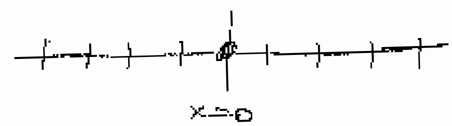
RANDOM WALK

– MODEL FOR PARTICLE(S) MOVING RANDOMLY THRU FLUID (DIFFUSION) (REIF gives other applications; tools are general)

1 DIM:

PARTICLE STARTS AT $X=0$



ASSUMPTIONS:

① GOES DISTANCE l BEFORE HITTING NEXT PARTICLE (≈ MEAN FREE PATH); REPEATS

② AFTER COLLISION, DIR. OF NEXT STEP INDEP. OF PREVIOUS ST

PROB NEXT STEP TO RT: p $\quad (0 < p < 1)$

" " " TO LT: $q = 1 - p$

(IF HOMOG: $p = q = \frac{1}{2}$; IF $p > q$ ~ FLUID FLOWS TO RT)

(treating w/ probability since don't want to get into details of each collision)

PROBLEM:

AFTER N STEPS, FIND PROB. $P(m)$ FOUND AT $X = ml$ (m AN INTEGER)

{CLASSIC FORMULATION: DRUNK AT LAMP POST ON SIDEWALK {eg $p=q=\frac{1}{2}$ the ave: $\bar{x}=0$ but std. $\neq 0$}}

PROBABILITY: IN LARGE ENSEMBLE OF IDENT SYSTEMS, AFTER N STEPS

$P(m) = (\#\text{ AT } ml)/N_{ENS}$.

REALIZATIONS:

(a) 1 PART. EACH IN N_{ENS} FLUIDS

(b) " " N_{ENS} TIMES IN 1 "

* (c) MANY PARTICLES IN 1 " ⇒ {ex DIFFUSION OF DYE, $P \propto$ DENSITY, CAN SEE DIRECTLY

skip {REIF: PROB. DEPENDS ON ENSEMBLE

ex prob seed yields red flower — diff of seed ∈ [tulips] vs seed ∈ [all plants, incl trees]

⇒ ie prob given the info or constraints on ensemble }

$\underline{p = q = \frac{1}{2}}$:

EXPECT:

- WILL USUALLY END CLOSE TO $x = 0$ (AND $\bar{x} = 0$ BY SYMM.)
- RARE FOR $m\ell = \pm N\ell$ (ESP FOR N LARGE) $\Rightarrow$ 1 WAY; BUT LOTS OF WAYS TO GET TO $x \sim 0$
- AVE DISTANCE $\neq 0$

$\underline{p > q}$: – ON AVE, TAKE MORE RT STEPS THAN LEFT $\Rightarrow$ MIGRATES TO RT

+ CAN GUESS: $\bar{x} = N(p-q)\ell$
(will confirm)

ex. $\underline{N = 3}$ (start simple)

m	ROUTE	PROB	
$+3$	$\overset{x=0}{\bullet}\xrightarrow{p}\xrightarrow{p}\xrightarrow{p}$	p^3	$\boxed{P(3) = p^3}$

just remind; use both here

TWO INDEPENDENT EVENTS A, B w/ PROB $P(A)$, $P(B)$

PROB BOTH HAPPEN: $P(A) \cdot P(B)$ $\quad (\leq P(A)$ OR $P(B))$

" EITHER " : $P(A) + P(B)$ $\quad (\geq P(A)$ OR $P(B))$

INDEP: VALUE GET FOR A HAS NO EFFECT ON B

ex 2 coins: prob of 2 heads $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
prob both same $\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

m	ROUTE	PROB	
$+2$	CAN'T DO IT	0	{ IF N ODD m MUST BE ODD }
$+1$	$\bullet\xrightarrow{p}\xrightarrow{p}$, $\xleftarrow{q}$	$p^2 q$	
	$\bullet\xrightarrow{p}$, $\xleftarrow{q}$, $\xrightarrow{p}$	pqp	
	$\xleftarrow{}\bullet$, $\xrightarrow{}\xrightarrow{}$	qp^2	TOTAL: $P(1) = 3p^2 q$

m	ROUTE	PROB
0	CAN'T	0
-1	ETC $\rightarrow$ JUST REPLACE $p \leftrightarrow q$	$\Rightarrow \boxed{P(-1) = 3pq^2}$
-3		q^3 $\quad \boxed{P(-3) = q^3}$

$\left\{ \text{ex } p = q = \tfrac{1}{2}: \quad \begin{array}{l} P(3) = P(-3) = \frac{1}{8} \\ P(1) = P(-1) = \frac{3}{8} \end{array} \right\}$

HAVE ALL CASES.

PROB THAT PARTICLE GOES ANYWHERE? 1

CHECK: $\sum_{m=-3}^{3} P(m) = 1$

$$\Rightarrow \quad p^3 + 3p^2 q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1 \checkmark$$

(any p, q)

IN GENERAL:

n_R, n_L = # RT, LT STEPS

$\boxed{\begin{array}{l} N = n_R + n_L \\ m = n_R - n_L = 2n_R - N \end{array}}$ OR $\boxed{\begin{array}{l} n_R = \frac{1}{2}(N+m) \\ n_L = \frac{1}{2}(N-m) \end{array}}$ put on side board

$n_{R,L} = 0, 1, 2 \ldots N$

N, m BOTH EVEN OR BOTH ODD

TYPICAL ROUTE TO m:

STEP	1	2	3					N
	R	R	L	R	L L		- -	R

$\left.\begin{array}{l} n_R \text{ R's} \\ n_L \text{ L's} \end{array}\right\}$

PROB FOR THIS SPECIFIC ROUTE $\quad p^{n_R} q^{n_L} = p^{\frac{1}{2}(N+m)} (1-p)^{\frac{1}{2}(N-m)}$

(note: if N, m fixed, so are n_R, n_L)

HOW MANY WITH THIS n_R & n_L ?

$\Rightarrow$ # WAYS TO ARRANGE n_R R'S & n_L L'S IN N SPACES / STEPS:

(1) ASSUME HAVE N DISTINCT OBJECTS | STEP $\underline{\ \ }_1$ $\underline{\ \ }_2$ $\underline{\ \ }_3$ . . . $\underline{\ \ }_N$

1ST : N CHOICES

2ND : $N-1$ " ETC { for each one of the N choices for 1ST

$\Rightarrow$ $N!$ WAYS TO ARRANGE

(2) WE OVER COUNTED:

NOW RECOGNIZE THAT R'S SAME, L'S SAME

R R L R L L L R R

SAME IF REARRANGE R'S AMONG SELVES; ALSO L'S

$n_R!$ WAYS TO ARRANGE R'S (counting same way)

$n_L!$ WAYS TO " L'S

$\Rightarrow$ NUMBER OF DISTINCT ARRANGEMENTS WITH n_R R'S & n_L L'S

$$\frac{N!}{n_L!\, n_R!} = \frac{N!}{(N-n_R)!\, n_R!} \equiv \binom{N}{n_R}$$

"BINOMIAL COEFFICIENT"

$$= \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\left[\frac{1}{2}(N-m)\right]!}$$

NOTE :

BINOMIAL EXPANSION:

$$(x+y)^N = \binom{N}{0} x^N + \binom{N}{1} x^{N-1} y^1 + \binom{N}{2} x^{N-2} y^2 + \cdots \binom{N}{N-1} x\, y^{N-1} + \binom{N}{N} y^N$$

$\{0! \equiv 1\}$

$$= \sum_{n=0}^{N} \binom{N}{n} x^{N-n} y^n$$

WHY ?

ex $(x+y)^3 = (x+y)(x+y)(x+y)$

$$= x^3 + \underbrace{(x^2y + xyx + yx^2)}_{3x^2y} + \cdots$$

ALL POSSIBLE TERMS W/ 2 x's AND 1 y ; $xxy + xyx + \cdots$

SAME COUNTING

FINALLY

$$\boxed{P(m) = \frac{N!}{\left[\frac{1}{2}(N+m)\right]!\left[\frac{1}{2}(N-m)\right]!}\, p^{\frac{1}{2}(N+m)}\, q^{\frac{1}{2}(N-m)}}$$

BINOMIAL DISTRIBUTION

- $q \equiv 1-p$
- m & N BOTH ODD OR BOTH EVEN ; ELSE 0

CHECK : q SMALL : $P(m)$ SMALL UNLESS $N \sim +m$

BINOM. COEFFS LARGEST WHEN $m \sim 0$

→ PICTURES $n_R \sim n_L \sim N/2$

* CAN ALSO EXPRESS AS PROB OF n_R RT. STEPS :

$$\boxed{W(n_R) \equiv P(m) = \frac{N!}{n_R!\,(N-n_R)!}\, p^{n_R}\, q^{(N-n_R)}}$$

(same info)

$$\sum_{m=-N}^{N} P(m) \stackrel{?}{=} 1$$

EASIEST IF USE FORM

$$W(n_R) = \frac{N!}{(N-n_R)!\, n_R!}\, p^{n_R} q^{N-n_R}$$

don't re-write if don't have to

skip (WITH $q = 1-p$ $\quad n_R + n_L = N$
$n_R - n_L = m$)

$$\sum_{n_R=0}^{N} W(n_R) = \sum_{n_R=0}^{N} \binom{N}{n_R} p^{n_R} q^{N-n_R}$$

$$= (p+q)^N = 1$$

skip WILL SEE SEVERAL TRICKS FOR SUMMING SERIES.
cf GRADSHTEYN & RYZHIK ; SYMBOLIC PROGRAMS ~ MATHEMATICA
MAPLE

skip *ex* USEFUL FOR PROB 1.5.c (cf A.1)

$$f(x) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \dots x^N$$

$$x f(x) = x + x^2 + \dots + x^N + x^{N+1}$$

SUBTR

$$(1-x) f(x) = 1 - x^{N+1}$$

$$f(x) = \frac{1 - x^{N+1}}{1-x}$$

(converges if $|x|<1$)

$N = \infty \Rightarrow 1 + x + x^2 + \dots = \frac{1}{1-x}$

also $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$

skip SPECIAL CASE: $\sum_{n=0}^{N} e^{ny} = \sum_{n=0}^{N} (\underbrace{e^y}_{\equiv x})^n \Rightarrow$ SAME AS ABOVE

$$= \frac{1 - e^{(N+1)y}}{1 - e^y}$$

1.5.1

$N = 20$
$p = q = \frac{1}{2}$

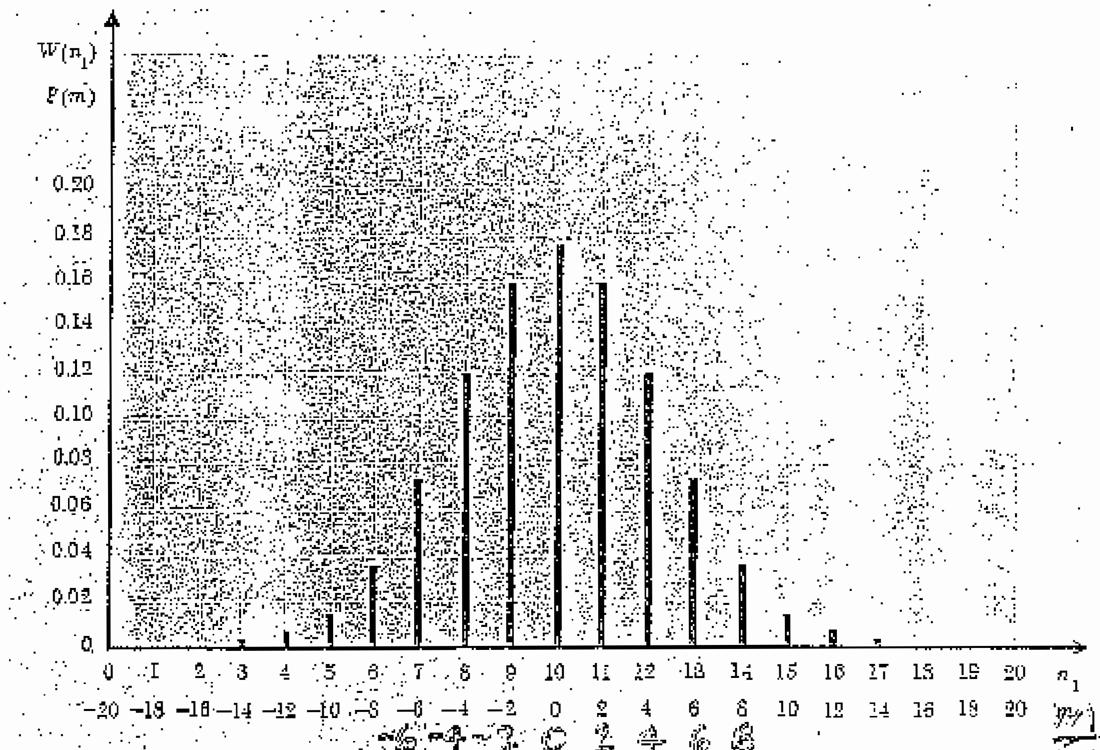
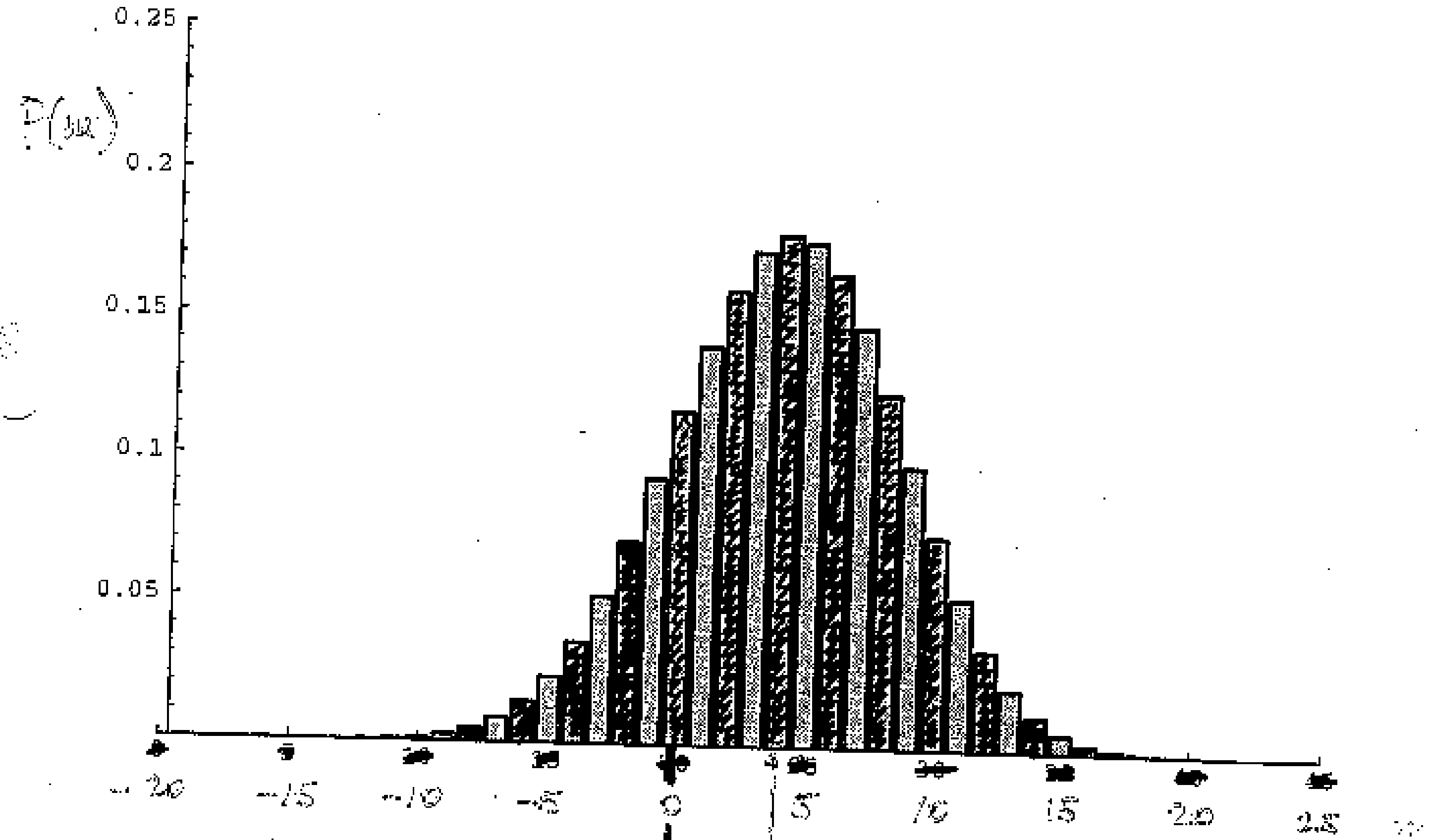


Fig. 1·2·3 Binomial probability distribution for $p = q = \frac{1}{2}$ when $N = 20$ steps. The graph shows the probability $W_N(n_1)$ of n_1 right steps, or equivalently the probability $P_N(m)$ of a net displacement of m units to the right.

$N = 20$

$p = .6 \quad q = .4$

(ONLY GOOD FOR m EVEN)



$\bar{m} = N(p-q) = 4$

$\sqrt{\overline{(\Delta m)^2}} = \sqrt{4pqN}$

COMPUTE SOME AVES w/ RAN WALK:

(EASIEST TO WORK w/ $W(n_R)$ $\Rightarrow$ DON'T HAVE TO WORRY ABOUT EVEN/ODD)

(1) HAVE $\sum_{n_R=0}^{N} W(n_R) = 1$

(2) $$\overline{n_R} = \sum_{n_R} W(n_R)\, n_R = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} \cdot n_R$$

TRICK:

(a) TREAT p, q AS INDEP. ; ie $\overline{n_R}(p,q)$ [ie $q \neq 1-p$ yet]

(b) NOTE $p \frac{\partial}{\partial p} p^{n_R} = n_R\, p^{n_R}$

(c) $$\overline{n_R}(p,q) = \sum_{n_R} \binom{N}{n_R} \left(p \frac{\partial}{\partial p} p^{n_R}\right) q^{N-n_R}$$

$$= p \frac{\partial}{\partial p} \Big\{ \underbrace{\sum_{n_R} \binom{N}{n_R} p^{n_R} q^{N-n_R}}_{(p+q)^N} \Big\}$$

CAN NOW SUM :

DERIV: $p N (p+q)^{N-1}$

$$\Rightarrow \overline{n_R}(p,q) = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} \underset{\text{q skip}}{n_R} = p N (p+q)^{N-1}$$

true for any p, q

(d) NOW TAKE $q = 1-p$

$$\overline{n_R} = \overline{n_R}(p,q)\Big|_{q=1-p} = Np$$

(YOU ~~WOULD~~ DID GUESS THIS)

IN SAME WAY $\boxed{\overline{n_R} = Np}$ $\boxed{\overline{n_L} = Nq}$ $\Rightarrow$ $\boxed{\overline{m} = \overline{n_R} - \overline{n_L} = N(p-q)}$ ($= n_R - n_L$)

(w/ $q = 1-p$)

(3) SPREAD : (a VARIANCE)

$$\overline{(\Delta m)^2} \underset{= (\Delta^* m)^2}{\overset{\checkmark}{=}} \overline{(m - \bar{m})^2} = \overline{m^2} - \bar{m}^2$$

DO USING n_R, $W(n_R)$

$$m = n_R - n_L = 2n_R - N \qquad \text{(on side ld)}$$

$$\Delta m = m - \bar{m} = (2n_R - N) - (2\bar{n}_R - N)$$

$$= 2(n_R - \bar{n}_R) = 2\Delta n_R$$

$$\Rightarrow \overline{(\Delta m)^2} = 4\overline{(\Delta n_R)^2} = 4(\overline{n_R^2} - \underbrace{\bar{n}_R^2}_{\hat{=} N^2 p^2})$$

USE SAME TRICK:

$$\overline{n_R^2} = \sum_{n_R} W(n_R)\, n_R^2 = \sum_{n_R} \binom{N}{n_R} p^{n_R} q^{N-n_R} n_R^2$$

$$= \left(p\frac{\partial}{\partial p}\right)\left(p\frac{\partial}{\partial p}\right)(p+q)^N \Big|_{q=1-p}$$

(SOME ALGEBRA, AFTER $q \equiv 1-p$)

$$\overline{n_R^2} = N^2p^2 + Npq$$

$$\overline{(\Delta m)^2} = 4(\overline{n_R^2} - \bar{n}_R^2) = 4Npq$$

$$\Delta^* m \equiv \sqrt{\overline{\Delta m^2}} = \sqrt{4Npq}$$

(can see them on previous picture)

$$\rightarrow \sqrt{N} \quad \text{IF } p = q = \tfrac{1}{2}$$

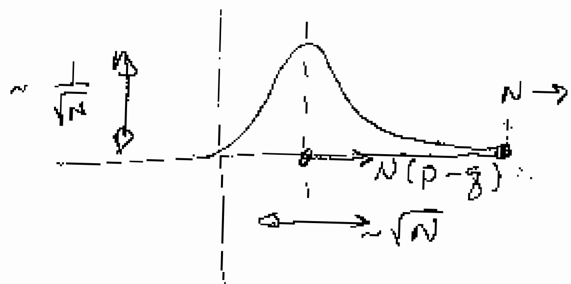
SOME GENERAL FEATURES: vs N

{ ALREADY: $\Delta^* m = \sqrt{4Npq}$ $\Delta^* n_R = \sqrt{Npq}$

PEAK (will see later)

$$P(m_{MAX}) \simeq \frac{1}{(2\pi Npq)^{1/2}} \xrightarrow[p=q=\frac{1}{2}]{\text{why?}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\left(\frac{1}{N}\right)^{\frac{1}{2}}$$

DISTRIBUTION:



(picture)

SHRINKS & SPREADS, BUT AREA CONST (why? prob dist.)

save for moving plot { (WILL SEE IN HW, CAN THINK OF AS DESCRIBING t DEVELOPMENT OF DIFFUSION PROCESS)

(THERE $N\ell = vt$
ℓ ave v)

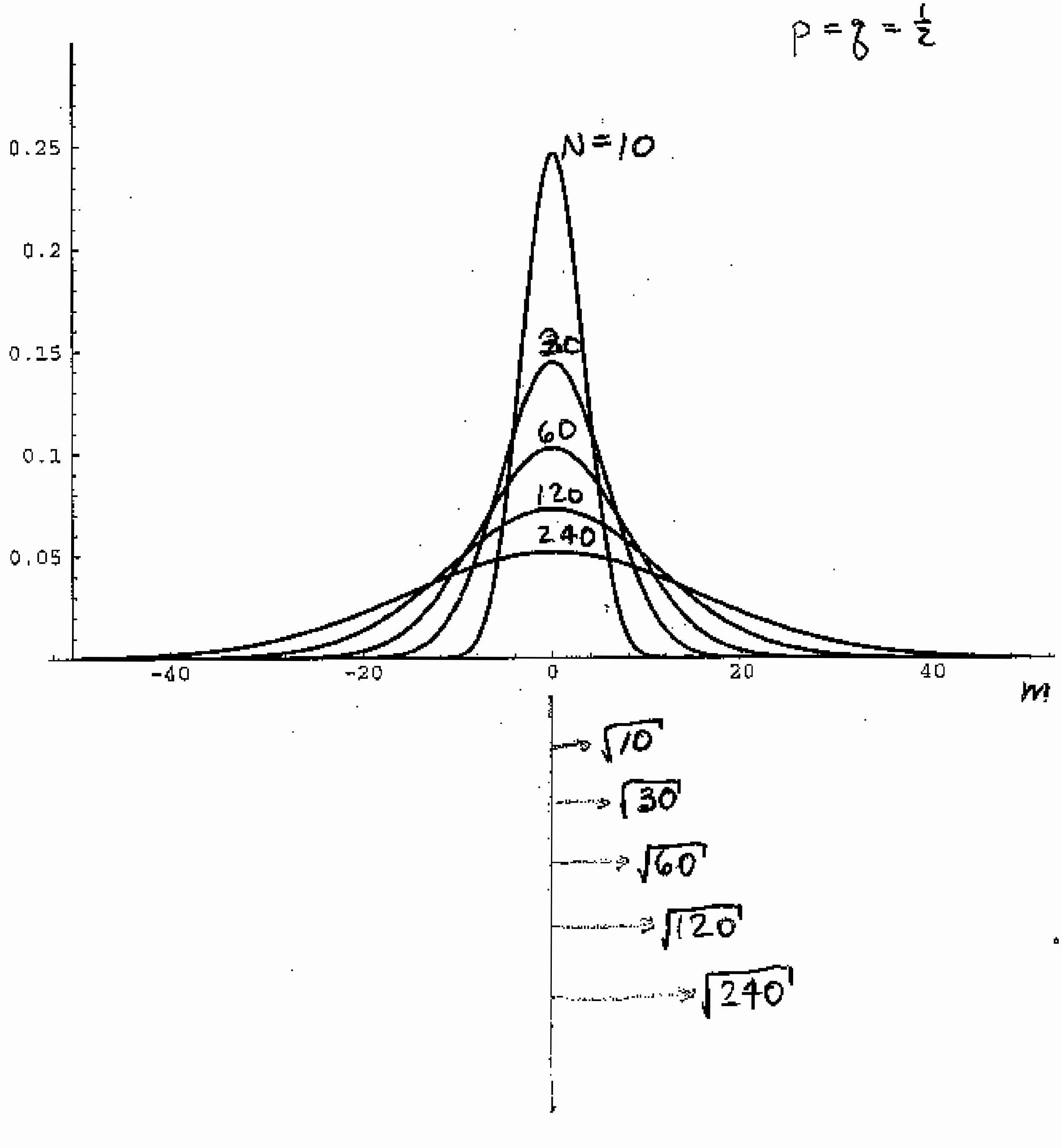
IN TERMS OF n_R & $W(n_R)$:

$$\overline{n_R} = Np$$

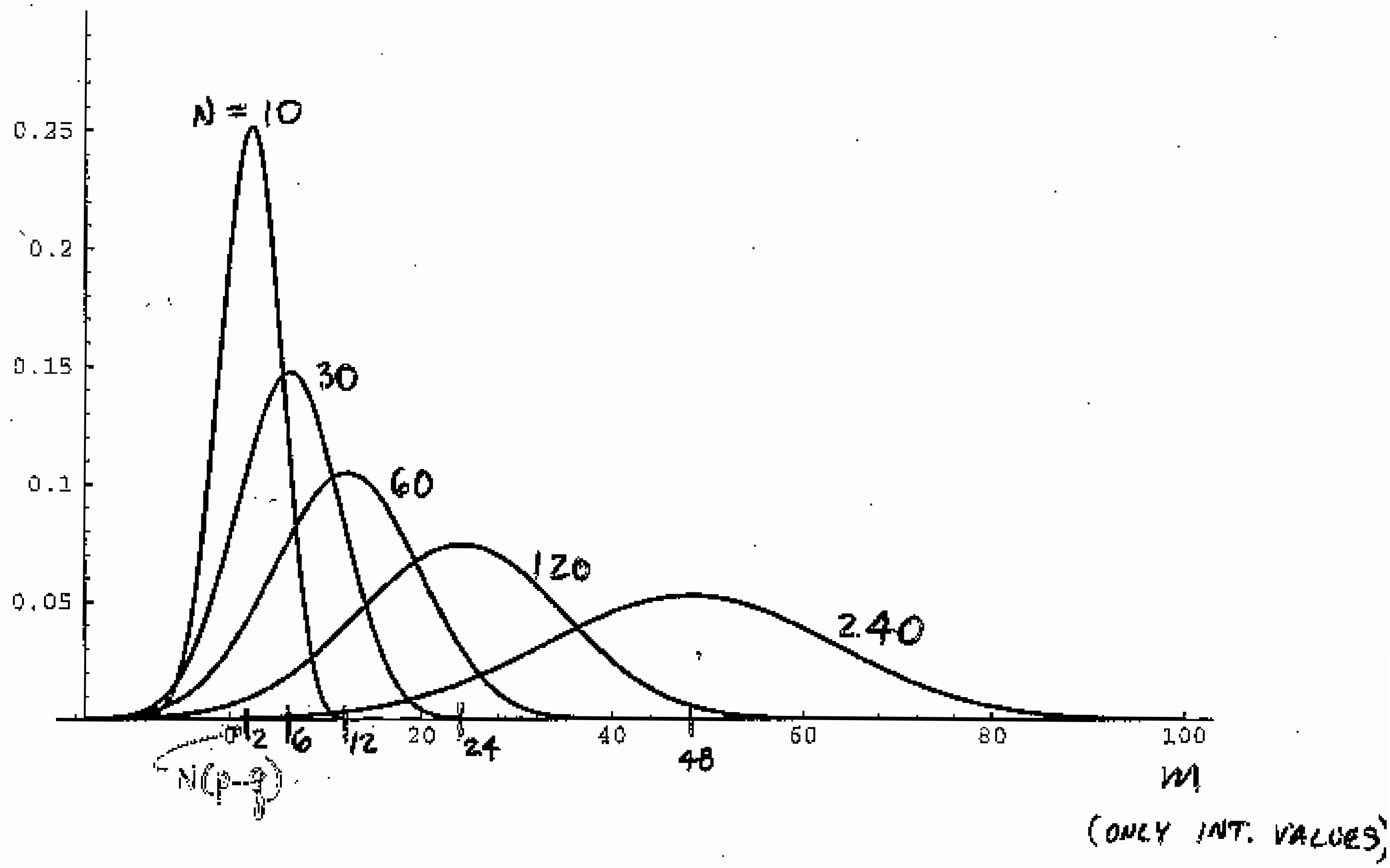
$$\Delta^* n_R = \frac{1}{2}\Delta^* m = \sqrt{Npq}$$

RELATIVE UNCERTAINTY: $$\frac{\Delta^* n_R}{\overline{n_R}} = \frac{1}{\sqrt{N}}\left(\frac{q}{p}\right)^{\frac{1}{2}}$$

SHRINKS w/ N $\Rightarrow$ % ACCURACY INCREASES
(ex $p=q=\frac{1}{2}$ $\overline{n_R}$ MORE FOCUSED AT $N/2$)


p = q = 1/2
N=10
30
60
120
240
0.25
0.2
0.15
0.1
0.05
-40
-20
0
20
40
m
√10
√30
√60
√120
√240

$p = .6 \quad q = .4$



N = 10
30
60
120
240
0.25
0.2
0.15
0.1
0.05
0
20
40
60
80
100
2
6
12
24
48
N(p-q)
M
(ONLY INT. VALUES)

CONTINUOUS APPROX { will work very hard to get only an approx soln from an exact soln   a. 1.

USEFUL FOR N LARGE

ex MOLECULES IN FLUID: $x \sim$ MACRO DIST ~ 1 cm

$\ell \sim$ ATOMIC " $\sim 1 \AA \sim 10^{-10}$ m $\ll x$

$\bar{x} = \bar{m}\ell = N(p-q)\ell \Rightarrow N \sim 10^{8}$ COLLISIONS BEFORE PEAK MOVES 1cm

INTERESTING LIMIT:

$\bar{x} \sim N \ell$ ← SMALL (ℓ), LARGE (N)

$\bar{x}$: REASONABLE (ie MACRO)

NOTE:

$P(m)$ (cf $N = 240$ PLOT)

(1) $P(m) \rightarrow$ SMOOTH (ie $\Delta P \ll P$) { jumps small rel. to what?

(2) FOR MACRO MEAS, UNCERTAINTY IN x IS $\gg \ell$

$\Rightarrow$ AVE. SOME x MEASMNTS: SOMETIMES FEW MORE, SOMETIMES LESS ℓ's

$\Rightarrow$ SMOOTHS OUT

$\Rightarrow$ MORE LIKELY MEAS. DENSITY IN $x \Rightarrow$ ALSO " "

$\Rightarrow$ UNCERT. IN $P \gg$ JUMPS

$\Rightarrow$ CONTINUOUS APPROX. SHOULD WORK OK

(also, this is a model; if meas. are acc. enough to see atoms, better treat it microscopically if more detailed approx; contin approx should be as good as discrete model)

REIF NOTATION:

dx = INFINITESIMAL BUT MACROSCOPIC

δx = SHORTEST (ATOMIC) DIST.

USUALLY $x \gg dx \gg \delta x$

(limit $dx \to 0$ really means make it small enough that $f(x)$ doesn't change much over dx)

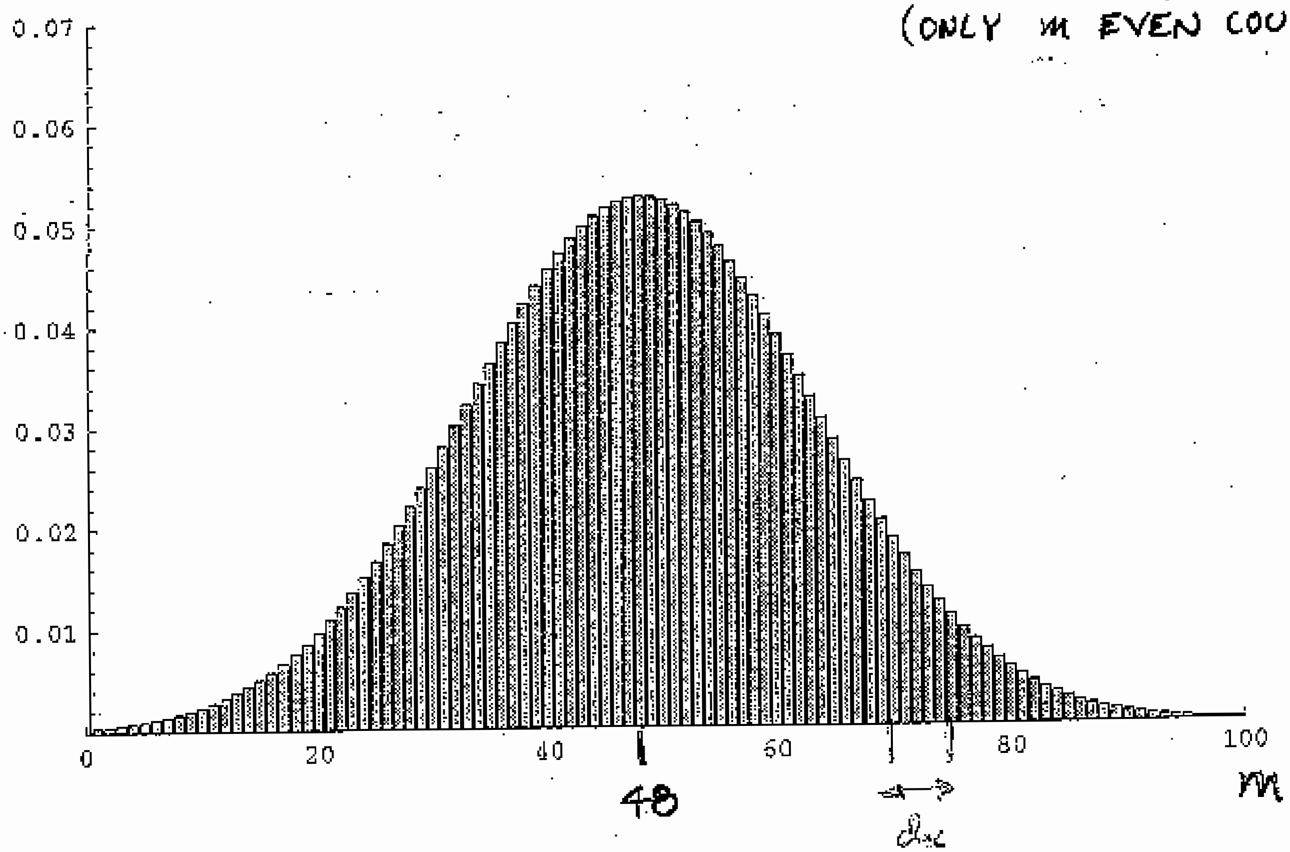
should only have values of 2's, or use W(n,r)

P(m)

$N = 240$

$p = .6 \quad q = .4$

(ONLY m EVEN COUNT)



DEFINE CONTINUOUS DISTR:

$x_m \equiv m\ell \qquad m = -N, -N+2, \ldots, N-2, N$

(recall m even or odd)

PROB DENSITY

$$\mathcal{P}(x_m) = \lim_{\ell \to 0} \frac{P(m)}{\underbrace{2\ell}_{\text{SMALLEST } \Delta x}}$$

PROBLEM:

P(m) HAS FACTORIALS:

- AWKWARD TO USE AS $N \to \infty$
- ONLY DEF'D FOR INTEGER VALUES: $\ell \to 0$ LIMIT PAINFUL

SOL'N:

REPLACE P(m) W/ SIMPLE CONTINUOUS APPROX:

WILL SEE:

EASY TO $\int$

CAN READ OFF $\bar{x}$, $\Delta^* x$

UNIVERSAL ⇒ GOOD APPROX FOR ALMOST ANY LARGE SUM OF RANDOM VARS

⇒ GAUSSIAN

PROBLEM:

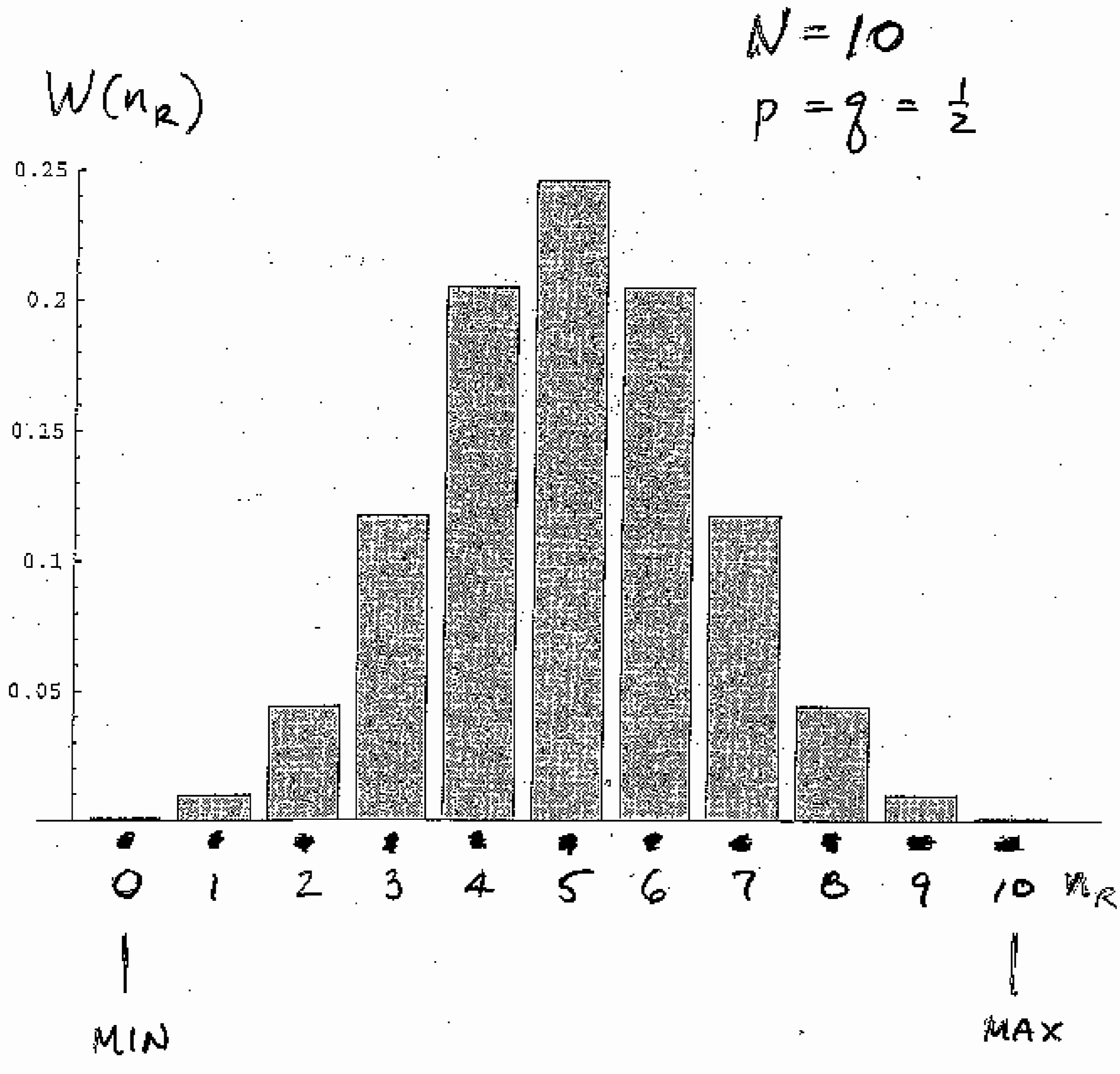
P(m) BECOMES SMOOTH BUT SHARPLY PEAKED:

$\Delta^* x \sim N^{1/2}\ell \qquad \bar{x} \sim N\ell$

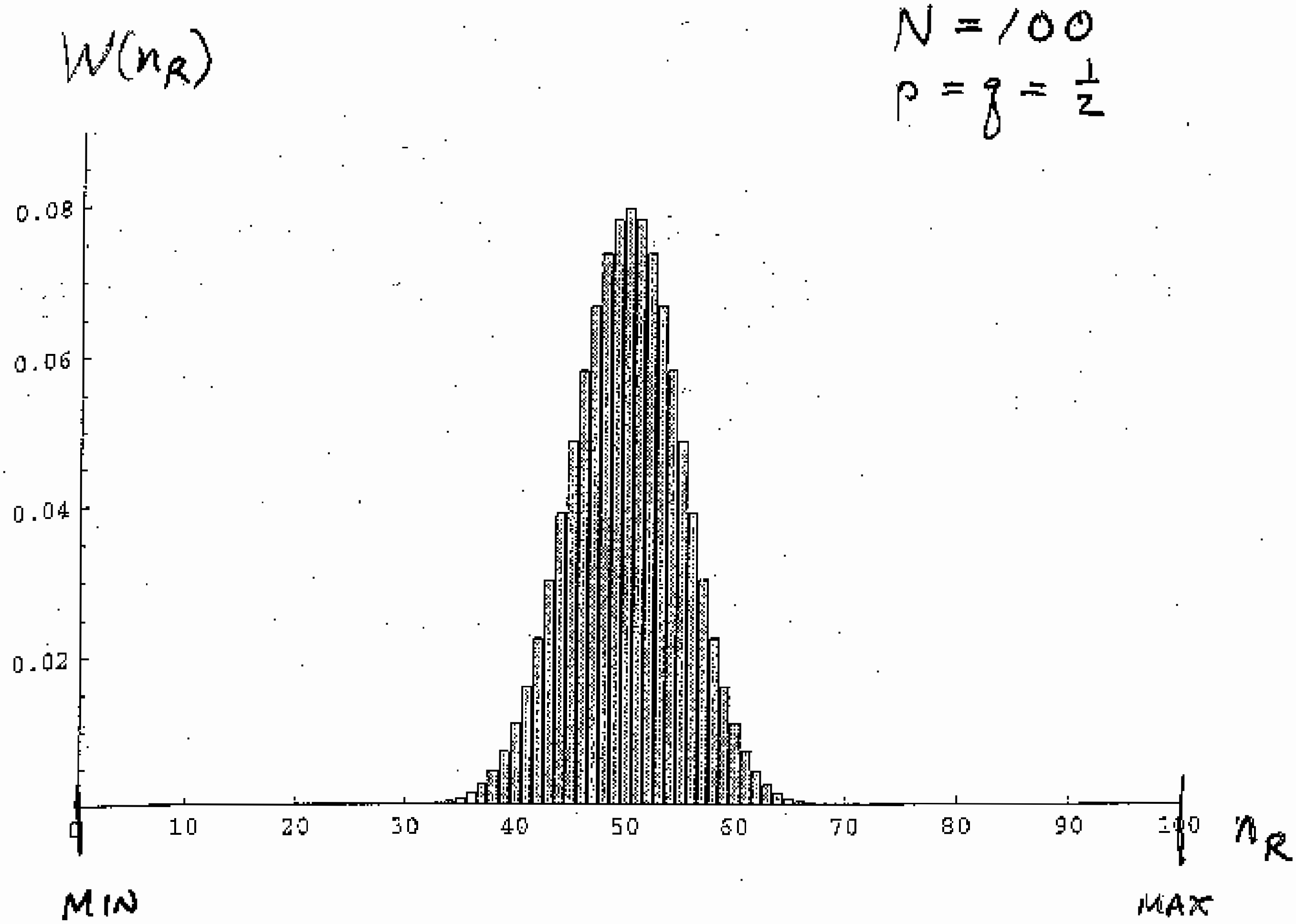
$$\frac{\Delta^* x}{\bar{x}} \sim \frac{1}{N^{1/2}}$$

(fn sits at 0 for most x, jumps quickly near peak)

cf plots


$N = 10$
$p = q = \frac{1}{2}$
$W(n_R)$
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7
8
9
10
n_R
MIN
MAX

$N = 100$
$p = q = \frac{1}{2}$



FN IS EXPONENTIAL:

POWERS ARE EXPONENTIAL: eg $p^{n_R} = e^{n_R \ln p}$

LARGE FACTORIALS $\approx$ EXPONENTIALS

(try 100! on calculator)

STIRLING APPROX (cf APPX)

$$n! \approx e^{n \ln n - n + \frac{1}{2}\ln(2\pi n) + \cdots}$$

terms which vanish as $n \to \infty$

$$= (2\pi n)^{\frac{1}{2}} e^{n(\ln n - 1)}$$

(uses method we'll talk about now)
"leading term in asymptotic expression"

ex $100! \sim e^{364}$

PLAN: TO FIND SIMPLE APPROX

(1) GOOD NEWS:

$P(x)$ ONLY $\gtrsim 0$ NEAR PEAK $\tilde{x}$

$\Rightarrow$ ONLY NEED GOOD APPROX NEAR MAX

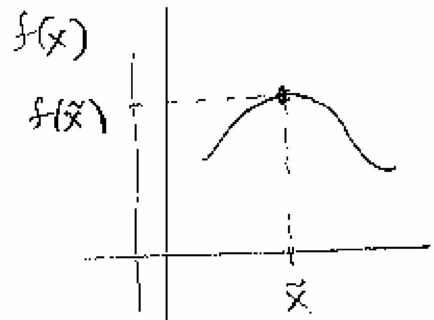
$\Rightarrow$ TAYLOR SERIES

(2) BAD NEWS:

$P(x)$ DROPS EXPONENTIALLY

$\Rightarrow$ BUILD INTO SERIES

SIMPLE
APPROX. NEAR PEAK:



TAYLOR:

$$f(x) \sim f(\tilde{x}) + \underbrace{f'(\tilde{x})}_{=0 \text{ (EXTREMUM)}}(x-\tilde{x}) + \underbrace{\tfrac{1}{2}f''(\tilde{x})}_{\leq 0 \text{ (CURVES DN)}}(x-\tilde{x})^2 + \cdots$$

DEFINE $\eta \equiv x - \tilde{x}$

(SO EXPAND AROUND $\eta = 0$)

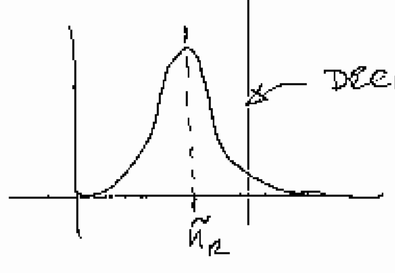
$$f(x) \equiv f(\tilde{x} + \eta) \approx f(\tilde{x}) + \tfrac{1}{2} f''(\tilde{x})\,\eta^2 + \cdots$$

$$= f(\tilde{x}) - \tfrac{1}{2} |f''(\tilde{x})|\,\eta^2 + \cdots$$

ACCURATE FOR η SMALL
TEST: NEW TERMS GET SMALLER

OUR CASE

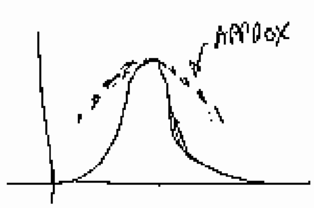
$W(n_R)$



DECREASE EXPONENTIALLY $\sim e^{-g(n_R)}$

TAYLOR SERIES: NO POLYNOMIAL CAN KEEP UP WITH EXPONENTIAL FOR LONG ⇒ ONLY GOOD FOR SMALL RANGE IN $\eta \equiv n_R - \tilde{n}_R$

⇒ GET



BE SMART:

KNOW $W(n_R) \sim e^{-g(n_R)}$

APPROX g, NOT W (builds in exp behavior)

ie $W(n_R) = e^{\underbrace{\ln W(n_R)}}$

→ MUCH SMOOTHER FN

→ EASIER TO APPROX

TAYLOR SERIES: (IN $\eta \equiv n_R - \tilde{n}_R$)

$$\ln W(n_R) \approx \ln W(\tilde{n}_R) + B_1 \eta + \frac{1}{2} B_2 \eta^2 + \frac{1}{3!} B_3 \eta^3 + \cdots$$

$$\boxed{B_k = \frac{d^k(\ln W(\tilde{n}_R))}{d n_R^k}}$$

KNOW

$B_1 = 0$

$B_2 < 0$

THEN

$$W(n_R) \approx e^{\left[\ln W(\tilde{n}_R) - \frac{1}{2}|B_2|\eta^2 + \cdots\right]}$$

$$\approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

"GAUSSIAN DISTR."

→ WILL CLEAN UP LATER

⇒ VERY GENERAL:

→ ONLY ASSUMED EXP. FALL OFF

→ VERY COMMON WHEN ADD LARGE # N OF RANDOM ~~STATIST.~~ VARIABLES (HERE N STEPS OF $s_i = \pm l$ WITH PROB p, q)

("CENTRAL LIMIT THM" → ALMOST ANY PROB FOR INDIV. s_i GIVES THIS FORM)

RANGE OF VALIDITY:

- GOOD APPROX FOR $\ln W$ IF

$$\left| \frac{1}{3!} B_3 \eta^3 \right| \ll \left| \frac{1}{2} B_2 \eta^2 \right|$$

it fails for large enough η

- WILL SEE: WHEN " $\sim$ "
  W NEGLIGIBLE

$\Rightarrow$ APPROX GOOD FOR ALL η

ex (REIF):

$$f(y) = \frac{1}{(1+y)^N} = e^{-N\ln(1+y)}$$

{ powers, exponentials ~ same

→ CHANGES SHARPLY IF N LARGE } see picture
→ $\ln f(y)$ MUCH SMOOTHER }

APPROX NEAR $y = 0$

DIRECTLY: $f(y) \sim 1 - Ny + \frac{1}{2}N(N+1)y^2 + \cdots$

N large → NOT SENSIBLE UNLESS $|Ny| \lesssim 1$

OR $|y| \lesssim 1/N$

LOG: $\ln f(y) \sim -N(y - \frac{1}{2}y^2 + \cdots)$

OK IF $y^2 \lesssim y$ OR $y \lesssim 1$ (indep of N)

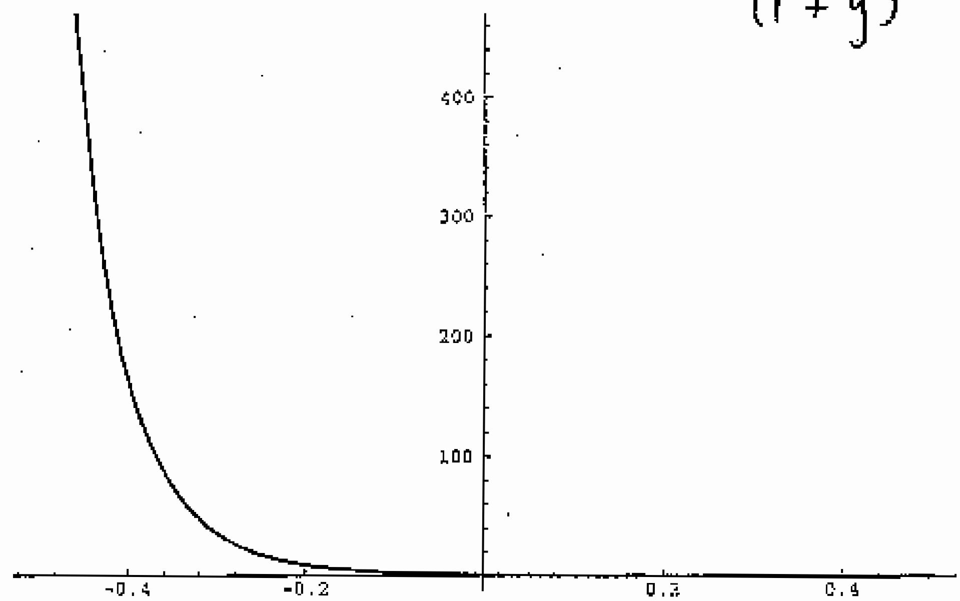
$$f(y) \sim e^{-N(y - \frac{1}{2}y^2 + \cdots)}$$

⇒ picture:

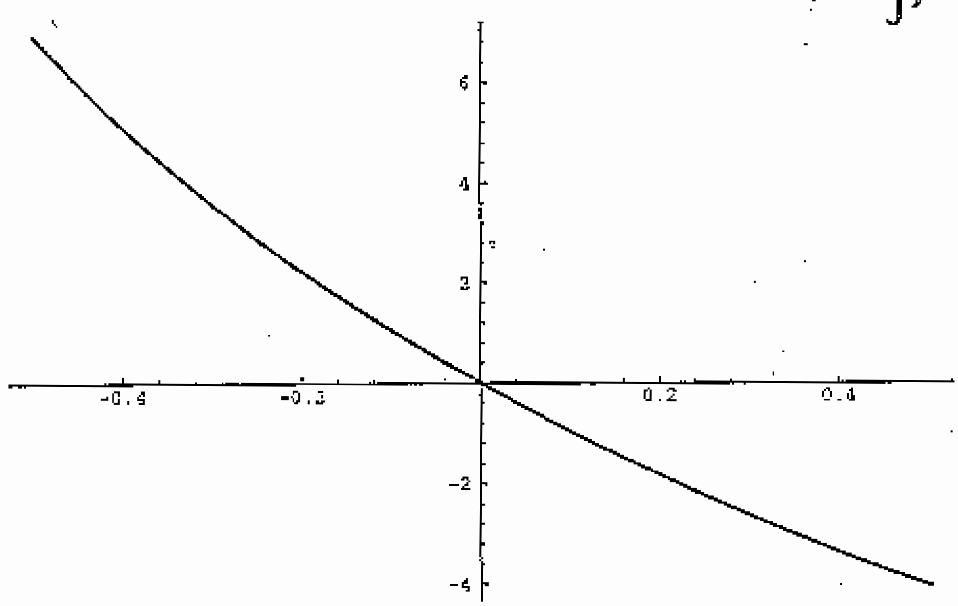
$N = 10$

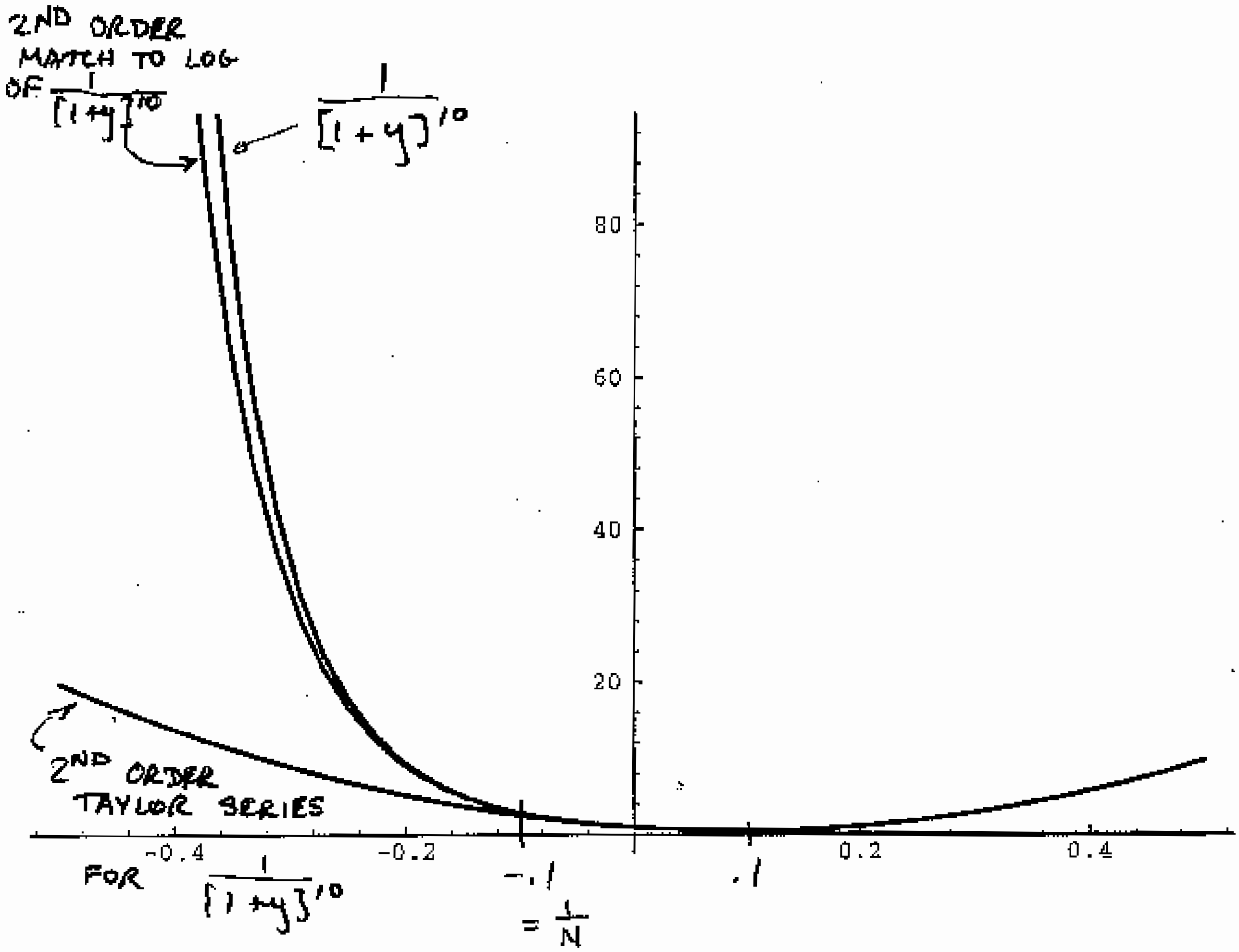
1^{st} pict dies at $y \sim \pm \frac{1}{10} = \pm .1$

$$\frac{1}{(1+y)^{10}}$$



$$\ln\left[\frac{1}{(1+y)^{10}}\right]$$





2ND ORDER MATCH TO LOG OF $\frac{1}{[1+y]^{10}}$
$\frac{1}{[1+y]^{10}}$
80
60
40
20
2ND ORDER TAYLOR SERIES FOR $\frac{1}{[1+y]^{10}}$
-0.4
-0.2
0.2
0.4
$-.1 = \frac{1}{N}$
$.1$

APPROX $\ln W(n_R)$: (cf REIF FOR DETAILS (or try yourself))

PROBLEM: NEED DERIVS

ONLY DEF'D AT INTEGER n_R'S

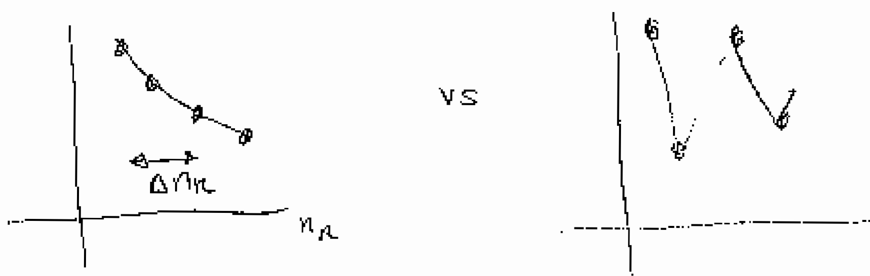
BEST APPROX:

$$\frac{d \ln W(n_R)}{d n_R} \approx \frac{\Delta \ln W(n_R)}{\Delta n_R}$$

SMALLEST $\Delta n_R = 1$

$$= \ln W(n_R+1) - \ln W(n_R)$$

OK IF $\ln W(n_R)$ SMOOTH:



(seen in plot that W becomes smoother as N incr.)

(a) MAX: $\frac{d \ln W}{d n_R} = 0$ (*)

SOLN: $\boxed{\tilde{n}_R = N_p}$ (note: same as $\overline{n_R}$)

(*) NEED THINGS LIKE

$$\frac{\Delta \ln n!}{\Delta n} \sim \frac{\ln (n+1)! - \ln n!}{1} = \ln \left[\frac{(n+1)!}{n!}\right] = \ln(n+1) \sim \ln n$$

$\{$ or $\ln(n+1) + \ln n) + \ln(n-1) + \cdots$
$- \ln(n) - \ln(n-1) + \cdots \}$

so $n \gg 1 \Rightarrow \frac{d \ln n!}{dn} \sim \ln n$ ETC

APPROX $\ln W(n_R)$: (cf REIF FOR DETAILS or better, try yourself)

FIND $\tilde{n}_R$ AT MAX:

(a) $$\frac{d \ln W(n_R)}{d n_R} = 0 \approx \ln W(n_R+1) - \ln W(n_R)$$

SOLN $$\boxed{\tilde{n}_R = Np}$$

(b) EXPAND AROUND $\tilde{n}_R$ $(\eta = n_R - \tilde{n}_R)$

COEFFICIENTS:

$B_1 = 0$

$\boxed{B_2 = -\frac{1}{Npq}}$ } AS EXPECTED, <0

$B_3 = \frac{q^2 - p^2}{N^2 p^2 q^2}$ ETC

EACH NEW TERM $B_k \eta^k$ HAS ADDITIONAL FACTOR $\frac{\eta}{Npq}$ TIMES PREVIOUS; SERIES OK IF $\frac{\eta}{Npq} \ll 1$ skip; done later

skip (N BIG; p, q NOT TOO SMALL, CF PROB 1.9)
(and when $\eta \sim Npq$, $W \sim 0$)

(c) $$W(\tilde{n}_R) \cong \frac{1}{(2\pi Npq)^{1/2}}$$ (STIRL. FORMULA) (~~1st lecture~~, APPX) (could also use norm cond)

FINALLY $$W(n_R) = W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

$$= \frac{1}{(2\pi Npq)^{1/2}} e^{-\frac{(n_R - Np)^2}{2Npq}} + \left[\text{CORRECTIONS WHICH VANISH AS } N \text{ LARGE}\right]$$

(using (a) & (b) above)

{ *for me:*

$W(\tilde{n}_R)$ using Stirling's formula:

$$W(n_R) = \frac{N!}{(N-n_R)!\, n_R!}\, p^{n_R} q^{N-n_R}$$

$$\tilde{n}_R = Np$$

$$W(\tilde{n}_R) = \frac{N!}{(N-Np)!\,(Np)!}\, p^{Np} q^{N-Np}$$

$$= \frac{N!}{(Nq)!\,(Np)!}\, p^{Np} q^{Nq}$$

$$\ln W(\tilde{n}_R) = \ln N! - \ln(Nq)! - \ln(Np)! + Np \ln p + Nq \ln q$$

Stirling:

$$\ln n! \underset{n \text{ large}}{\sim} n \ln n - n + \tfrac{1}{2}\ln(2\pi n) + \cdots$$

$$\begin{aligned}\ln W(\tilde{n}_R) \sim\ & N \ln N - N + \tfrac{1}{2}\ln(2\pi N) \\ & - Nq \ln Nq + Nq - \tfrac{1}{2}\ln(2\pi Nq) \\ & - Np \ln Np + Np - \tfrac{1}{2}\ln(2\pi Np) \\ & + Np \ln p + Nq \ln q\end{aligned}$$

from $p+q=1 \Rightarrow -Nq \ln Nq - Np \ln Np = \underbrace{-Nq \ln N - Np \ln N}_{-N \ln N} - Nq \ln q - Np \ln p$

everything cancels but

$$\tfrac{1}{2}\left[\ln(2\pi N) - \ln(2\pi Nq) - \ln(2\pi Np)\right] = \tfrac{1}{2}\ln\frac{2\pi N}{(2\pi Nq)(2\pi Np)}$$

$$= \tfrac{1}{2}\ln\frac{1}{2\pi Npq}$$

$$\Rightarrow W(\tilde{n}_R) = \left(\frac{1}{2\pi Npq}\right)^{\frac{1}{2}}$$

}

THEN $W(n_R) \approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$

USE $W(\tilde{n}_R) \cong \dfrac{1}{(2\pi Npq)^{1/2}}$ (STIRLING'S FORMULA)

$|B_2| = \dfrac{1}{Npq}$

AND $\eta \equiv n_R - \tilde{n}_R = n_R - Np$

$$W(n_R) = \frac{1}{(2\pi Npq)^{1/2}}\, e^{-\frac{(n_R - Np)^2}{2Npq}} + \text{CORRECTIONS} \sim \frac{1}{N}$$

CONVERT TO $\mathcal{P}(x) = \dfrac{P(m)}{2l}$

USE $n_R = \frac{1}{2}(N+m)$, $x = ml$

ALSO
$$\boxed{\begin{aligned} \mu &\equiv (p-q)Nl \\ \sigma^2 &\equiv 4Npq\,l^2 \end{aligned}}$$
(just defns, but we'll see $\mu = \bar{x}$, $\sigma^2 = \overline{(\Delta x)^2}$
(put on side)

THEN
$$\boxed{\mathcal{P}(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}}\, e^{-(x-\mu)^2/2\sigma^2}}$$

GAUSSIAN DISTRIBUTION
(discuss – peaked at $x \sim \mu$, falls off for $|x-\mu| > \sigma$)

skip
→ ALMOST ALWAYS GET THIS WHEN ADD LARGE # RAND. VARS.
(CENTRAL LIMIT THM — cf * SECTIONS FOR PROOF)

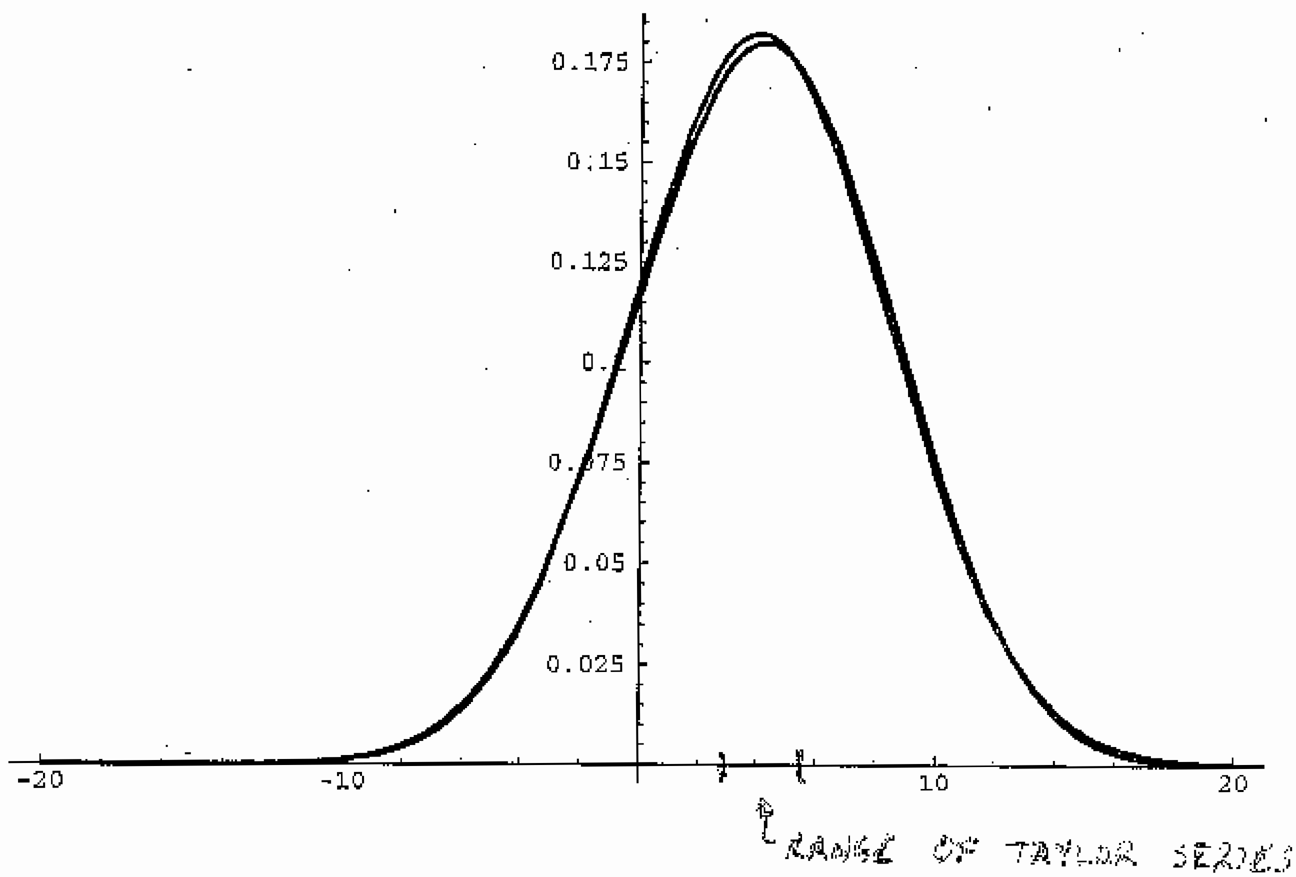
→ μ & σ HAVE SIMPLE INTERPRETATION

incl plot

N=20 p=.6

GAUSSIAN VS BINOMIAL DISTRIBUTIONS

(PLOTTED AS CONTINUOUS FNS)



CHECK NORMALIZATION:

$$\int_{x_{MIN}}^{x_{MAX}} P(x)\,dx$$

LIMITS: $\pm N\ell$

$P(x) \approx 0$

REPLACE w/ $\pm\infty$ (only affects by terms which vanish exp. w/ N)

NEED

$$\int_{-\infty}^{\infty} P(x)\,dx = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \underbrace{\int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2}\,dx}_{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\,dy}$$

$y \equiv x - \mu$

2 TRICKS: DO ALL GAUSS. INTS (one reason they're so popular - ints are easy)

CONSIDER $I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha z^2}\,dz$

COULD DO IF HAD $z\,dz$

$$I^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha z_1^2}\,dz_1 \int_{-\infty}^{\infty} e^{-\alpha z_2^2}\,dz_2$$

$$= \int_{-\infty}^{\infty} dz_1\,dz_2\; e^{-\alpha(z_1^2 + z_2^2)}$$

(ONLY DISCUSS IF NEED IMMEDIATELY FOR PROB 15)

INSERT: (put this here to summarize for HW; can drop next time)

NORM:

$$\int_{x_{MIN} = -N\ell}^{x_{MAX} = +N\ell} \underbrace{\mathcal{P}(x)}_{\sim 0 \text{ long before MIN, MAX}} dx$$

$$\simeq \int_{-\infty}^{\infty} \mathcal{P}(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2} dx$$

CAN SEE $= 1$ (later?)

PROB. $x < y$?

$$\int_{-\infty}^{y} \mathcal{P}(x)\, dx \Rightarrow \text{ERROR FN, APPX A.5}$$

PROBLEM 15: FOR WHAT y IS PROB. $> C$?

cf TABLES, MATHEMATICA (ask Scalise)

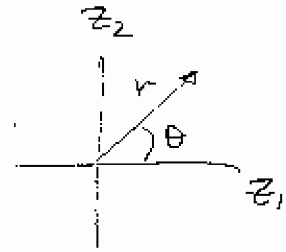
WILL SEE:

$$\int \mathcal{P}(x)\, dx = 1$$

$$\left.\begin{aligned} \bar{x} &= \int \mathcal{P}(x) \cdot x\, dx = \mu \\ \overline{(\Delta x)^2} &= \int \mathcal{P}(x)(x - \bar{x})^2 dx = \sigma^2 \end{aligned}\right\} \text{PARAMS IN } \mathcal{P}(x)$$

(later)

POLAR COORDS



skip $\begin{cases} r = (z_1^2 + z_2^2)^{1/2} \\ \tan\theta = z_2/z_1 \end{cases}$

$$dz_1\, dz_2 \to r\,dr\,d\theta \qquad z_1^2 + z_2^2 = r^2$$

$$I^2(\alpha) = \underbrace{\int_0^{2\pi} d\theta}_{2\pi} \underbrace{\int_0^{\infty} r\,dr\; e^{-\alpha r^2}}_{\frac{1}{2\alpha}}$$

$u \equiv r^2$

$du = 2r\,dr$

$$I^2(\alpha) = \frac{\pi}{\alpha}$$

$$\boxed{I(\alpha) = \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}}$$

CAN USE THIS TO COMPUTE RELATED INTS:

$$\frac{\partial}{\partial\alpha} I(\alpha) = \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty} e^{-\alpha z^2} dz = \frac{\partial}{\partial\alpha}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}$$

$$= -\int_{-\infty}^{\infty} e^{-\alpha z^2} z^2\, dz = -\frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}$$

$$\frac{\partial^n}{\partial\alpha^n} I(\alpha) = \boxed{(-1)^n \int_{-\infty}^{\infty} e^{-\alpha z^2} z^{2n}\, dz = \frac{\partial^n}{\partial\alpha^n}\left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}}}$$

ALSO $\boxed{\int_{-\infty}^{\infty} e^{-\alpha z^2} z^{2n+1}\, dz = 0}$ (z ODD $\Rightarrow \pm z$ GIVE OPP CONTRIB)

USE SIMILAR TRICK FOR:

$$\int_0^\infty e^{-\alpha z^2} z^{2n+1}\, dz$$

(EVEN ARE JUST $\frac{1}{2}$ OF $\int_{-\infty}^{\infty}$ INTS)

START WITH $\int_0^\infty e^{-\alpha z^2} z\, dz = \frac{1}{2\alpha}$

THEN $\boxed{(-1)^n \int_0^\infty e^{-\alpha z^2} z^{2n+1}\, dz = \frac{\partial^n}{\partial \alpha^n} \frac{1}{2\alpha}}$ $n=0,1,2,\ldots$

IN PARTICULAR:

$$\int_{-\infty}^{\infty} P(x)\, dx = \frac{1}{(2\pi\sigma^2)^{1/2}} \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2\sigma^2}\, dy}$$

here $\alpha = \frac{1}{2\sigma^2}$ $\quad \left(\frac{\pi}{\alpha}\right)^{\frac{1}{2}} = (2\pi\sigma^2)^{1/2}$

$= 1$ ✓

ALSO $\bar{x} = \int_{-\infty}^{\infty} P(x) \cdot x\, dx$ $\left\{ \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-(x-\mu)^2/2\sigma^2} x\, dx \right.$ (a) $y = x - \mu$ (b) $\alpha = \frac{1}{2\sigma^2}$ $(y+\mu)$ no contrib, norm. =

$\Rightarrow \boxed{\bar{x} = \mu}$ (~ AS EXPECTED)

$$\overline{(\Delta x)^2} = \overline{(x - \bar{x})^2} = \int_{-\infty}^{\infty} \underbrace{\underbrace{(x-\mu)^2}_{=y^2} \underbrace{P(x)}_{\frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-y^2/2\sigma^2}}\, dx}_{\frac{1}{2}\left(\frac{\pi}{\alpha^3}\right)^{\frac{1}{2}}} \quad \alpha = \frac{1}{2\sigma^2}$$

$$\boxed{\overline{(\Delta x)^2} = \sigma^2}$$

$\therefore$ (1) CAN READ OFF $\bar{x}$ & $\overline{(\Delta x)^2}$ FROM GAUSS DISTR: (ie μ AND σ^2)

(2) HERE $\bar{x} = (p-q)N\ell = \bar{m}\ell$

$\overline{(\Delta x)^2} = 4Npq\,\ell^2 = \overline{(\Delta m)^2}\,\ell^2$ $\Big\{$ consistent w/ results from discr. distr

$\therefore$ (AS IT MUST) (& suppress up to $\sim \frac{1}{N}$)

ONE LAST INTEGRAL: (NOT ALL FUN & GAMES)

ERROR FN: (cf prob 15)

slig FOR $P(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2}$

ex FIND PROB $x < y$: (OR $y_1 < x < y_2$)

$$\int_{-\infty}^{y} P(x)\,dx$$

RELATED TO ($\int_{-\infty}^{0}$ part is easy)

$$\mathrm{erf}(y) \equiv \frac{2}{\sqrt{\pi}} \int_0^y e^{-x^2} dx$$

cf APPX A.5 (DISCUSSION & APPROX. FORMULA)

$\Rightarrow$ NO CLOSED FORM; DO NUMERICALLY OR

LOOK UP IN TABLES, MATHEMATICA (Erf[y]), C

VALIDITY OF GAUSSIAN APPROX:

$$W(n_R) \approx e^{\left[\ln W(\tilde{n}_R) + \frac{1}{2} B_2 \eta^2 + \frac{1}{3!} B_3 \eta^3 + \cdots\right]}$$

skip $\left\{ \begin{array}{l} \tilde{n}_R \text{ IS PT WHERE } W \text{ IS MAX} \qquad \eta \equiv n_R - \tilde{n}_R \\ B_k \equiv \left.\dfrac{d^k(\ln W)}{d n_R^{\,k}}\right|_{n_R = \tilde{n}_R} \end{array} \right.$

$B_1 = 0$ (MAX)

$B_2 = -\dfrac{1}{Npq}$ ($< 0 \rightarrow$ MAX)

EACH NEW TERM:

EXTRA $\dfrac{\eta}{Npq}$ (ex $B_3 = \dfrac{q^2 - p^2}{N^2 p^2 q^2}$)

(a) SERIES OK WHILE $\boxed{\dfrac{\eta}{Npq} \ll 1}$ OR $\boxed{\eta \ll Npq}$

ie: N large, p, q NOT TOO SMALL (cf p. 9 for diff approx when p or q small)

skip $\Rightarrow W(n_R) = W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$ GOOD APPROX

(b) FAILS WHEN $\eta \approx Npq$

HOW BIG IS W?

$$W \approx W(\tilde{n}_R)\, \underbrace{e^{-\frac{1}{2}\frac{1}{Npq}(Npq)^2}}_{e^{-\frac{1}{2}Npq}}$$

IF $\boxed{Npq \gg 1}$, $W \sim 0$, APPROX GOOD <u>FOR ALL</u> η

(ie it's already zero when fails)

USEFUL GENERAL RESULTS (SECT 1.9)

(more general than ran. walk, Gauss. approx)

CONSIDER SUM OF RAN. #'s s_i (N NOT NECESS LARGE)

$$x = s_1 + s_2 + s_3 \cdots + s_N = \sum_{i=1}^{N} s_i$$

{* INDEX ON s_i HERE IS WHICH STEP, NOT WHAT VALUES s CAN TAKE

ex FINAL POSITION AFTER N STEPS

$$s_1 = 1^{ST} \text{ STEP } = +\ell \text{ w/ PROB } p$$
$$-\ell \quad " \quad " \quad q$$

($w \equiv$ doublеu, not omega)

HERE: ASSUME ONLY SAME DISTR. $w(s)$ FOR EACH s_i's & INDEP.

– DISCRETE OR CONTINUOUS

WHAT IS $\overline{x}$ & HOW DOES IT FLUCTUATE?

{ because s_i's random x is also random; means it has some distr. $P(x)$

(a) AVE OF SUM = SUM OF AVES (have seen already)

AGAIN:

skip

$$\overline{x} = \frac{1}{M}\sum_{m=1}^{M} x_m \qquad \underline{M \to \infty}$$

$$= \frac{1}{M}\Big\{ (s_1 + s_2 + \cdots + s_N) + \qquad \{ m=1$$

$$+ (s_1' + s_2' + \cdots + s_N') + \qquad \{ m=2$$

$$\vdots$$

$$+ (s_1'''' + s_2'''' + \cdots + s_N'''') \Big\} \qquad \{ m=M$$

$$= \overline{s_1} + \overline{s_2} + \cdots + \overline{s_N}$$

{ here: N is just a sum - could be 3, for ex, but M is to average $\to \infty$ }

ALSO, EACH STEP HAS SAME PROB DIST: $\bar{s}_1 = \bar{s}_2 = \cdots$

$$\boxed{\bar{x} \ \text{"="} \ \overline{\textstyle\sum s_i} \ \sum_{i=1}^{N} \bar{s}_i = N\bar{s}}$$

{emphasize diff between $\sum_i^N$ and sum used in ave

w/ $\bar{s} = \int ds\, w(s) \cdot s$ {if $w(s)$ continuous
else sum: ex ran walk
$\bar{s} = p \cdot \ell + q(-\ell)$
$= (p-q)\ell$

(b) SPREAD ?

$$\Delta x = x - \bar{x} \ \overset{\sum_{i=1}^{N} s_i - N\bar{s}}{=} \ \sum_{i=1}^{N} (s_i - \bar{s}) = \sum_{i=1}^{N} \Delta s_i$$

$$\overline{(\Delta x)^2} = \overline{\underbrace{\left(\sum_{i=1}^{N} \Delta s_i\right)}_{\Delta x}\underbrace{\left(\sum_{j=1}^{N} \Delta s_j\right)}_{\Delta x}}$$

(need diff i, j to keep two Δx's straight; write out if necessary)

SEPARATE OUT DIAG., CROSS TERMS

$$= \overline{\sum_{i=j=1}^{N} (\Delta s_i)^2} + \overline{\sum_{i \neq j} (\Delta s_i)(\Delta s_j)}$$

ave of sum = sum of aves

$$\sum \overline{(\Delta s_i)^2} \qquad \sum_{i \neq j} \overline{(\Delta s_i)(\Delta s_j)}$$

each same $\quad = N\overline{(\Delta s)^2}$

5

INSIDE 2ND SUM: IS AVE OF PROD = PROD OF AVE ?

$\Rightarrow$ ONLY IF 2 TERMS STATIST. INDEP:

IF a, b INDEP RANDOM #S w/ DISTR. $P(a)$, $Q(b)$

PROB a & b: $P(a) \cdot Q(b)$

(NOT INDEP? SOME OTHER GEN'L DISTR. $R(a,b)$
GIVES PROB OF BOTH, TAKES AFFECT OF a ON b INTO ACCT)

THEN:

$$\overline{ab} = \underbrace{\sum_{k,\ell}}_{\substack{\text{all possible} \\ \text{values for } a \cdot b}} P(a_k) Q(b_\ell)\,(a_k b_\ell)$$

$$= \left(\Sigma_k P(a_k)\, a_k\right)\left(\Sigma_\ell\, Q(b_\ell)\, b_\ell\right)$$

$$= \bar{a} \cdot \bar{b}$$

OUR PROBLEM: EACH TERM $(i \neq j)$

$$\overline{(\Delta S_i)(\Delta S_j)} = \underbrace{\overline{(\Delta S_i)}}_{=0} \cdot \underbrace{\overline{(\Delta S_j)}}_{=0} = 0$$

(wouldn't work for $i = j \Rightarrow$ certainly not indep.)

FINALLY

$$\overline{(\Delta x)^2} = N\,\overline{(\Delta s)^2} \quad \text{OR} \quad \boxed{\Delta^* x = \sqrt{N}\,\Delta^* s}$$

MAKES SENSE: MORE TERMS IN SUM, MORE TOTAL MOVES AROUND

BUT: BIGGER N, SMALLER RELATIVE CHG:

$$\boxed{\frac{\Delta^* x}{\bar{x}} = \frac{1}{\sqrt{N}}\,\frac{\Delta^* s}{\bar{s}}}$$

WHY IS THIS USEFUL?

(0) CAN TELL AVE, STD DEV FOR SUM ONLY KNOWING " , " FOR PIECES; TELLS A LOT ABOUT DISTR.

(1) N LARGE: (CAN SAY MORE)

SPS s HAS ARBITRARY DISTR $w(s)$ & N LARGE

(eg RANDOM WALK)

skip

→ COMPUTE $\bar{s}$, $\Delta^* s = \sqrt{\overline{(\Delta s)^2}}$

ALSO $X \equiv s_1 + \cdots + s_N$, N LARGE

THEN: KNOW DISTRIB. $\mathcal{P}(x)$ FOR X COMPLETELY

HOW?

(a) $\bar{x} = N\bar{s} \equiv \mu$

(b) ~~$\overline{(\Delta X)^2} = N\overline{(\Delta s)^2}$ OR~~ $\Delta^* x = \sqrt{N}\,\Delta^* s = \sigma$

(c) N LARGE ⇒ $\mathcal{P}(x)$ GAUSSIAN } cf optional sections 10 & 11 "central limit thm"

$$\Rightarrow \mathcal{P}(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2}$$

⇒ CAN COMPUTE ANYTHING ABOUT X

⇒ TRUE FOR ANY $w(s)$ (w/ some rare exceptions)

(2) STATISTICAL ERRORS:

ACCURACY OF MEAS. IMPROVES AS $\frac{1}{\sqrt{N}}$

IF REPEAT N TIMES: (if errors are statistical)

WHY:

MEAS. QTY s N TIMES

BEST VALUE: $s_{BEST} = \frac{1}{N}\sum_{i=1}^{N} s_i \equiv \frac{1}{N} X$

(i = # OF MEASMT.)

WHAT SHOULD QUOTE FOR ERROR?

$\cancel{\Rightarrow \frac{1}{\sqrt{N}} \Delta^* S}$

~~WHY?~~

HOW MUCH DOES S_{BEST} FLUCT?

$$\Delta^* S_{BEST} = \frac{1}{N} \Delta^* X = \frac{1}{N} \sqrt{N}\, \Delta^* S = \frac{1}{\sqrt{N}} \Delta^* S$$

ie: gives idea of what std dev. would be if repeated (N meas, then ave) large # times.

DISTRIB'N FOR S_{BEST}?

IF LARGE N: $\Rightarrow$ GAUSSIAN

WORK 10 TIMES HARDER $\Rightarrow$ DO ~3 TIMES BETTER
(ie more accurate)