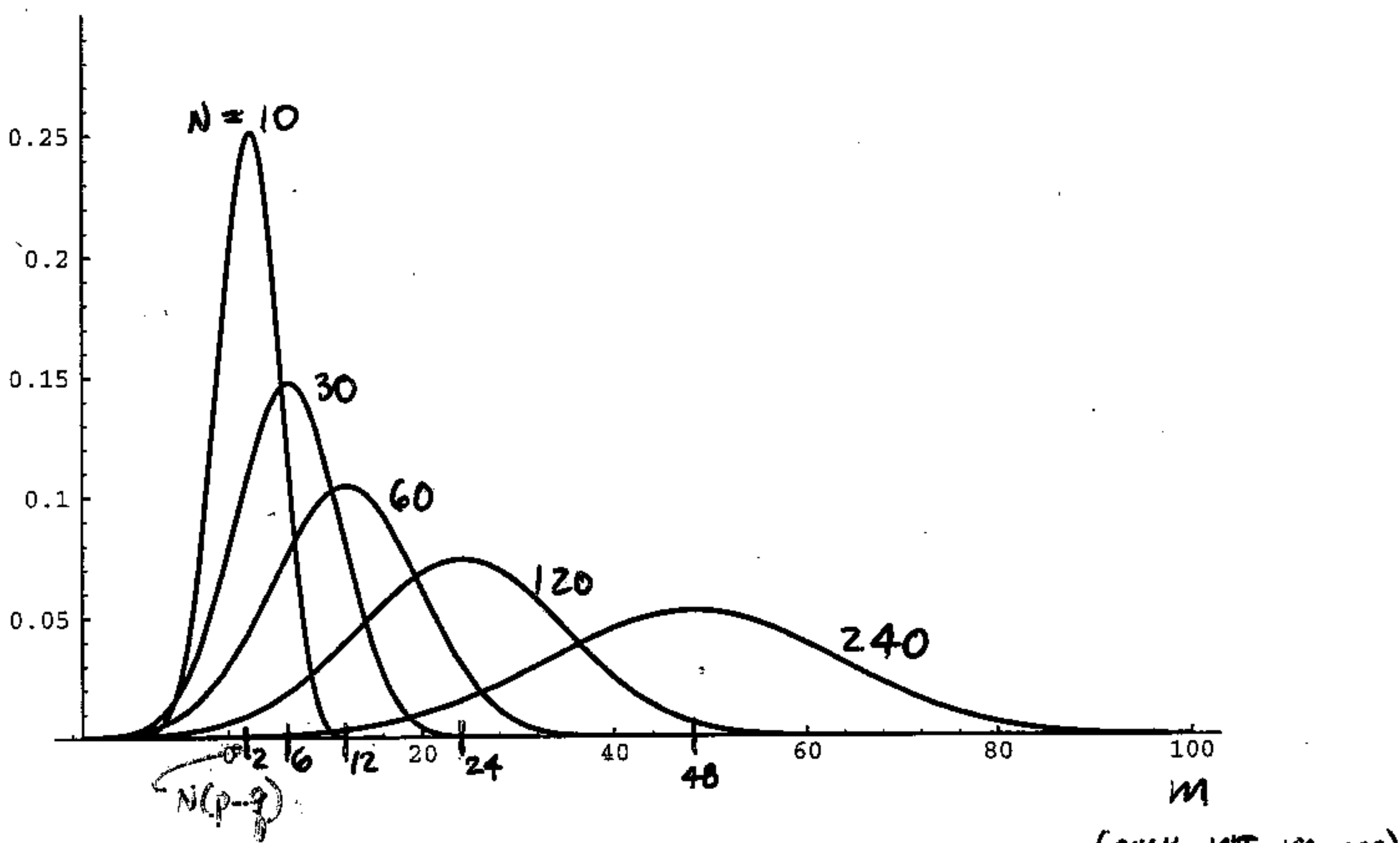
1.12.1

skip



p = q = 1/2
N=10
30
60
120
240
0.25
0.2
0.15
0.1
0.05
-40
-20
0
20
40
m
√10
√30
√60
√120
√240

$p = .6 \quad q = .4$



N = 10
30
60
120
240
0.25
0.2
0.15
0.1
0.05
0
20
40
60
80
100
2
6
12
24
48
N(p-q)
m
(ONLY INT. VALUES)

WE'LL NEED VERY LARGE N:

MOLS. IN FLUID:

$x \sim$ MACRO DIST ~ 1 cm

$\ell \sim$ ATOMIC " $\sim 1\,\text{Å} \sim 10^{-10}$ m $\ll x$

$\bar{x} = \bar{m}\ell = N(p-q)\ell \Rightarrow N \sim 10^{8}$ COLLISIONS BEFORE PEAK MOVES 1 cm

INTERESTING LIMIT:

$\underbrace{\bar{x}}_{\text{REASONABLE (ie MACRO)}} \propto \underbrace{N}_{\text{LARGE}} \; \ell \leftarrow$ SMALL

CONTINUOUS APPROX:
(cf N = 240 PLOT)

P(m) → SMOOTH (ie $\Delta P \ll P$)

CAN REPLACE EXACT SOLN w/ CONTINUOUS PROB DENSITY $\mathcal{P}(x)$

(SEEMS COUNTERPRODUCTIVE ⇒ WORK HARD TO GET APPROX ?)

WHY?

(1) EASIER TO WORK WITH (INTEGRALS VS SUMS)

(2) MORE APPROPRIATE FOR MACRO MEAS:
TREATS AS CONTINUOUS FLUIDS

(3) AS ACCURATE:

(a) UNCERTAINTY IN SIZE OF INSTRUMENT $\gg \ell$
" " # PARTICLES $\gg \Delta P$
⇒ CAN'T SEE BUMPS ⇒ INFINITESIMAL ON MACRO SCALE

(b) IF COULD SEE BUMPS (ie MEAS. ON ATOMIC LEVEL)
NEED A BETTER MODEL (DON'T REALLY BELIEVE MICRO DETAILS OF THIS ONE)

(4) ALL MICRO MODELS GIVE SAME MACRO $P(x)$ (WILL SEE)

$\Rightarrow$ ATOMIC DETAILS ENCAPSULATED IN 2 PARAMETERS

skip SIMILAR TO E & M IN MEDIA

$\Rightarrow$ APPROX. ATOMIC DETAILS IN DIELECTRIC CONST ϵ

REPLACE $\vec{E}(x)$ w/ $\vec{D} = \epsilon \vec{E}$

REIF NOTATION:

dx = INFINITESIMAL BUT MACROSCOPIC

δx = SHORTEST (ATOMIC) DIST.

USUALLY $x \gg dx \gg \delta x$

(limit $dx \to 0$ really means small enough that $f(x)$ doesn't change much over dx, but not so small can see atoms)

DEFINE CONTINUOUS DISTR:

USUAL CALCULUS APPROACH: DEF. VAR, FN AT DISCRETE SET OF PTS & LET PTS GET CLOSER;

CONTINUOUS VAR:

$x_m \equiv m\ell \qquad m = -N, -N+2, \ldots, N-2, N$

$\Delta x = \Delta m \cdot \ell = 2\ell \qquad$ (recall m even or odd)

PROB. DENSITY:

$$\mathcal{P}(x_m) = \lim_{\ell \to 0} \frac{P(m)}{\underbrace{2\ell}_{\Delta x}}$$

(WELL-DEF'D AS $\Delta x = 2\ell \to 0$)

(IF $P(m)$ SIMPLE FN)
JUST REPLACE m w/ x/ℓ, TAKE LIMIT → DONE

PROBLEM:

$P(m)$ HAS FACTORIALS:
- AWKWARD TO USE AS $N \to \infty$
- ONLY DEF'D FOR INTEGER VALUES:
  $\ell \to 0$ LIMIT PAINFUL

SOL'N:

REPLACE $P(m)$ w/ SIMPLE CONTINUOUS APPROX:

WILL SEE:
- EASY TO $\int$
- CAN READ OFF $\bar{x}$, $\Delta^* x$
- UNIVERSAL ⇒ GOOD APPROX FOR ALMOST ANY LARGE SUM OF RANDOM VARS

pos

⇒ GAUSSIAN

PROBLEM:

$P(m)$ BECOMES SMOOTH BUT SHARPLY PEAKED:

skip; already covered

$\Delta^* x \sim N^{1/2}\ell, \quad x_{MAX} \sim N\ell$

$$\frac{\Delta^* x}{x_{MAX}} \sim \frac{1}{N^{1/2}}$$

already discussed on pg 1.12

(fn sits at 0 for most x, jumps quickly near peak)

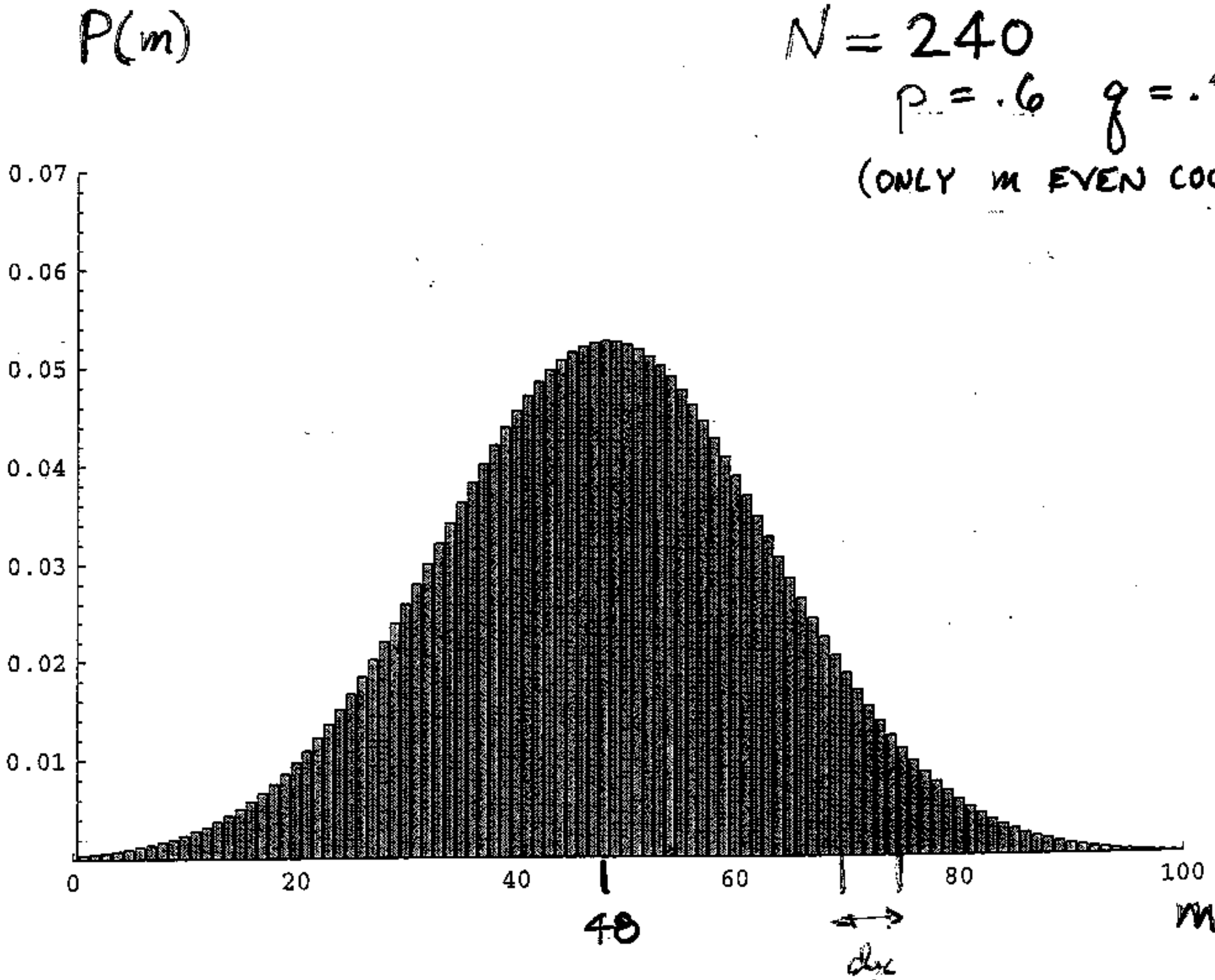
cf plots

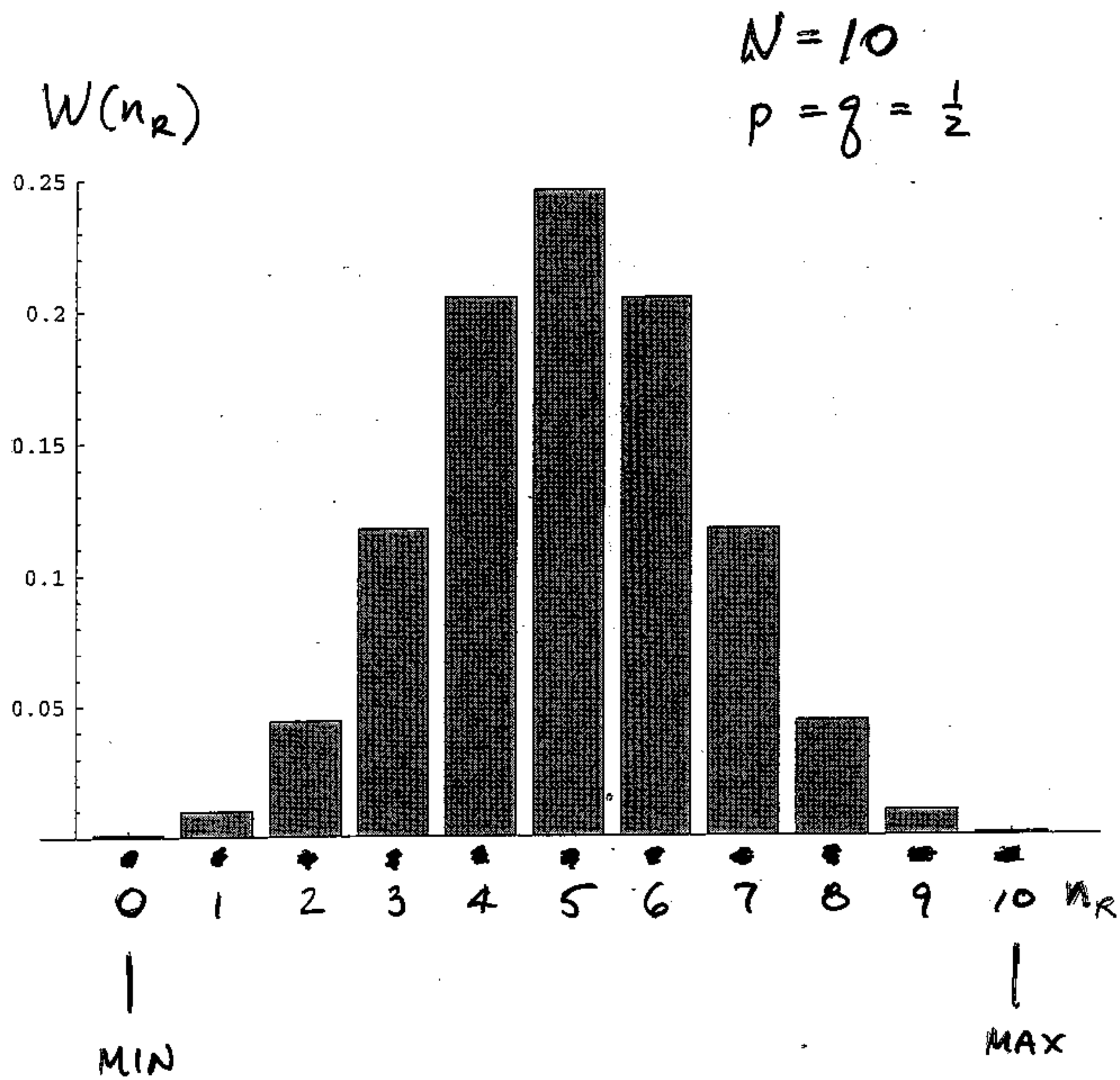
should only have values at 2's, or use $W(n_R)$

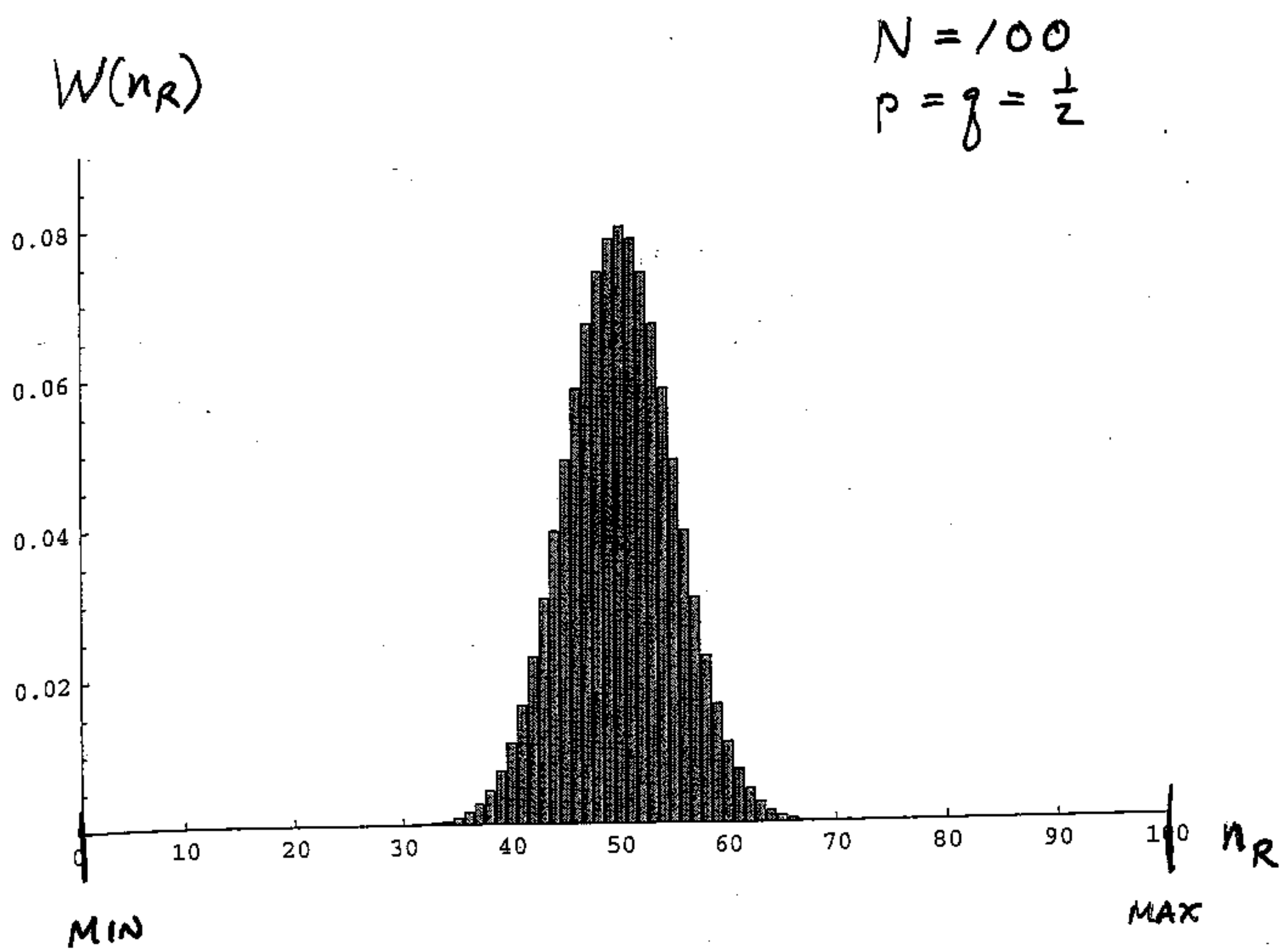
$N = 240$

$p = .6 \quad q = .4$

(ONLY m EVEN COUNT)




N = 10
p = q = 1/2
W(n_R)
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7
8
9
10
n_R
MIN
MAX


N = 100
p = q = 1/2
W(n_R)
0.08
0.06
0.04
0.02
0
10
20
30
40
50
60
70
80
90
100
n_R
MIN
MAX

(MORE SPECIFICALLY) FN IS EXPONENTIAL (w/ LARGE ARG.)

POWERS ARE EXPONENTIAL: $p^{n_R} = e^{n_R \ln p}$

LARGE FACTORIALS $\approx$ EXPONENTIALS:

(try 100! on calculator)

STIRLING APPROX (cf APPX)

$$n! \approx e^{n \ln n - n + \frac{1}{2}\ln(2\pi n) + \cdots}$$

terms which vanish as $n \to \infty$

$$= (2\pi n)^{\frac{1}{2}} e^{n(\ln n - 1)}$$

& appx (uses method we'll talk about now)

"leading term in asymptotic expansion as $n \to \infty$"

ex $100! \sim e^{364}$

PLAN: TO FIND SIMPLE APPROX

(1) GOOD NEWS:

$P(m)$ ONLY $\gtrsim 0$ NEAR PEAK $\tilde{x}$

$\Rightarrow$ ONLY NEED GOOD APPROX NEAR MAX

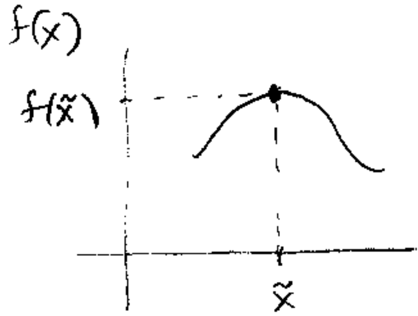
$\Rightarrow$ TAYLOR SERIES

(2) BAD NEWS:

$P(m)$ DROPS EXPONENTIALLY

$\Rightarrow$ BUILD INTO SERIES

SIMPLE
APPROX. NEAR PEAK:



TAYLOR:

$$f(x) \sim f(\tilde{x}) + \underbrace{f'(\tilde{x})}_{\substack{=0 \\ \text{(EXTREMUM)}}}(x-\tilde{x}) + \underbrace{\tfrac{1}{2}f''(\tilde{x})}_{\substack{\leq 0 \\ \text{(CURVES DN)}}}(x-\tilde{x})^2 + \cdots$$

7

DEFINE $\eta \equiv x - \tilde{x}$

(SO EXPAND AROUND $\eta = 0$)

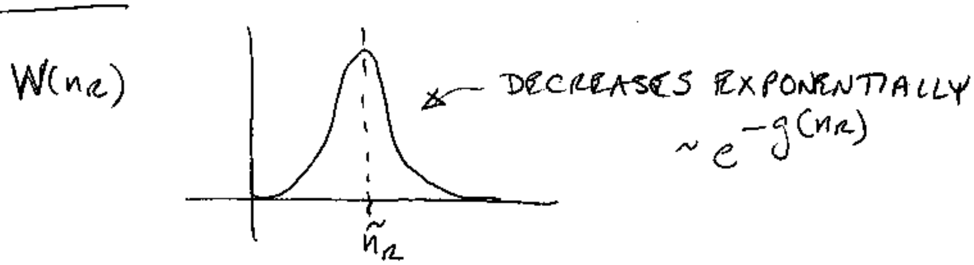
$$f(x) \equiv f(\tilde{x} + \eta) \approx f(\tilde{x}) + \tfrac{1}{2} f''(\tilde{x})\,\eta^2 + \cdots$$

$$= f(\tilde{x}) - \tfrac{1}{2}\,|f''(\tilde{x})|\,\eta^2 + \cdots$$

ACCURATE FOR η SMALL

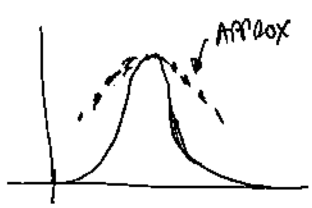
TEST: NEW TERMS GET SMALLER

OUR CASE



TAYLOR SERIES: POLYNOMIAL CAN'T KEEP UP w/ EXPONENTIAL

$\Rightarrow$ GET



$\Rightarrow$ ONLY GOOD FOR SMALL RANGE IN $\eta \equiv n_R - \tilde{n}_R$

BE SMART:

KNOW $W(n_R) \sim e^{-g(n_R)}$

APPROX g, NOT W (builds in exp behavior)

ie $W(n_R) = e^{\underbrace{\ln W(n_R)}}$

→ MUCH SMOOTHER FN

→ EASIER TO APPROX

TAYLOR SERIES: (IN $\eta \equiv n_R - \tilde{n}_R$ AROUND ZERO)

$$\ln W(n_R) \approx \ln W(\tilde{n}_R) + B_1 \eta + \tfrac{1}{2} B_2 \eta^2 + \tfrac{1}{3!} B_3 \eta^3 + \cdots$$

$$\boxed{B_k = \frac{d^k(\ln W(\tilde{n}_R))}{d n_R^k}}$$

KNOW

$B_1 = 0$

$B_2 < 0$

THEN

$$W(n_R) \approx e^{\left[\ln W(\tilde{n}_R) - \frac{1}{2}|B_2|\eta^2 + \cdots\right]}$$

$$\approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

"GAUSSIAN DISTR."

→ WILL CLEAN UP LATER

⇒ VERY GENERAL:

→ ONLY ASSUMED EXP. FALL OFF

→ VERY COMMON WHEN VARIABLE IS SUM OF LARGE # N OF RANDOM VARIABLES (HERE N STEPS OF $s_i = \pm l$ WITH PROB p, q)

("CENTRAL LIMIT THM" → ALMOST ANY PROB FOR INDIV. s_i GIVES THIS FORM)

RANGE OF VALIDITY:

- GOOD APPROX FOR $\ln W$ IF

$$\left| \frac{1}{3!} B_3 \eta^3 \right| \ll \left| \frac{1}{2} B_2 \eta^2 \right|$$

ie fails for large enough η

say { - WILL SEE: WHEN " ~ "
W NEGLIGIBLE

$\Rightarrow$ APPROX GOOD FOR ALL η }

ex (REIF):

$$f(y) = \frac{1}{(1+y)^N} = e^{-N \ln(1+y)}$$

{ powers, exponentials ~same

→ CHANGES SHARPLY IF N LARGE } see picture

→ $\ln f(y)$ MUCH SMOOTHER } (power of 10 becomes factor of 10)

APPROX NEAR $y=0$

DIRECTLY: $f(y) \approx 1 - Ny + \frac{1}{2}N(N+1)y^2 + \dots$

N large → NOT SENSIBLE UNLESS $|Ny| \lesssim 1$

OR $|y| \lesssim 1/N$

LOG: $\ln f(y) \approx -N(y - \frac{1}{2}y^2 + \dots)$

FAILS UNLESS $|y^2| \lesssim |y|$ OR $|y| \lesssim 1$ (indep of N)

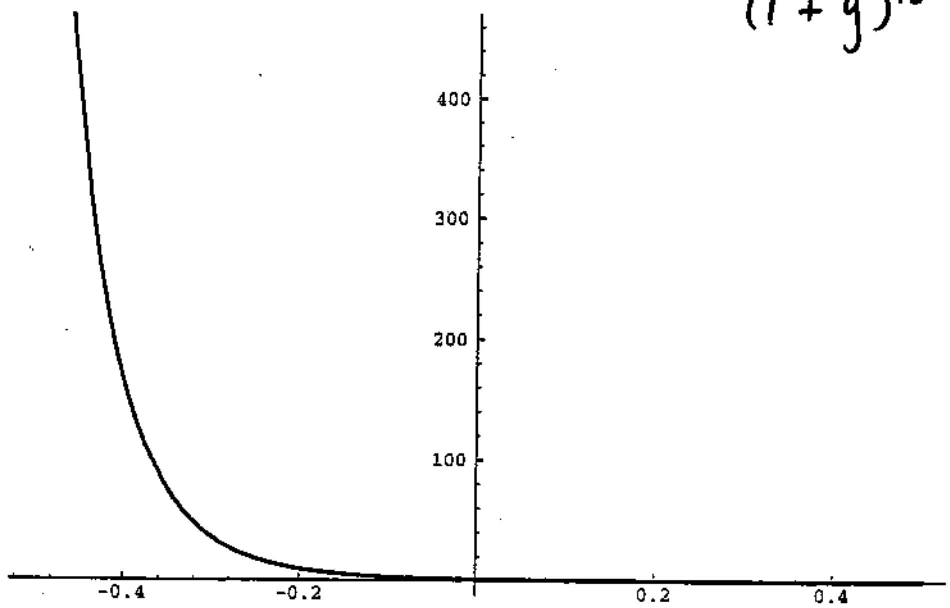
$$f(y) \approx e^{-N(y - \frac{1}{2}y^2 + \dots)}$$

⇒ picture:

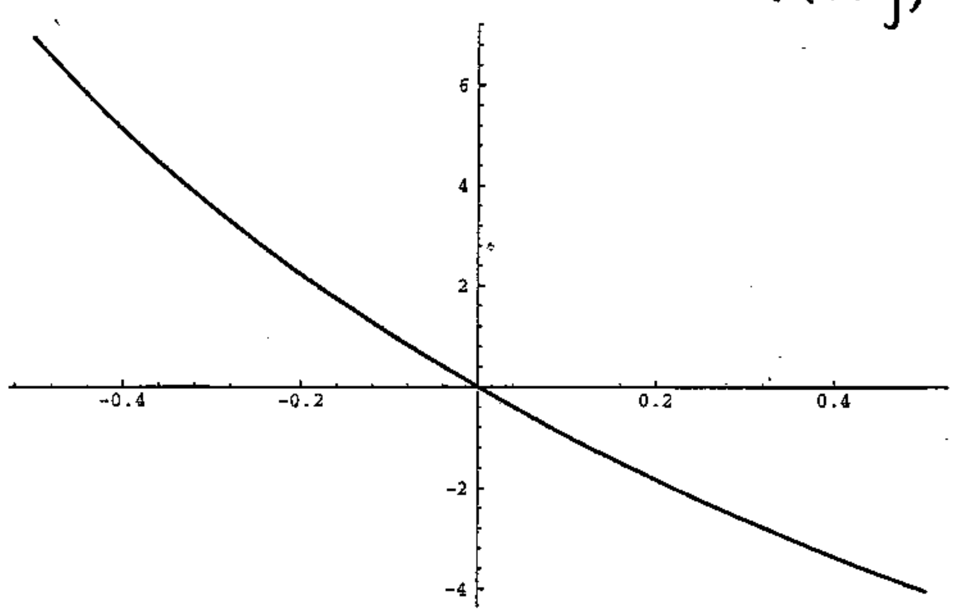
$N = 10$

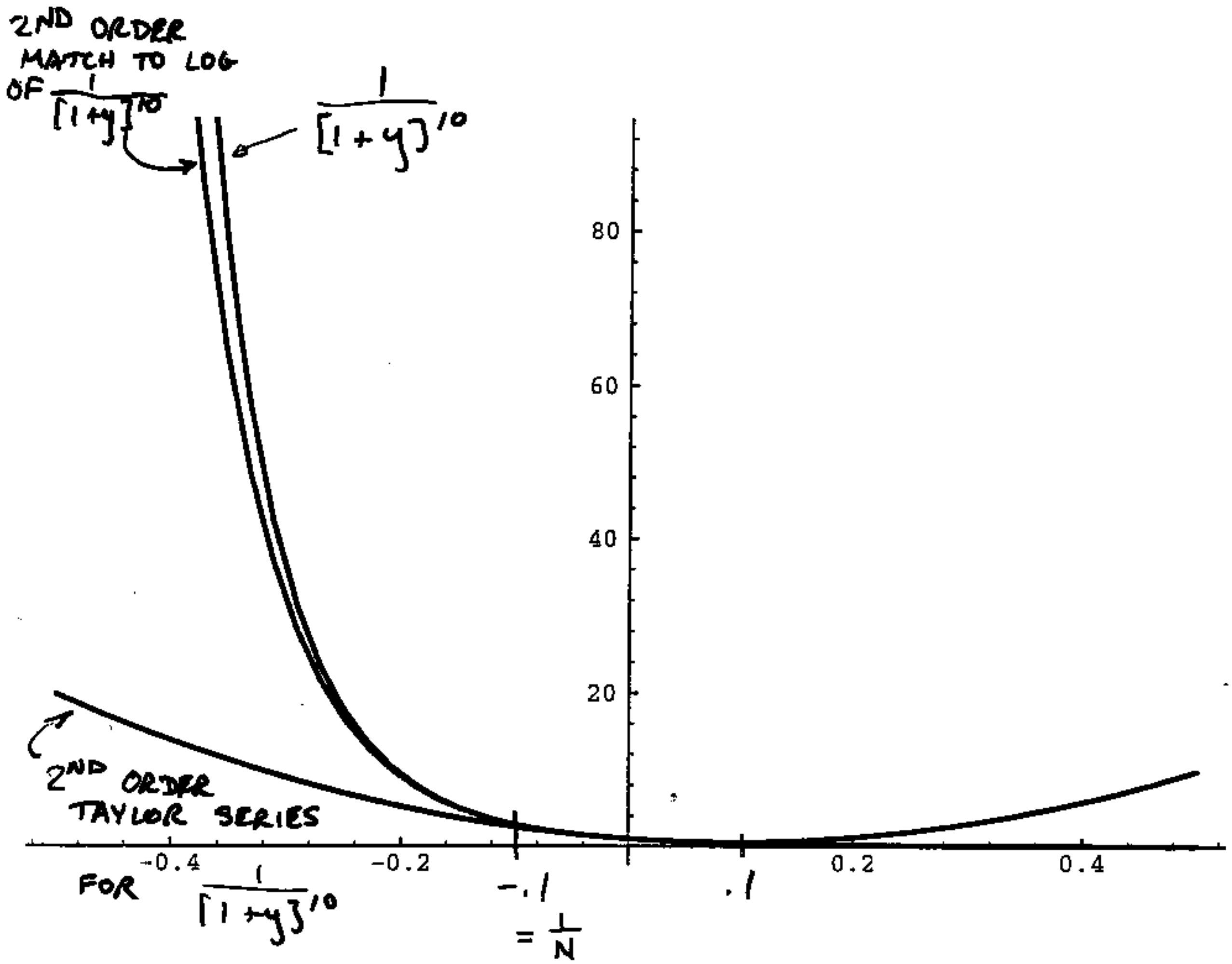
1st pict dies at $y \sim \pm\frac{1}{10} = \pm .1$

$$\frac{1}{(1+y)^{10}}$$



$$\ln\left[\frac{1}{(1+y)^{10}}\right]$$





2ND ORDER MATCH TO LOG OF $\frac{1}{[1+y]^{10}}$
$\frac{1}{[1+y]^{10}}$
80
60
40
20
2ND ORDER TAYLOR SERIES FOR $\frac{1}{[1+y]^{10}}$
-0.4
-0.2
-.1 $= \frac{1}{N}$
.1
0.2
0.4

APPROX $\ln W(n_R)$: (cf REIF & following notes FOR DETAILS (or try yourself))

PROBLEM: NEED DERIVS

ONLY DEF'D AT INTEGER n_R'S

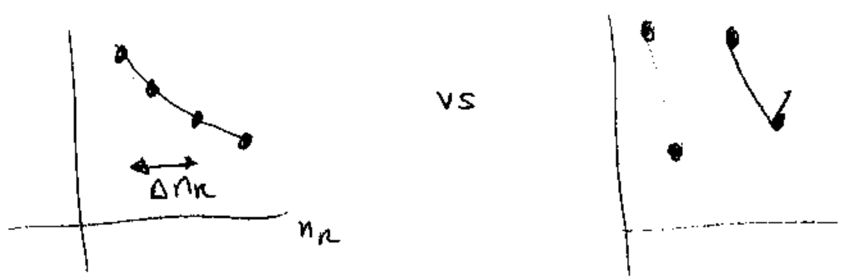
BEST APPROX:

$$\frac{d \ln W(n_R)}{d n_R} \approx \frac{\Delta \ln W(n_R)}{\Delta n_R}$$

SMALLEST $\Delta n_R = 1$

$$= \ln W(n_R+1) - \ln W(n_R)$$

OK IF $\ln W(n_R)$ SMOOTH: (saw in plot that W becomes smoother as N incr.)



(a) MAX: $\frac{d \ln W}{d n_R} = 0$ ($*$)

SOLN: $\boxed{\tilde{n}_R = N_p}$ (note: same as $\overline{n_R}$)

($*$) NEED THINGS LIKE (for n large)

$$\frac{\Delta \ln n!}{\Delta n} \sim \frac{\ln (n+1)! - \ln n!}{1} = \ln\left[\frac{(n+1)!}{n!}\right] = \ln(n+1) \sim \ln n$$

$\{$ or $\ln(n+1) + \ln(n) + \ln(n-1) + \cdots - \ln(n) - \ln(n-1) + \cdots]$

so $n \gg 1 \Rightarrow \frac{d \ln n!}{dn} \sim \ln n$ ETC

$\{$ IN DETAIL:

$W(\tilde{n}_R)$ using Stirling's formula:

$$W(n_R) = \frac{N!}{(N-n_R)!\, n_R!}\, p^{n_R}\, q^{N-n_R}$$

$$\tilde{n}_R = Np$$

$$W(\tilde{n}_R) = \frac{N!}{(N-Np)!\,(Np)!}\, p^{Np}\, q^{N-Np}$$

$$= \frac{N!}{(Nq)!\,(Np)!}\, p^{Np}\, q^{Nq}$$

$$\ln W(\tilde{n}_R) = \ln N! - \ln(Nq)! - \ln(Np)! + Np\ln p + Nq\ln q$$

Stirling:

$$\ln n! \underset{n \text{ large}}{\sim} n\ln n - n + \tfrac{1}{2}\ln(2\pi n) + \cdots$$

$$\begin{aligned} \ln W(\tilde{n}_R) \sim\ & N\ln N - N + \tfrac{1}{2}\ln(2\pi N) \\ & - Nq\ln Nq + Nq - \tfrac{1}{2}\ln(2\pi Nq) \\ & - Np\ln Np + Np - \tfrac{1}{2}\ln(2\pi Np) \\ & + Np\ln p + Nq\ln q \end{aligned}$$

from $p+q=1 \Rightarrow -Nq\ln Nq - Np\ln Np = \underbrace{-Nq\ln N - Np\ln N}_{-N\ln N} - Nq\ln q - Np\ln p$

everything cancels but

$$\tfrac{1}{2}\left[\ln(2\pi N) - \ln(2\pi Nq) - \ln(2\pi Np)\right] = \tfrac{1}{2}\ln\frac{2\pi N}{(2\pi Nq)(2\pi Np)}$$

$$= \tfrac{1}{2}\ln\frac{1}{2\pi Npq}$$

$$\Rightarrow\ W(\tilde{n}_R) = \left(\frac{1}{2\pi Npq}\right)^{\frac{1}{2}} \quad \}$$

APPROX $\ln W(n_R)$: (cf REIF FOR DETAILS or better, try yourself)

FIND $\tilde{n}_R$ AT MAX:

(a) $$\frac{d \ln W(n_R)}{d n_R} = 0 \approx \ln W(n_R+1) - \ln W(n_R)$$

SOLN $\boxed{\tilde{n}_R = Np}$

(b) EXPAND AROUND $\tilde{n}_R$ $\quad (\eta = n_R - \tilde{n}_R)$

COEFFICIENTS:

$B_1 = 0$

$$\boxed{B_2 = -\frac{1}{Npq}}$$ < 0, AS EXPECTED

$$B_3 = \frac{q^2-p^2}{N^2p^2q^2}$$ ETC

$\Rightarrow$ OBSERVE:
EACH NEW TERM $B_k \eta^k$ HAS ADDITIONAL FACTOR

$\frac{\eta}{Npq}$ TIMES PREVIOUS. SERIES OK IF $\frac{\eta}{Npq} \ll 1$ skip; done later

later (N BIG; p, q NOT TOO SMALL; CF PROB 1.9)
(and when $\eta \sim Npq$, $W \sim 0$)

(c) $$W(\tilde{n}_R) \cong \frac{1}{(2\pi Npq)^{1/2}}$$ (STIRLING FORMULA) (cf TEXT OR NOTES)
(could also use norm cond)

COMBINE: $$W(n_R) = W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

$$= \frac{1}{(2\pi Npq)^{1/2}} e^{-\frac{(n_R - Np)^2}{2Npq}} + \left[\begin{array}{c}\text{CORRECTIONS}\\ \text{WHICH VANISH}\\ \text{AS } N \to \infty\end{array}\right]$$

THEN $W(n_R) \approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$

USE $W(\tilde{n}_R) \cong \dfrac{1}{(2\pi Npq)^{1/2}}$ (STIRLING'S FORMULA)

$$|B_2| = \frac{1}{Npq}$$

AND $\eta \equiv n_R - \tilde{n}_R = n_R - Np$

$$W(n_R) = \frac{1}{(2\pi Npq)^{1/2}}\, e^{-\frac{(n_R - Np)^2}{2Npq}} + \text{CORRECTIONS} \sim \frac{1}{N}$$

CONVERT TO $\mathcal{P}(x) = \dfrac{P(m)}{2\ell}$

USE $n_R = \frac{1}{2}(N+m)$, $x = m\ell$

ALSO
$$\boxed{\begin{aligned} \mu &\equiv (p-q)N\ell \\ \sigma^2 &\equiv 4Npq\,\ell^2 \end{aligned}}$$
(put on side)

{micro information encapsulated in them 2 parameters

THEN
$$\boxed{\mathcal{P}(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\, e^{-(x-\mu)^2/2\sigma^2}}$$

GAUSSIAN DISTRIBUTION

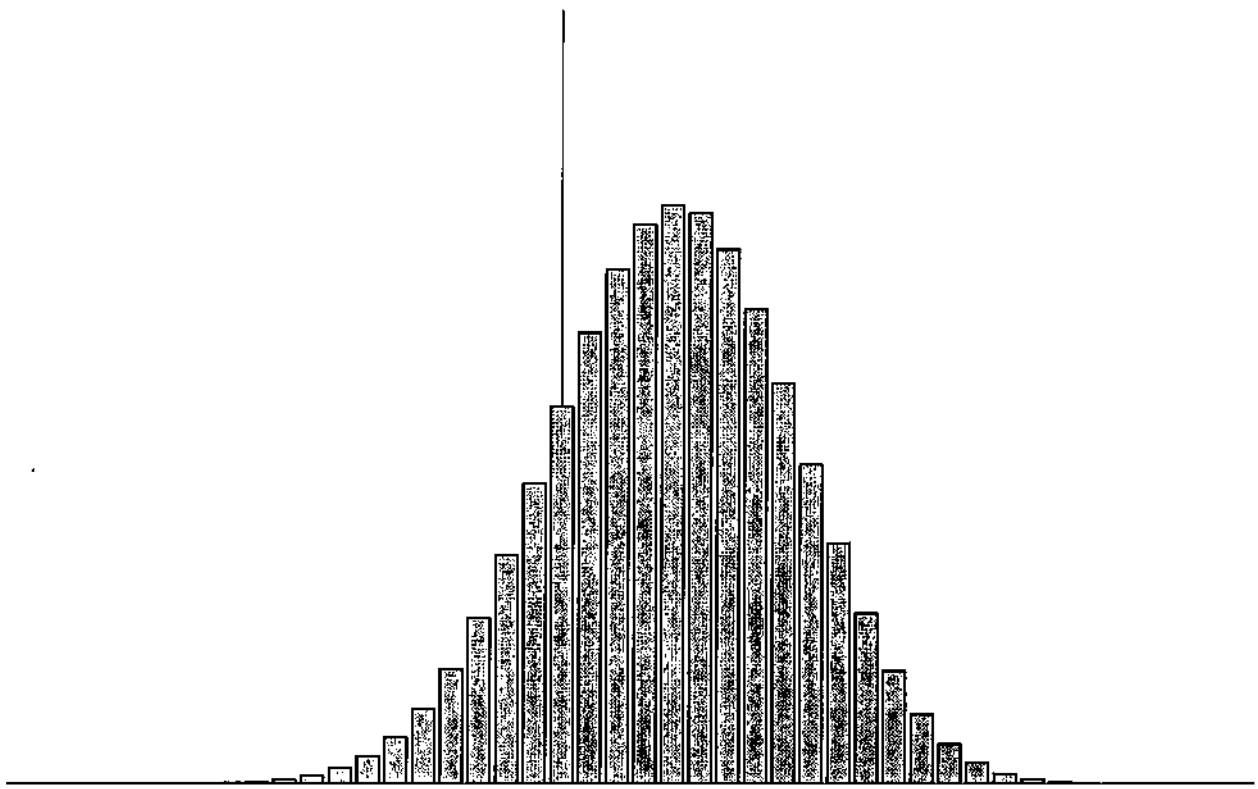
cf PLOT

skip
→ ALMOST ALWAYS GET THIS
WHEN ADD LARGE # RAND. VARS.
(CENTRAL LIMIT THM — cf * SECTIONS FOR PROOF)

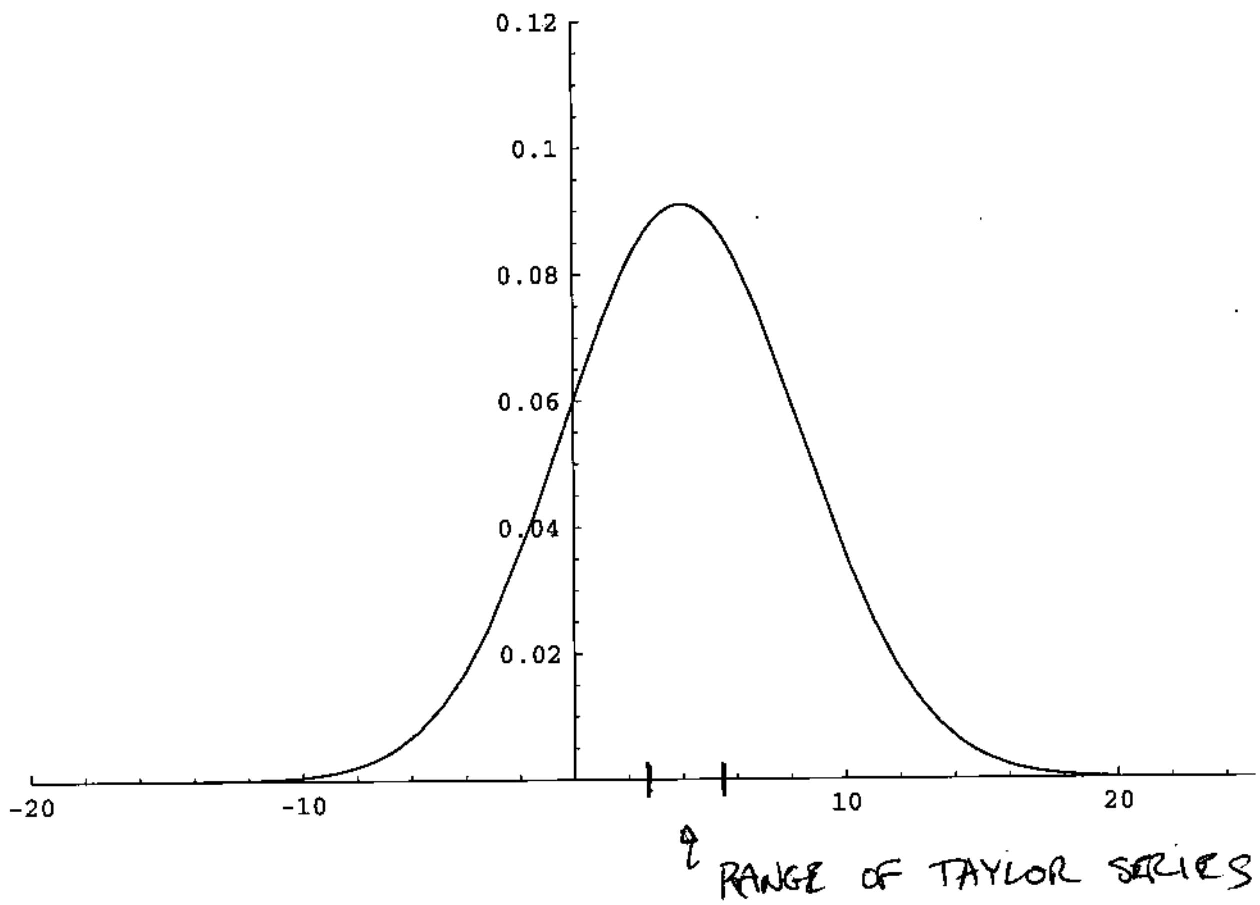
** DISCUSS: μ, σ JUST DEFNS, BUT HAVE SIMPLE INTERP:
PEAKED AT $x = \mu$
FALLS OFF FOR $|x-\mu| \gtrsim \sigma$ $\Rightarrow \sigma \sim$ WIDTH

N = 20 $p = 0.6$ $\ell = 1$

GAUSSIAN vs BINOMIAL DISTRIBUTIONS



1.23-B
OVERLAY

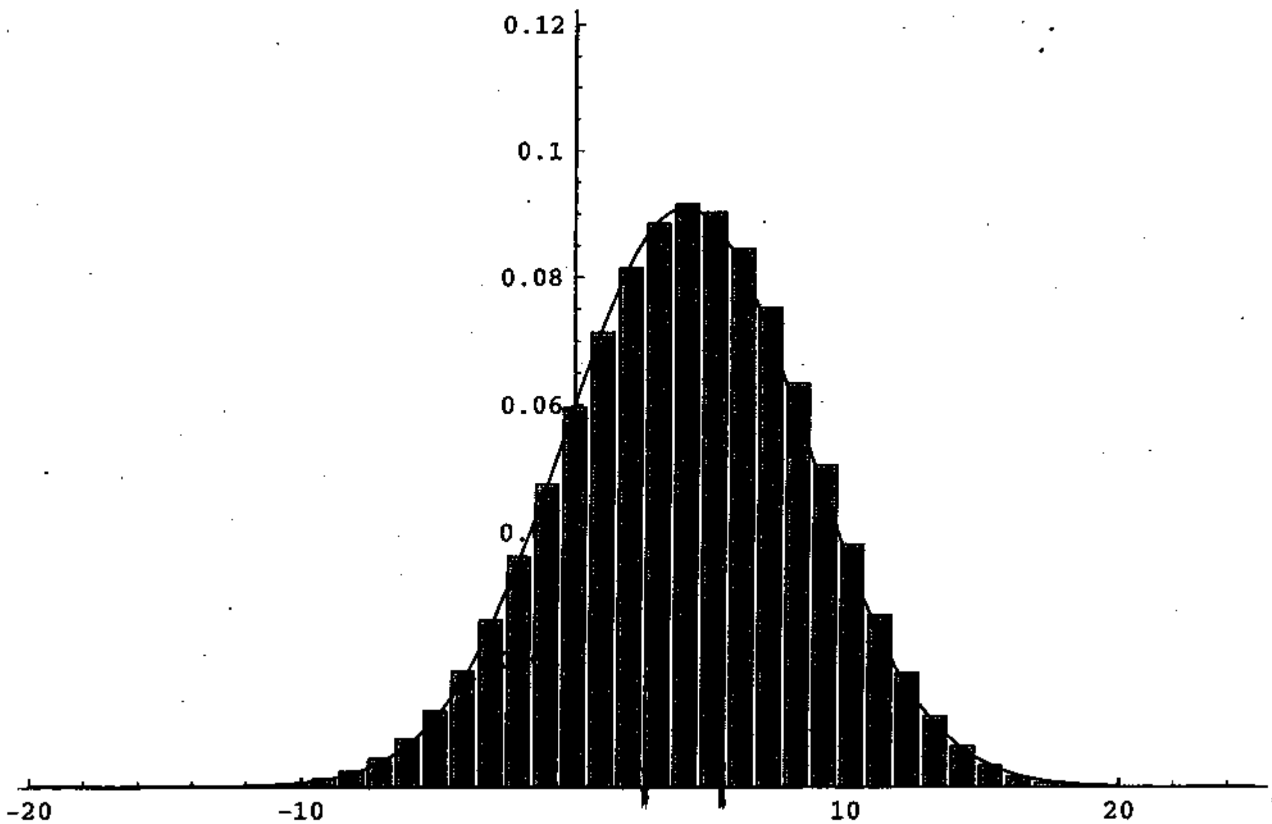


$P(x)$, $\ell=1$

note. even this approx fails for some x; looks good for all x here; why?
{⇒ at x where fails, $P \sim 0$}

$N = 20 \quad p = 0.6 \quad \ell = 1$

GAUSSIAN vs BINOMIAL DISTRIBUTIONS



RANGE OF TAYLOR SERIES

CHECK NORMALIZATION:

$$\int_{x_{MIN}}^{x_{MAX}} P(x)\,dx$$

LIMITS: $\pm N\ell$

$P(x) \approx 0$

REPLACE w/ $\pm\infty$ (can check: only affects by terms which vanish exp. w/ N)

NEED

$$\int_{-\infty}^{\infty} P(x)\,dx = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \underbrace{\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx}_{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy}$$

$y \equiv x - \mu$

2 TRICKS: DO (ALMOST) ALL GAUSS. INTS (one reason they're so popular - ints are easy)

cf REIF app A.2,4,5

CONSIDER $I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha z^2}\,dz$

COULD DO IF HAD $z\,dz$

$$I^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha z_1^2}\,dz_1 \int_{-\infty}^{\infty} e^{-\alpha z_2^2}\,dz_2$$

$$= \int_{-\infty}^{\infty} dz_1\,dz_2\; e^{-\alpha(z_1^2+z_2^2)}$$

(ONLY DISCUSS IF NEED
IMMEDIATELY FOR PROB 15)

INSERT: (put this here to summarize for HW; can drop next time)

NORM:

$$\int_{x_{MIN} = -N\ell}^{x_{MAX} = +N\ell} \underbrace{\mathcal{P}(x)}_{\sim 0 \text{ long before MIN, MAX}} \, dx$$

$$\simeq \int_{-\infty}^{\infty} \mathcal{P}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2} dx$$

CAN SEE $= 1$ (later?)

PROB. $x < y$?

$$\int_{-\infty}^{y} \mathcal{P}(x)\,dx \Rightarrow \text{ERROR FN, APPX A.5}$$

PROBLEM 15: FOR WHAT y IS PROB. $> C$?

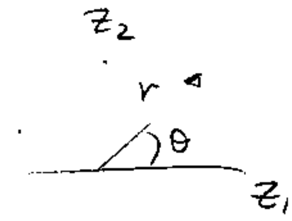
cf TABLES, MATHEMATICA (ask Scalise)

WILL SEE:

$$\int \mathcal{P}(x)\,dx = 1$$

$$\left.\begin{aligned} \bar{x} &= \int \mathcal{P}(x) \cdot x \, dx = \mu \\ \overline{(\Delta x)^2} &= \int \mathcal{P}(x)(x-\bar{x})^2 dx = \sigma^2 \end{aligned}\right\} \text{PARAMS IN } \mathcal{P}(x)$$

(later)



POLAR COORDS

skip $\left\{ \begin{array}{l} r = (z_1^2 + z_2^2)^{1/2} \\ \tan\theta = z_2/z_1 \end{array} \right.$

$$dz_1\, dz_2 \rightarrow r\,dr\,d\theta \qquad z_1^2 + z_2^2 = r^2$$

$$I^2(\alpha) = \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\infty} r\,dr\, e^{-\alpha r^2}}$$

$$\begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array} \Rightarrow \int_0^{\infty} \tfrac{1}{2} du\, e^{-\alpha u} = \frac{1}{2\alpha}$$

$$I^2(\alpha) = \frac{\pi}{\alpha}$$

$$\boxed{I(\alpha) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}} \qquad \text{know } I > 0$$

CAN USE THIS TO COMPUTE RELATED INTS:

$$\frac{\partial}{\partial\alpha} I(\alpha) = \frac{\partial}{\partial\alpha} \int_{-\infty}^{\infty} e^{-\alpha z^2} dz = \frac{\partial}{\partial\alpha} \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}$$

$$= -\int_{-\infty}^{\infty} e^{-\alpha z^2} z^2\, dz = -\frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}$$

$$\frac{\partial^n}{\partial\alpha^n} I(\alpha) = \boxed{(-1)^n \int_{-\infty}^{\infty} e^{-\alpha z^2} z^{2n}\, dz = \frac{\partial^n}{\partial\alpha^n}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}}$$

ALSO $$\boxed{\int_{-\infty}^{\infty} e^{-\alpha z^2} z^{2n+1}\, dz = 0}$$ (z ODD $\Rightarrow \pm z$ GIVE OPP CONTRIB)

USE SIMILAR TRICK FOR:

$$\int_0^\infty e^{-\alpha z^2} z^{2n+1}\, dz \qquad \left(\text{EVEN ARE JUST } \tfrac{1}{2} \text{ OF } \int_{-\infty}^{\infty} \text{ INTS}\right)$$

START WITH $\displaystyle\int_0^\infty e^{-\alpha z^2} z\, dz = \frac{1}{2\alpha}$

THEN $$\boxed{(-1)^n \int_0^\infty e^{-\alpha z^2} z^{2n+1}\, dz = \frac{\partial^n}{\partial \alpha^n} \frac{1}{2\alpha}}_{\,n=0,1,2,\dots}$$

IN PARTICULAR:

$$\int_{-\infty}^{\infty} P(x)\, dx = \frac{1}{(2\pi\sigma^2)^{1/2}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy}$$

here $\alpha = \frac{1}{2\sigma^2} \Rightarrow \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} = (2\pi\sigma^2)^{1/2}$

$$= 1 \quad \checkmark$$

ALSO $$\bar{x} = \int_{-\infty}^{\infty} P(x) \cdot x\, dx \quad \left\{ \begin{array}{l} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\, x\, dx \\ \text{CHG VARS} \quad (y+\mu) \\ (a)\ y \equiv x - \mu \quad \text{no contrib} \\ (b)\ \alpha = \frac{1}{2\sigma^2} \quad \text{norm.} = 1 \end{array} \right.$$

$\Rightarrow \boxed{\bar{x} = \mu}$ (~ AS EXPECTED)

$$\overline{(\Delta x)^2} = \overline{(x - \bar{x})^2} = \underbrace{\int_{-\infty}^{\infty} \underbrace{(x-\mu)^2}_{= y^2} \underbrace{P(x)\, dx}_{\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-y^2/2\sigma^2}}}_{\frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}} \quad \alpha \equiv \frac{1}{2\sigma^2}}$$

$$\boxed{\overline{(\Delta x)^2} = \sigma^2}$$

$\therefore$ (1) CAN READ OFF $\bar{x}$ & $\overline{(\Delta x)^2} = (\Delta^* x)^2$ FROM GAUSS DISTR (ie μ AND σ^2)

(2) HERE $\bar{x} = (p-q)N\ell \quad = \bar{m}\,\ell$

$\overline{(\Delta x)^2} = 4Npq\,\ell^2 \quad = \overline{(\Delta m)^2}\,\ell^2$ $\left\{ \text{consistent w/ results from discr. distr} \right.$

? (AS IT MUST) (I suppose up to $\sim \frac{1}{N}$)

ONE LAST INTEGRAL: (NOT ALL FUN & GAMES)

ERROR FN: (cf prob 1.15)

slip FOR $P(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2}$

ex FIND PROB $x < y$: (OR $y_1 < x < y_2$)

$$\int_{-\infty}^{y} P(x)\,dx$$

RELATED TO ($\int_{-\infty}^{0}$ part is easy)

$$\text{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx$$

cf APPX A.5 (DISCUSSION & APPROX. FORMULA)

$\Rightarrow$ NO CLOSED FORM; DO NUMERICALLY OR

LOOK UP IN TABLES, MATHEMATICA (Erf[y]), C

VALIDITY OF GAUSSIAN APPROX:

$$W(n_R) \approx e^{\left[\ln W(\tilde{n}_R) + \frac{1}{2} B_2 \eta^2 + \frac{1}{3!} B_3 \eta^3 + \cdots\right]}$$

$\eta \equiv n_R - \tilde{n}_R$

skip $\left\{ \begin{array}{l} \tilde{n}_R \text{ IS PT WHERE } W \text{ IS MAX} \\ B_k \equiv \left.\dfrac{d^k(\ln W)}{d n_R^{\,k}}\right|_{n_R = \tilde{n}_R} \end{array} \right.$

$B_1 = 0$ (MAX)

$B_2 = -\frac{1}{Npq}$ ($<0 \rightarrow$ MAX)

$B_3 = \frac{q^2 - p^2}{N^2 p^2 q^2}$

EACH NEW TERM:

EXTRA $\frac{\eta}{Npq}$

(a) SERIES OK WHILE $\boxed{\frac{\eta}{Npq} \ll 1}$ OR $\boxed{\eta \ll Npq}$

ie: N large
p, q NOT TOO SMALL
(cf prob 9 for diff approx when p or q small)

skip $\Rightarrow W(n_R) = W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$ GOOD APPROX

(b) FAILS WHEN $\eta \approx Npq$

HOW BIG IS W?

$$W \approx W(\tilde{n}_R)\, \underbrace{e^{-\frac{1}{2}\frac{1}{Npq}\overset{\eta^2}{(Npq)^2}}}_{e^{-\frac{1}{2}Npq}}$$

13

IF $\boxed{Npq \gg 1}$, $W \sim 0$, APPROX GOOD FOR ALL η
(ie it's already zero when fails)

USEFUL GENERAL RESULTS (SECT 1.9)

(more general than ran. walk, Gauss. approx)

CONSIDER SUM OF N RAN #'s S_i (N NOT NECESS LARGE)

$$X = S_1 + S_2 + S_3 \cdots + S_N = \sum_{i=1}^{N} S_i$$

SUM IS NOT AVG'ING; JUST DEFN OF X

{* INDEX ON S_i HERE IS WHICH STEP, NOT WHAT VALUES S CAN TAKE

ex FINAL POSITION AFTER N STEPS

S_1 = 1^{ST} STEP $= +l$ w/ PROB p

$-l$ " " q

($\omega \equiv$ double-u, not omega)

HERE: ASSUME ONLY SAME DISTR. $w(s)$ FOR EACH S_i, & INDEP.

- DISCRETE OR CONTINUOUS

WHAT IS $\bar{X}$ & HOW DOES IT FLUCTUATE?

{ because s_i's random, x is also random; means it has some distr. $\mathcal{P}(x)$

(a) AVE OF SUM = SUM OF AVES (have seen already)

skip

AGAIN:

$$\bar{X} = \frac{1}{M}\sum_{m=1}^{M} X_m \qquad \underline{M \to \infty}$$

$$= \frac{1}{M}\Big\{ (S_1 + S_2 + \cdots + S_N) + \qquad \{ m=1$$

$$+ (S_1' + S_2' + \cdots + S_N') + \qquad \{ m=2$$

$$\vdots$$

$$+ (S_1'''' + S_2'''' + \cdots + S_N'''')\Big\} \qquad \{ m=M$$

$$= \bar{S_1} + \bar{S_2} + \cdots + \bar{S_N}$$

{ here: N is just a sum - could be 3, for ex, but M is to average $\Rightarrow \infty$ }

EACH STEP HAS SAME PROB DIST: $\bar{s}_1 = \bar{s}_2 = \cdots \equiv \bar{s}$

$$\boxed{\bar{X} = \overline{\textstyle\sum_{i=1}^{N} s_i} = \sum_{i=1}^{N} \bar{s}_i = N\bar{s}}$$

{emphasize diff between Σ^N and sum ...ed in ave

w/ $\bar{s} = \int ds\, \omega(s) \cdot s$ {if $\omega(s)$ continuous
else sum: ex ran walk
$\bar{s} = p \cdot \ell + q(-\ell)$
$= (p-q)\ell$

(b) SPREAD ?

$$\Delta X = X - \bar{X} = \underbrace{\sum_{i=1}^{N} s_i - N\bar{s}}_{} = \sum_{i=1}^{N} (s_i - \bar{s}) = \sum_{i=1}^{N} \Delta s_i$$

$$\overline{(\Delta X)^2} = \overline{\underbrace{\left(\sum_{i=1}^{N} \Delta s_i\right)}_{\Delta X}\underbrace{\left(\sum_{j=1}^{N} \Delta s_j\right)}_{\Delta X}}$$

(need diff i, j to keep two Δx's straight; write out if necessary)
(recall $I^2(\alpha) \Rightarrow$ need z_1 & z_2 to replace z)

SEPARATE OUT DIAG., CROSS TERMS

$$= \overline{\sum_{i=j=1}^{N} (\Delta s_i)^2} + \overline{\sum_{i \neq j} (\Delta s_i)(\Delta s_j)}$$

ave of sum = sum of aves

$$\sum_i \overline{(\Delta s_i)^2} \qquad\qquad \sum_{i \neq j} \overline{(\Delta s_i)(\Delta s_j)}$$

each same $\quad N\overline{(\Delta s)^2}$

INSIDE 2^{ND} SUM: IS AVE OF PROD = PROD OF AVE ?

$\Rightarrow$ ONLY IF 2 TERMS STATIST. INDEP:

IF a, b INDEP RANDOM. #S w/ DISTR. $P(a)$, $Q(b)$

PROB a & b: $P(a) \cdot Q(b)$

(NOT INDEP? SOME OTHER GEN'L DISTR. $R(a,b)$ GIVES PROB OF BOTH; TAKES EFFECT OF a ON b INTO ACCT)

THEN:

$$\overline{ab} = \underbrace{\sum_{k,l}}_{\substack{\text{all possible} \\ \text{values for } a\cdot b}} P(a_k)Q(b_l)\,(a_k b_l) \qquad = \sum_k \sum_l$$

$$= \left(\Sigma_k P(a_k)\, a_k\right)\left(\Sigma_l\, Q(b_l)\, b_l\right)$$

$$= \overline{a}\cdot\overline{b}$$

(same for continuous distr.)

OUR PROBLEM: EACH TERM $(i \neq j)$

$$\overline{(\Delta S_i)(\Delta S_j)} = \underbrace{\overline{(\Delta S_i)}}_{=0} \cdot \underbrace{\overline{(\Delta S_j)}}_{=0} = 0$$

(wouldn't work for $i=j \Rightarrow$ certainly not indep)

FINALLY

$$\overline{(\Delta x)^2} = N\,\overline{(\Delta s)^2} \qquad \text{OR} \qquad \boxed{\Delta^* x = \sqrt{N}\,\Delta^* s}$$

MAKES SENSE: MORE TERMS IN SUM, MORE TOTAL MOVES AROUND

BUT: BIGGER N, <u>SMALLER</u> RELATIVE CHG:

$$\boxed{\frac{\Delta^* x}{\overline{x}} = \frac{1}{\sqrt{N}}\,\frac{\Delta^* s}{\overline{s}}}$$

WHY IS THIS USEFUL ?

(0) CAN TELL AVE, STD DEV FOR SUM ONLY KNOWING ", " FOR PIECES; TELLS A LOT ABOUT DISTR.

(1) N LARGE: CAN SAY MORE

skip

(ex RANDOM WALK)

→ COMPUTE $\bar{s}$, $\Delta^* s = \sqrt{\overline{(\Delta s)^2}}$

ALSO $X \equiv s_1 + \cdots + s_N$, N LARGE

⇒ FOR $x = \Sigma_{i=1}^N s_i$, GIVEN $w(s)$

KNOW DISTRIB. $\mathcal{P}(x)$ FOR x COMPLETELY

HOW?

(a) $\bar{x} = N\bar{s} \equiv \mu$

(b) $\Delta^* x = \sqrt{N}\, \Delta^* s \equiv \sigma$

(c) N LARGE ⇒ $\mathcal{P}(x)$ GAUSSIAN } cf optional sections 10 & 11

$$\Rightarrow \mathcal{P}(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2}$$

"central limit thm"

⇒ CAN COMPUTE ANYTHING ABOUT X

⇒ TRUE FOR ALMOST ANY $w(s)$ (w/ some rare exceptions) (all moments must exist; ie $w(s) \to 0$ as $|s| \to \infty$ fast enough)

(2) STATISTICAL ERRORS:

REPEAT MEAS N TIMES ⇒

ERROR DECREASES AS $\frac{1}{\sqrt{N}}$

WHY:

14 ~~17~~

MEAS. QTY S N TIMES

BEST VALUE: $S_{BEST} = \frac{1}{N} \sum_{i=1}^N S_i \equiv \frac{1}{N} X$

(i = # OF MEASMT.)

WHAT SHOULD QUOTE FOR ERROR?

~~$\Rightarrow \frac{1}{\sqrt{N}} \Delta^* S$~~

~~WHY?~~

HOW MUCH DOES S_{BEST} FLUCT?

$$\Delta^* S_{BEST} = \frac{1}{N} \Delta^* X = \frac{1}{N} \sqrt{N}\, \Delta^* S = \frac{1}{\sqrt{N}} \Delta^* S$$

ie: gives idea of what std dev. would be of repeated (N meas, then ave) large # times.

DISTRIB'N FOR S_{BEST}?

IF LARGE N $\Rightarrow$ GAUSSIAN

WORK 10 TIMES HARDER $\Rightarrow$ DO ~3 TIMES BETTER (ie more accurate)