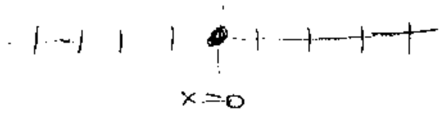
- a chance to put to use some prob.
  in an interesting physics problem
- build some tools useful for rest of course

RANDOM WALK

- MODEL FOR PARTICLE(S) MOVING RANDOMLY THRU FLUID (DIFFUSION) (REIF gives other applications; tools are general)

1 DIM:

PARTICLE STARTS AT $x = 0$



ASSUMPTIONS:

① GOES DISTANCE l BEFORE HITTING NEXT PARTICLE (≃ MEAN FREE PATH); REPEATS

② AFTER COLLISION, DIR. OF NEXT STEP INDEP. OF PREVIOUS STEP

PROB NEXT STEP TO RT: p $(0 < p < 1)$

" " " TO LT: $q = 1 - p$

{ particle is at the mercy of the fluid particles }

(IF ISOTROPIC: $p = q = \frac{1}{2}$; IF $p > q$ ~ FLUID FLOWS TO RT)

(treating w/ probability, since don't want to get into details of each collision)

PROBLEM:

AFTER N STEPS, FIND PROB. $P(m)$ FOUND AT $x = ml$ (m AN INTEGER)

{CLASSIC FORMULATION: DRUNK AT LAMP POST ON SIDEWALK}

PROBABILITY: IN LARGE ENSEMBLE OF IDENT SYSTEMS, AFTER N STEPS

$$P(m) = (\# \text{ AT } ml)/N_{ENS}.$$

REALIZATIONS:

(a) 1 PART. EACH IN N_{ENS} FLUIDS

(b) " " N_{ENS} TIMES IN 1 "

* (c) MANY PARTICLES IN 1 " ⇒ { eg DIFFUSION OF DYE
$P \propto$ DENSITY
CAN SEE DIRECTLY }

skip

{REIF: PROB. DEPENDS ON ENSEMBLE

ex prob seed yields red flower - diff if seed ∈ {tulips}
vs seed ∈ {all plants incl trees}

⇒ "prob given the info or constraints on ensemble"

4

EXPECT:

IF $p = q = \frac{1}{2}$

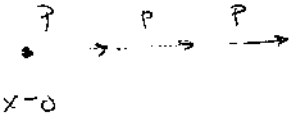
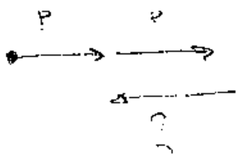
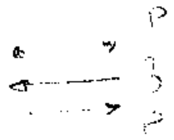
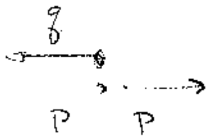
- USUALLY: END CLOSE TO $x = 0$
- $\bar{x} = 0$ BY SYMM BUT AVE DIST $\neq 0$
- RARE: $m\ell = \pm N\ell$ (ESP FOR N LARGE)
  - CAN ONLY BE DONE 1 WAY (ALL R OR L STEPS)
  - LOTS OF WAYS TO GET TO $x \sim 0$

IF $p > q$:

- MORE R THAN L STEPS ON AVE ⇒ MIGRATES R
- GUESS: $\bar{x} = N(p-q)\ell$ (WILL CONFIRM)

START SIMPLE:

N = 3

m	ROUTE	PROB
+3		p^3 ⇒ $\boxed{P(3) = p^3}$ R and R and R in 3 steps ⇒ mult. probs
+2	CAN'T DO IT	0
+1		$p^2 q$
OR		$p q p$
OR		$q p^2$

path 1 or 2 or 3
⇒ add

TOTAL: $\boxed{P(1) = 3p^2 q}$

r-3
1,3

m	ROUTE	PROB
0	CAN'T	0
-1 ETC	→ JUST REPLACE $p \leftrightarrow q$	$\to P(-1) = 3pq^2$
-3		q^3 $\quad P(-3) = q^3$

$\left\{ \text{eg } p = q = \frac{1}{2}: \quad \begin{array}{l} P(3) = P(-3) = \frac{1}{8} \\ P(1) = P(-1) = \frac{3}{8} \end{array} \right\}$

HAVE ALL CASES.

PROB THAT PARTICLE GOES ANYWHERE? 1

CHECK: $\displaystyle\sum_{m=-3}^{3} P(m) = 1$

$$\Rightarrow \quad p^3 + 3p^2 q + 3pq^2 + q^3 = (p+q)^3 = 1^3 = 1 \checkmark$$

(any p, q)

IN GENERAL:

$n_R, n_L \equiv$ # RT, LT STEPS (Reif uses n_1, n_2)

$$\boxed{\begin{array}{l} N = n_R + n_L \\ m = n_R - n_L = 2n_R - N \end{array}} \quad \text{OR} \quad \boxed{\begin{array}{l} n_R = \frac{1}{2}(N+m) \\ n_L = \frac{1}{2}(N-m) \end{array}}$$

{same as found

$n_{R,L} = 0, 1, 2 \ldots N$

N, m BOTH EVEN OR BOTH ODD

TYPICAL ROUTE TO m:

STEP 1 2 3

R R L R L L $\quad$ N R $\quad \left.\begin{array}{l} n_R \text{ R'S} \\ n_L \text{ L'S} \end{array}\right\}$

PROB FOR THIS SPECIFIC ROUTE $\quad p^{n_R} q^{n_L} = p^{\frac{1}{2}(N+m)} q^{\frac{1}{2}(N-m)}$

(note: if N, m fixed, so are n_R, n_L)

→)

HOW MANY WITH THIS n_R & n_L ?

NEED # WAYS TO ARRANGE n_R R'S AND n_L L'S IN N STEPS / SPACES.

STEP: $\overline{\ \ }_1$ $\overline{\ \ }_2$ $\overline{\ \ }_3$ ... $\overline{\ \ }_N$

(1) ASSUME HAVE N DISTINCT OBJECTS

1ST : N CHOICES

2ND : N − 1 " ETC { for each one of the N choices for 1ST

⇒ N! WAYS TO ARRANGE

(2) WE OVER COUNTED:

NOW RECOGNIZE THAT R'S SAME, L'S SAME

R R L R L L L R R

SAME IF REARRANGE R'S AMONG SELVES, ALSO L'S

$n_R!$ WAYS TO ARRANGE R'S (counting same way)

$n_L!$ WAYS TO " L'S

⇒ NUMBER OF DISTINCT ARRANGEMENTS WITH n_R R'S & n_L L'S:

$$\frac{N!}{n_L!\,n_R!} = \frac{N!}{(N-n_R)!\,n_R!} \equiv \binom{N}{n_R}$$

"BINOMIAL COEFFICIENT"

$$= \frac{N!}{\{\frac{1}{2}(N+m)\}!\,\{\frac{1}{2}N-m)\}!}$$

cf

BINOMIAL EXPANSION:

$$(x+y)^N = \binom{N}{0}x^N + \binom{N}{1}x^{N-1}y^1 + \binom{N}{2}x^{N-2}y^2 +$$

$$\cdots + \binom{N}{N-1}xy^{N-1} + \underset{\uparrow\,1}{\binom{N}{N}}y^N$$

$$= \sum_{n=0}^{N}\binom{N}{n}x^{N-n}y^n \qquad (\text{NOTE } 0! \equiv 1)$$

WHY?

$$\text{ex} \quad (x+y)^3 = (x+y)(x+y)(x+y)$$

$$= x^3 + (\underbrace{x^2y}_{3x^2y} + xyx + yx^2) + \cdots$$

⇒ ALL POSSIBLE TERMS w/ 2x's & 1 y

⇒ FOR EACH OF 3 FACTORS, PICK EITHER AN x OR y SUCH THAT HAVE 2 x's & 1 y

SAME COUNTING AS PICK L & R's FROM 3 STEPS

FINALLY

$$P(m) = \frac{N!}{[\frac{1}{2}(N+m)]!\,[\frac{1}{2}(N-m)]!}\,p^{\frac{1}{2}(N+m)}\,q^{\frac{1}{2}(N+m)}$$

"BINOMIAL DISTRIBUTION"

$q \equiv 1-p$

m, N BOTH ODD OR BOTH EVEN (ELSE $P=0$)

EQUIVALENTLY IN TERMS OF n_R :

$$W(n_R) \equiv P(m) = \frac{N!}{n_R!\,(N-n_R)!}\,p^{n_R}\,q^{(N-n_R)}$$

SAME DISTR. w/ CHG OF VARS

CHECK:

(1) SYMM IF $p \leftrightarrow q$ AND $m \leftrightarrow -m$

(2) q SMALL : $P(m)$ SMALL UNLESS $N \approx +m$ (IN POWER)

(3) BIN. COEFF LARGE WHEN $m \sim 0$ $n_R \sim n_L \sim N/2$

⇒ PICTURES

COMMENTS ON PLOTS: (say)

<u>$p = q = \frac{1}{2}$</u>

- GIVES BOTH $P(m)$ & $W(n_R)$ { since m & n_R tied together, corresp plots same
- PEAKED AT $m = 0$
- UNLIKELY FOR $m = \pm 20 = \pm N$
- SHAPE DUE ENTIRELY TO BIN. COEFF (ie # ROUTES)
  → PROB. OF EACH ROUTE SAME

<u>$p = 0.6 \quad q = 0.4$</u> (note - only good for m even)

- DRIFTS RT.
- SEE INTERPLAY BETW. PREF. TO MOVE RT
  VS MANY MORE WAYS TO END AT ORIGIN

  <u>so</u> STILL UNLIKELY TO END AT $m = +20$

IN BOTH - WILL SEE THAT (AS EXPECT):
MEAN NEAR PEAK
$\Delta x \sim$ WIDTH { same

1.5.1

$N = 20$

$p = q = \frac{1}{2}$

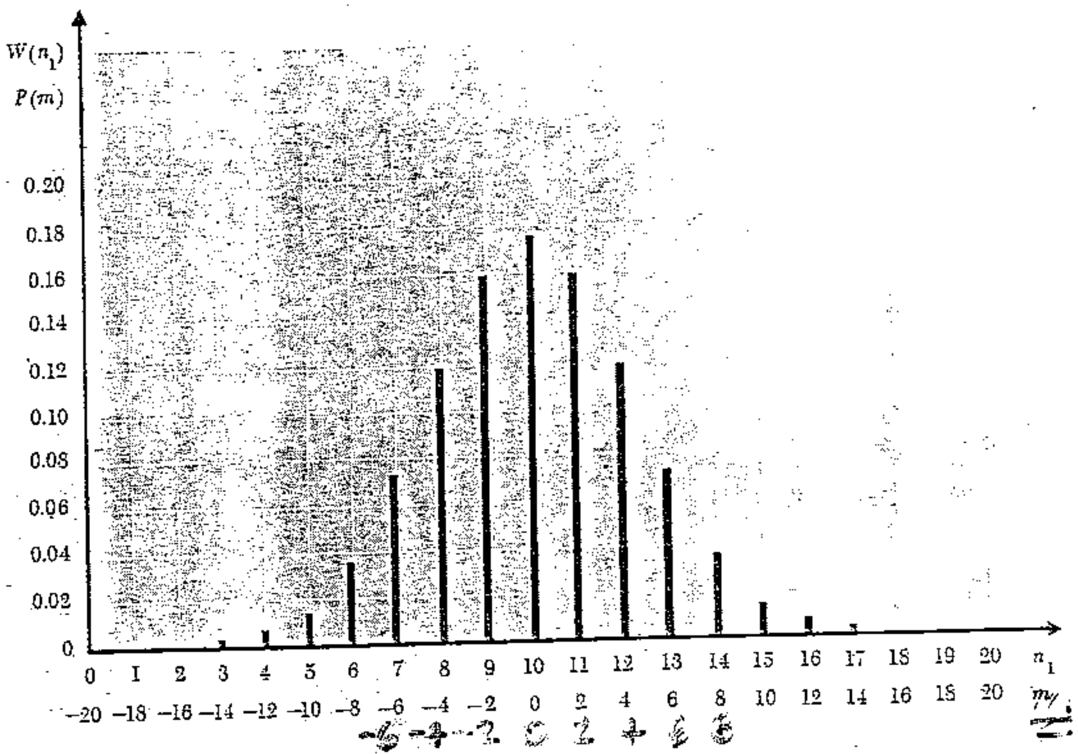
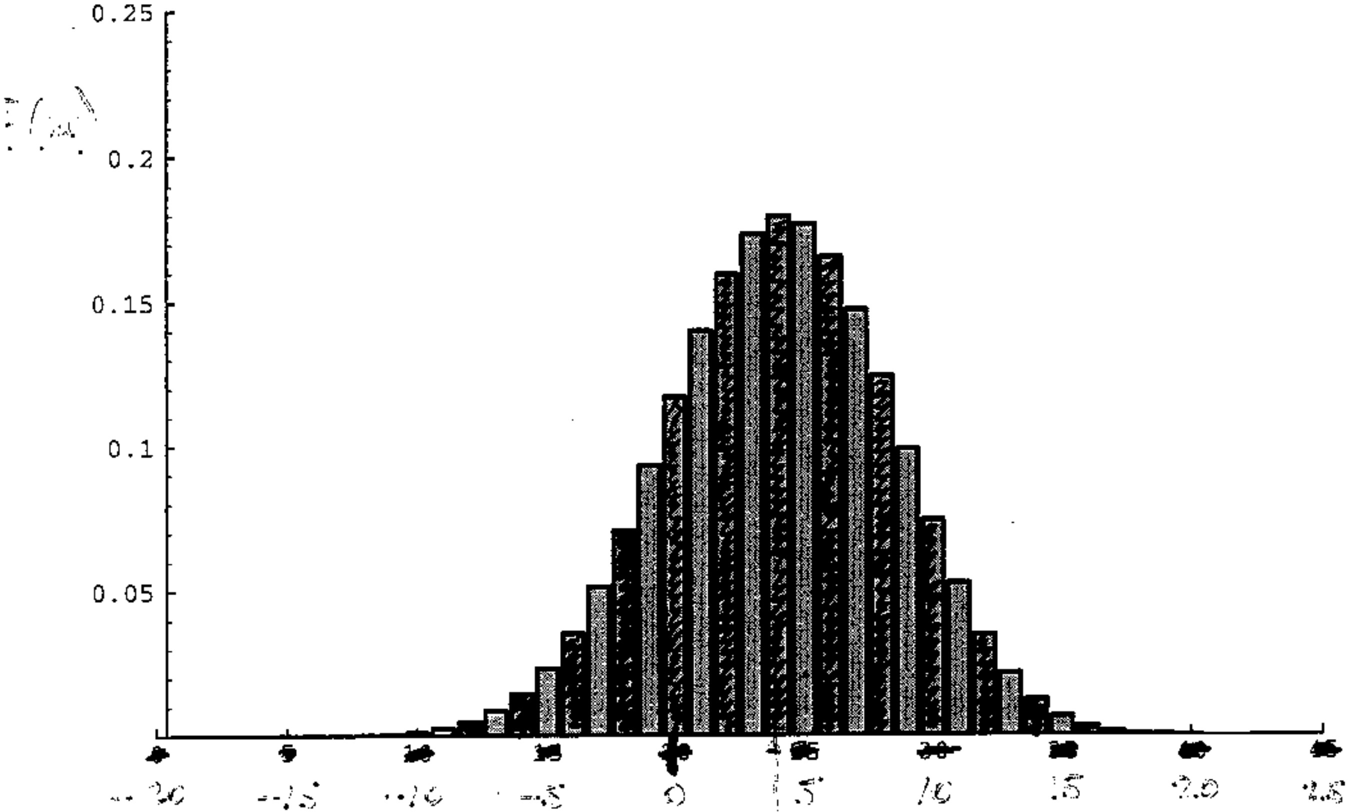


Fig. 1·2·3 Binomial probability distribution for $p = q = \frac{1}{2}$ when $N = 20$ steps. The graph shows the probability $W_N(n_1)$ of n_1 right steps, or equivalently the probability $P_N(m)$ of a net displacement of m units to the right.

$N = 20$

$p = .6 \quad q = .4$

(ONLY GOOD FOR m EVEN)



$\overline{m} = N(p-q) = 4$

$\sqrt{\overline{(\Delta m)^2}} = \sqrt{4pqN}$

p.6
1.6

$$\sum_{m=-N}^{N} P(m) \overset{?}{=} 1$$

EASIEST IF USE FORM

$$W(n_R) = \frac{N!}{(N-n_R)!\, n_R!} p^{n_R} q^{N-n_R}$$

[illegible]

skip , (WITH $q = 1 - p$ $\quad n_R + n_L = N$
$n_R - n_L = m$)

$$\sum_{n_R=0}^{N} W(n_R) = \sum_{n_R=0}^{N} \binom{N}{n_R} p^{n_R} q^{N-n_R}$$

$$= (p+q)^N = 1 \quad ✓$$

skip {WILL SEE SEVERAL TRICKS FOR SUMMING SERIES.
cf GRADSHTEYN & RYZHIK ; SYMBOLIC PROGRAMS - MATHEMATICA,
MAPLE)

skip *ex* USEFUL FOR PROB 1.5.c (cf A.1)

$$f(x) = \sum_{n=0}^{N} x^n = 1 + x + x^2 + \quad x^N$$

$$x f(x) = x + x^2 + \quad + x^N + x^{N+1}$$

SUBTR

$$(1-x) f(x) = 1 - x^{N+1}$$

$$f(x) = \frac{1 - x^{N+1}}{1-x}$$

$N = \infty \Rightarrow 1 + x + x^2 + \cdots = \frac{1}{1-x}$ (converges if $|x|<1$)

also $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

skip SPECIAL CASE: $\sum_{n=0}^{N} e^{ny} = \sum_{n=0}^{N} (\underbrace{e^{y}}_{\equiv x})^n \Rightarrow$ SAME AS ABOVE

$$= \frac{1 - e^{(N+1)y}}{1 - e^{y}}$$

COMPUTE SOME AVGS W/ RAN WALK:

(EASIEST TO WORK W/ $W(n_R)$ → DON'T HAVE TO WORRY ABOUT EVEN/ODD)

(1) HAVE $\sum_{n_R=0}^{N} W(n_R) = 1$

(2) $\bar{n}_R = \sum_{n_R} W(n_R)\, n_R = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} \cdot n_R$

TRICK:

(a) TREAT p, q AS INDEP. ; ie $\bar{n}_R(p,q)$, ie $q \neq 1-p$ yet

(b) NOTE $p\frac{\partial}{\partial p} p^{n_R} = n_R\, p^{n_R}$

(c) $\bar{n}_R(p,q) = \sum_{n_R}\binom{N}{n_R}\left(p\frac{\partial}{\partial p}p^{n_R}\right) q^{N-n_R}$

$$= p\frac{\partial}{\partial p}\left\{\sum_{n_R}\binom{N}{n_R}p^{n_R}q^{N-n_R}\right\}$$

CAN NOW SUM :

$$(p+q)^N$$

DERIV: $pN(p+q)^{N-1} = \bar{n}_R(p,q)$ FOR ANY p, q

$\Rightarrow \bar{n}_R(p,q) = \sum\binom{N}{n_R}p^{n_R}q^{N-n_R}\, n_R = pN(p+q)^{N-1}$

true for any p, q

(d) NOW TAKE $q = 1-p$

$$\bar{n}_R = \bar{n}_R(p,q)\Big|_{q=1-p} = Np$$

$\boxed{\bar{n}_R = Np}$ (YOU ~~WOULD~~ DID GUESS THIS)

$m = n_R - n_L$

IN SAME WAY $\boxed{\bar{n}_L = Nq}$ $\Rightarrow \boxed{\bar{m} = \bar{n}_R - \bar{n}_L = N(p-q)}$

(w/ $q = 1-p$)

(3) VARIANCE

$$\overline{(\Delta m)^2} \equiv (\Delta^* m)^2 = \overline{(m-\bar{m})^2}$$

USE n_R, $W(n_R)$:

$$\Delta m = m - \bar{m} = (2n_R - N) - (2\bar{n}_R - N) = 2\Delta n_R$$

$$\overline{(\Delta m)^2} = 4\overline{(\Delta n_R)^2} = 4(\overline{n_R^2} - \underset{\wr\; N^2p^2}{\bar{n}_R^2})$$

SAME TRICK:

$$\overline{n_R^2} = \sum_{n_R} W(n_R)\, n_R^2 = \sum \binom{N}{n_R} p^{n_R} q^{N-n_R} n_R^2$$

$$= \left(p\frac{\partial}{\partial p}\right)\left(p\frac{\partial}{\partial p}\right)(p+q)^N \Big|_{q=1-p}$$

SOME ALGEBRA, AFTER $q = 1-p$

$$\overline{n_R^2} = N^2p^2 + Npq$$

$$\overline{(\Delta m)^2} = 4(\overline{n_R^2} - \bar{n}_R^2) = 4Npq$$

$$\Delta^* m = \sqrt{4Npq} \qquad \text{note} \sim N^{\frac{1}{2}}$$

cf previous plot

WILL WANT N LARGE (AND INCR. w/ t) FOR DIFFUSION MODEL :

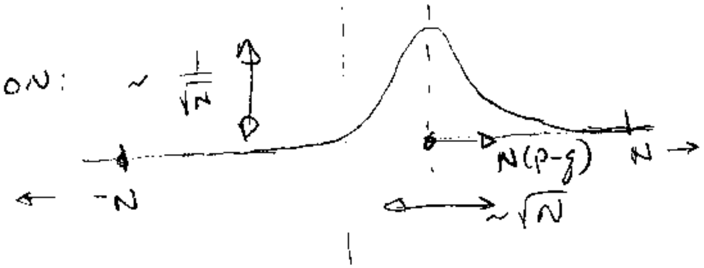
GEN'L FEATURES VS N:

AVE : $\bar{m} = N(p-q)$ ≈ LOC OF PEAK

WIDTH : $\Delta^* m = \sqrt{4Npq}$ WIDTH REL TO EXTENT: $\frac{\Delta^* m}{2N} - \left(\frac{pq}{N}\right)^{\frac{1}{2}}$

HEIGHT: (will see). $P_{max} \approx \left(\frac{1}{2\pi Npq}\right)^{\frac{1}{2}}$

DISTRIBUTION:



(picture - moving plot)

SHRINKS & SPREADS, BUT AREA CONST (why? prob distr)

(WILL SEE IN HW, CAN THINK OF AS DESCRIBING t DEVELOPMENT OF DIFFUSION PROCESS)

(THERE $N\ell = vt$
ℓ ave v)

① also, cp width to max extent = 2N
② mention ⇒ sharply peaked

IN TERMS OF n_R & $W(n_R)$:

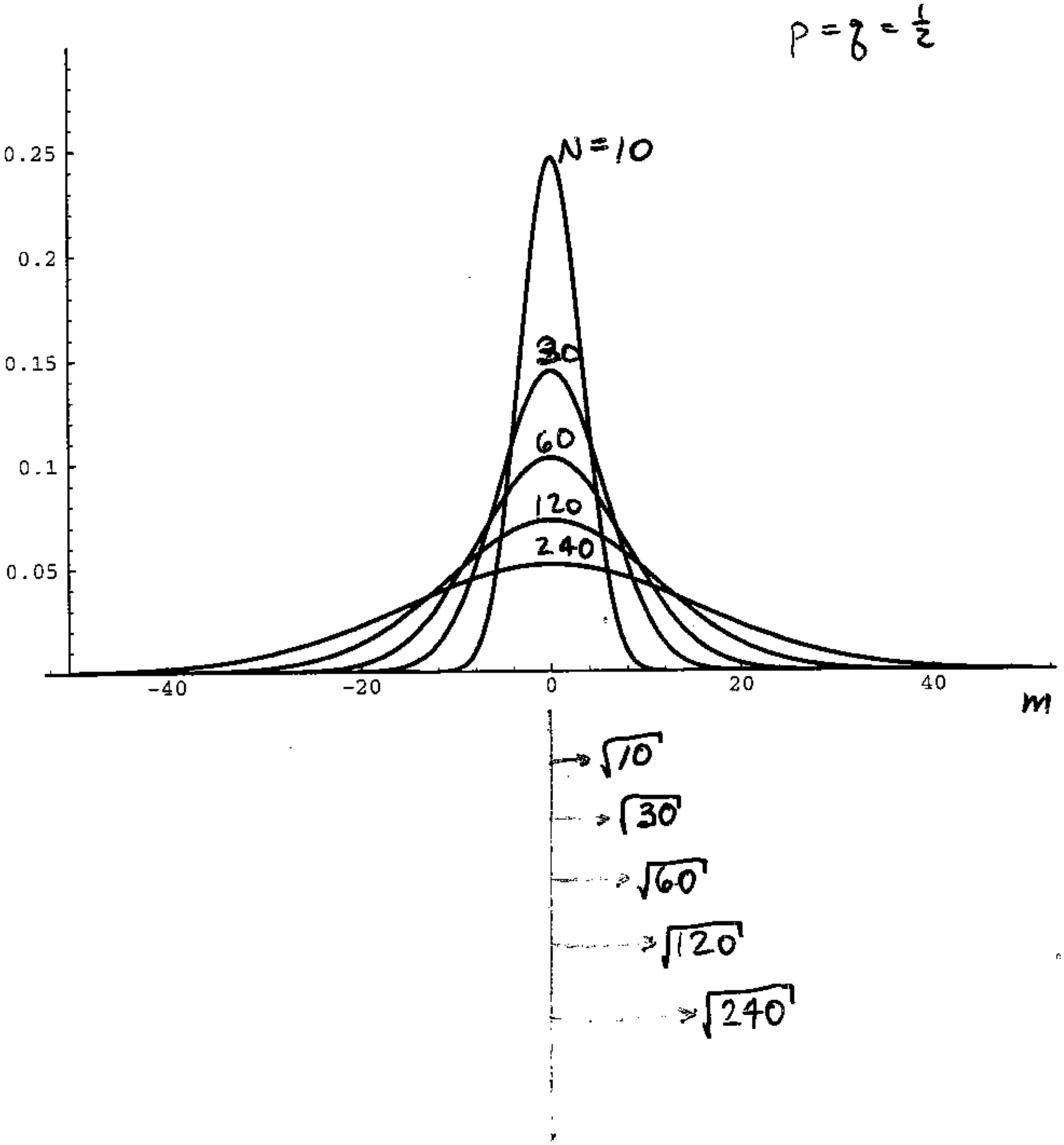
$$\bar{n}_R = Np$$

$$\Delta^* n_R = \frac{1}{2} \Delta^* m = \sqrt{Npq}$$

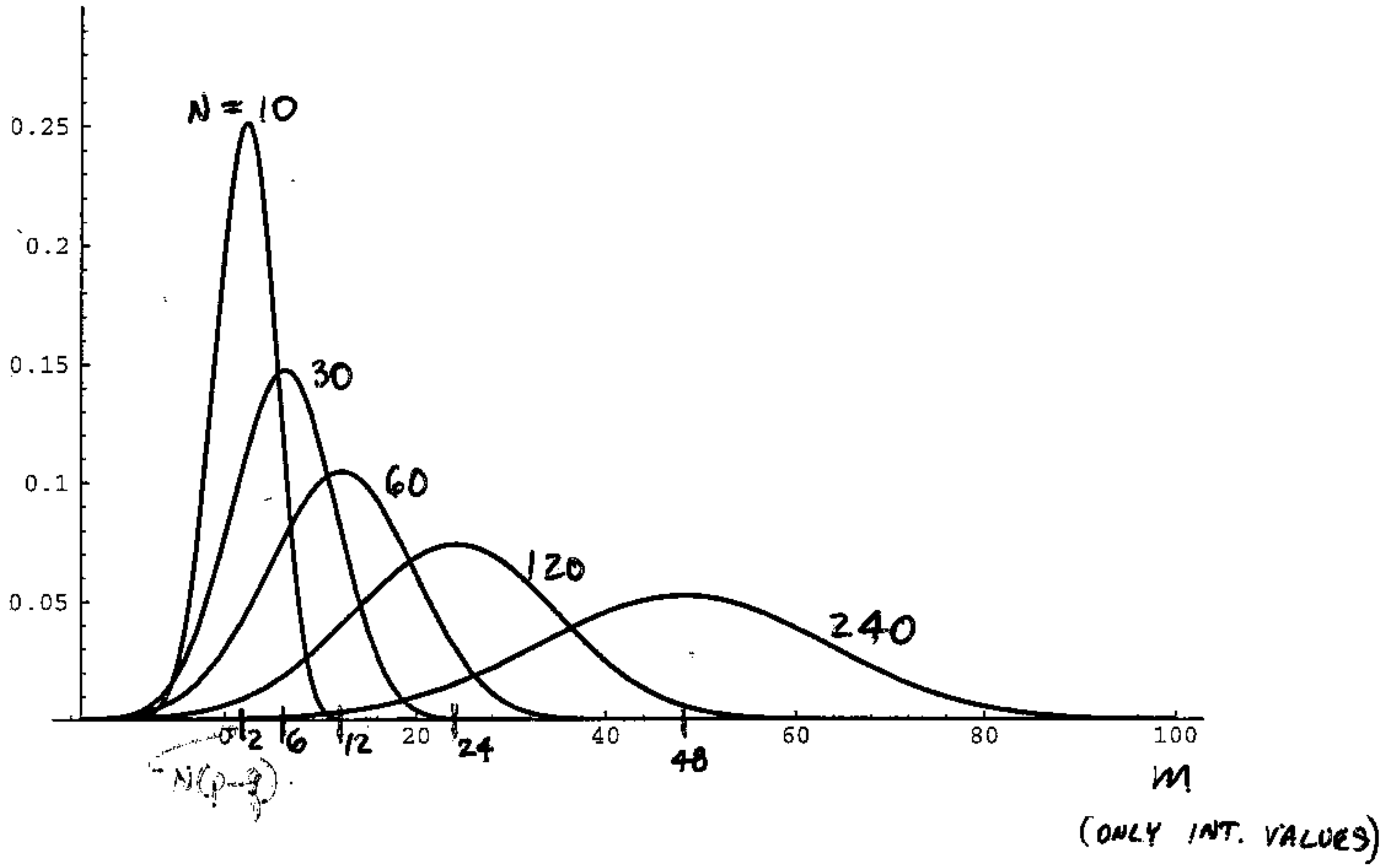
RELATIVE UNCERTAINTY : $\frac{\Delta^* n_R}{\bar{n}_R} = \frac{1}{\sqrt{N}}\left(\frac{q}{p}\right)^{\frac{1}{2}}$

SHRINKS w/ N ⇒ % ACCURACY INCREASES
(ex $p=q=\frac{1}{2}$ $\bar{n}_R$ MORE FOCUSED AT $N/2$)

1.12.1


p = q = 1/2
0.25
0.2
0.15
0.1
0.05
-40
-20
0
20
40
m
N=10
30
60
120
240
√10
√30
√60
√120
√240

$p = .6 \quad q = .4$


N = 10
30
60
120
240
0.25
0.2
0.15
0.1
0.05
0
20
40
60
80
100
2
6
12
24
48
N(p−q)
m
(ONLY INT. VALUES)

WE'LL NEED VERY LARGE N:

MOLS. IN FLUID:

$x \sim$ MACRO DIST $\sim 1\ \text{cm}$

$\ell \sim$ ATOMIC " $\sim 1\ \text{Å} \sim 10^{-10}\ \text{m} \ll x$

$\bar{x} = \bar{m}\ell = N(p-q)\ell \Rightarrow N \sim 10^{8}$ COLLISIONS BEFORE PEAK MOVES 1 CM

INTERESTING LIMIT:

$$\underbrace{\bar{x}}_{\text{REASONABLE (ie MACRO)}} \propto \underbrace{N}_{\text{LARGE}}\,\ell \leftarrow \text{SMALL}$$

CONTINUOUS APPROX:

(cf N = 240 PLOT)

$P(m) \rightarrow$ SMOOTH (ie $\Delta P \ll P$)

CAN REPLACE EXACT SOLN w/ CONTINUOUS PROB DENSITY $\mathcal{P}(x)$

(SEEMS COUNTERPRODUCTIVE ⇒ WORK HARD TO GET APPROX ?)

WHY?

(1) EASIER TO WORK WITH (INTEGRALS VS SUMS)

(2) MORE APPROPRIATE FOR MACRO MEAS:
TREATS AS CONTINUOUS FLUIDS

(3) AS ACCURATE:

(a) UNCERTAINTY IN SIZE OF INSTRUMENT $\gg \ell$
" " # PARTICLES $\gg \Delta\mathcal{P}$
⇒ CAN'T SEE BUMPS ⇒ INFINITESIMAL ON MACRO SCALE

(b) IF COULD SEE BUMPS (ie MEAS. ON ATOMIC LEVEL)
NEED A BETTER MODEL (DON'T REALLY BELIEVE MICRO DETAILS OF THIS ONE)

(4) ALL MICRO MODELS GIVE SAME MACRO $P(x)$ (WILL SEE)

⇒ ATOMIC DETAILS ENCAPSULATED IN 2 PARAMETERS

skip

SIMILAR TO E & M IN MEDIA

⇒ APPROX. ATOMIC DETAILS IN DIELECTRIC CONST ϵ

REPLACE $\vec{E}(x)$ w/ $\vec{D} = \epsilon \vec{E}$

REIF NOTATION:

dx = INFINITESIMAL BUT MACROSCOPIC

δx = SHORTEST (ATOMIC) DIST.

USUALLY $x \gg dx \gg \delta x$

(limit $dx \to 0$ really means small enough that $f(x)$ doesn't change much over dx, but not so small can see atoms)

DEFINE CONTINUOUS DISTR:

USUAL CALCULUS APPROACH: DEF. VAR, FN AT DISCRETE SET OF PTS & LET PTS GET CLOSER:

CONTINUOUS VAR:

$x_m \equiv m\ell \qquad m = -N, -N+2, \ldots, N-2, N$

$\Delta x = \Delta m\, \ell = 2\ell$ (recall m even or odd)

PROB. DENSITY:

$$P(x_m) = \lim_{\ell \to 0} \frac{P(m)}{\underbrace{2\ell}_{\Delta x}}$$

(WELL-DEF'D AS $\Delta x = 2\ell \to 0$)

(IF P(m) SIMPLE FN)

JUST REPLACE m w/ x/ℓ, TAKE LIMIT → DONE

PROBLEM:

P(m) HAS FACTORIALS:

- AWKWARD TO USE AS $N \to \infty$
- ONLY DEF'D FOR INTEGER VALUES: $\ell \to 0$ LIMIT PAINFUL

SOL'N:

REPLACE P(m) w/ $\overset{\text{SIMPLE}}{\wedge}$ CONTINUOUS APPROX:

WILL SEE:

EASY TO $\int$

CAN READ OFF $\overline{x}$, $\Delta^* x$

UNIVERSAL ⇒ GOOD APPROX FOR $\overset{\text{ALMOST}}{\wedge}$ANY LARGE SUM OF RANDOM VARS

⇒ GAUSSIAN

PROBLEM:

P(m) BECOMES SMOOTH BUT SHARPLY PEAKED!

$\Delta^* x \sim N^{1/2}\ell, \quad x_{MAX} \sim N\ell$

already discussed on pg 1.12

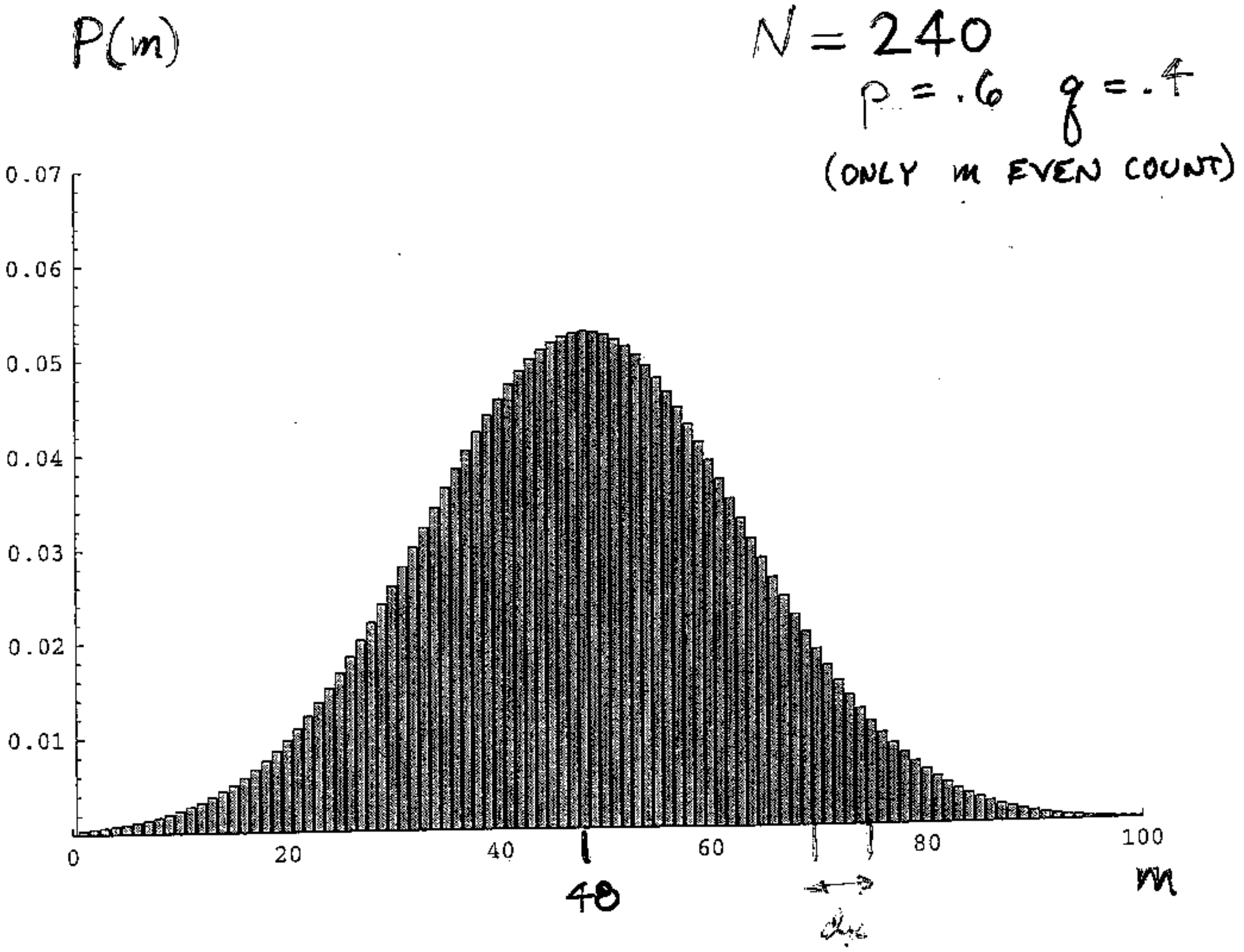
$$\frac{\Delta^* x}{x_{MAX}} \sim \frac{1}{N^{1/2}}$$

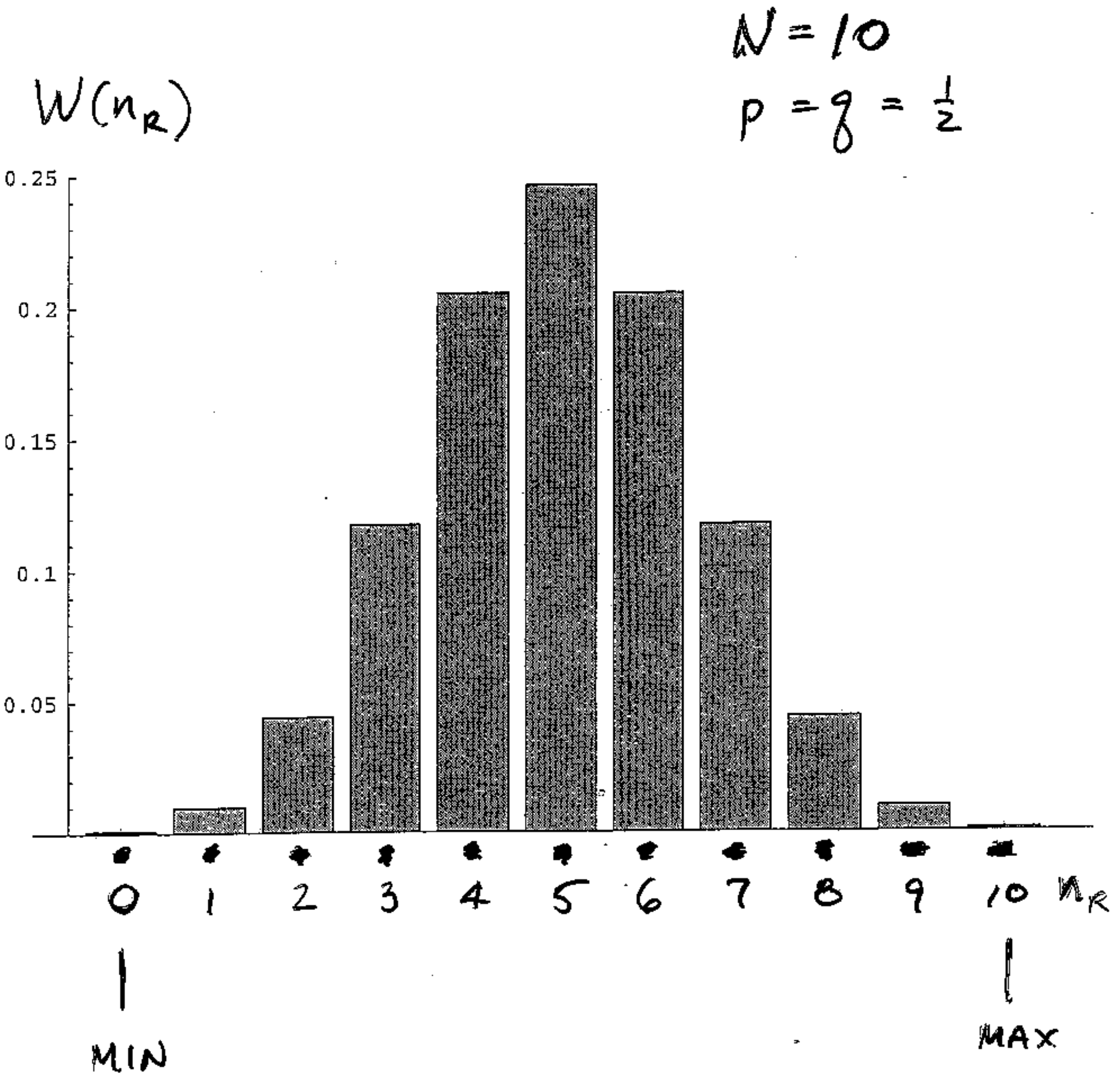
(fn sits at 0 for most x, jumps quickly near peak)

of plots

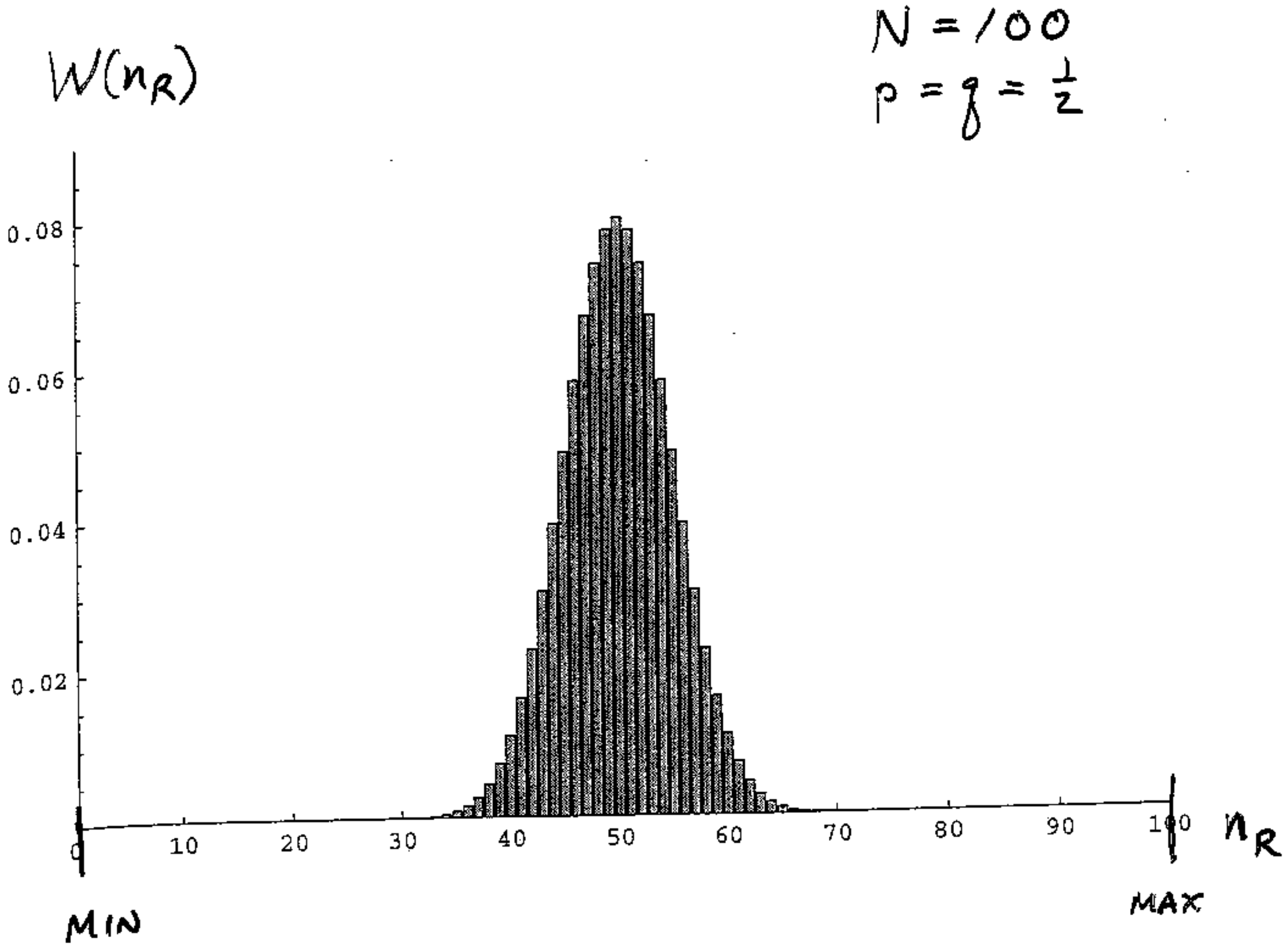
a.1.2

should only have values at 2's, or use W(n,k)





N = 10
p = q = ½
W(n_R)
0.25
0.2
0.15
0.1
0.05
0
1
2
3
4
5
6
7
8
9
10
n_R
MIN
MAX


$W(n_R)$
$N = 100$
$p = q = \frac{1}{2}$
0.08
0.06
0.04
0.02
0
10
20
30
40
50
60
70
80
90
100
n_R
MIN
MAX

(MORE SPECIFICALLY)
FN IS EXPONENTIAL:

POWERS ARE EXPONENTIAL: $p^{n_R} = e^{n_R \cdot \ln p}$

LARGE FACTORIALS ≈ EXPONENTIALS:

(try 100! on calculator)

<u>STIRLING APPROX</u> (cf APPX)

terms which vanish as n → ∞

$$n! \approx e^{n \ln n - n + \frac{1}{2} \ln(2\pi n) + \cdots}$$

$$= (2\pi n)^{\frac{1}{2}} e^{n(\ln n - 1)}$$

(uses method we'll talk about now)
"leading term in asymptotic expansion"

ex $100! \sim e^{364}$

PLAN: TO FIND SIMPLE APPROX

(1) GOOD NEWS:

P(m) ONLY ≳ 0 NEAR PEAK $\tilde{X}$

⇒ ONLY NEED GOOD APPROX NEAR MAX

⇒ TAYLOR SERIES

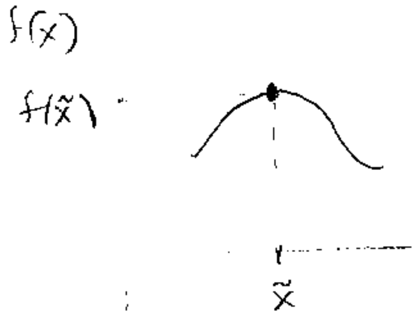
(2) BAD NEWS:

P(m) DROPS EXPONENTIALLY

⇒ BUILD INTO SERIES

1.17-18

SIMPLE
APPROX. NEAR PEAK:



TAYLOR:

$$f(x) \sim f(\tilde{x}) + \underbrace{f'(\tilde{x})}_{\substack{=0 \\ (\text{EXTREMUM})}}(x-\tilde{x}) + \underbrace{\tfrac{1}{2}f''(\tilde{x})}_{\substack{\leq 0 \\ (\text{CURVES DN})}}(x-\tilde{x})^2 + \cdots$$

DEFINE $\eta \equiv x - \tilde{x}$

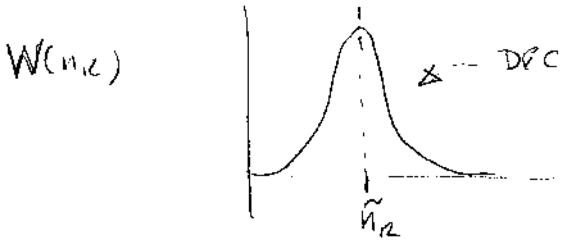
(SO EXPAND AROUND $\eta = 0$)

$$f(x) \equiv f(\tilde{x}+\eta) \approx f(\tilde{x}) + \tfrac{1}{2}f''(\tilde{x})\eta^2 + \cdots$$

$$= f(\tilde{x}) - \tfrac{1}{2}|f''(\tilde{x})|\eta^2 + \cdots$$

ACCURATE FOR η SMALL
TEST: NEW TERMS GET SMALLER

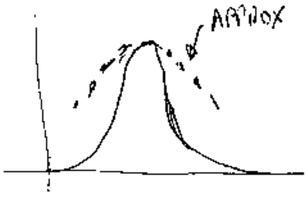
OUR CASE



← DECREASES EXPONENTIALLY $\sim e^{-g(n_R)}$

TAYLOR SERIES: POLYNOMIAL CAN'T KEEP UP W/ EXPONENTIAL

⇒ GET



⇒ ONLY GOOD FOR SMALL RANGE IN $\eta \equiv n_R - \tilde{n}_R$

BE SMART:

KNOW $W(n_R) \sim e^{-g(n_R)}$

APPROX g, NOT W (builds in exp behavior)

ie $$W(n_R) = e^{\underline{\ln W(n_R)}}$$

→ MUCH SMOOTHER FN

→ EASIER TO APPROX

TAYLOR SERIES: (IN $\eta = n_R - \tilde{n}_R$ AROUND ZERO)

$$\ln W(n_R) \approx \ln W(\tilde{n}_R) + B_1 \eta + \frac{1}{2} B_2 \eta^2 + \frac{1}{3!} B_3 \eta^3 + \cdots$$

$$\boxed{B_k = \frac{d^k(\ln W(\tilde{n}_R))}{d n_R^k}}$$

KNOW

$B_1 = 0$

$B_2 < 0$

THEN

$$W(n_R) \approx e^{\left[\ln W(\tilde{n}_R) - \frac{1}{2}|B_2|\eta^2 + \cdots\right]}$$

$$\approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

"GAUSSIAN DISTR."

→ WILL CLEAN UP LATER

⇒ VERY GENERAL:

→ ONLY ASSUMED EXP. FALL OFF

→ VERY COMMON WHEN VARIABLE IS SUM OF LARGE # N OF RANDOM VARIABLES (HERE N STEPS OF $s_i = \pm l$ WITH PROB p, q)

("CENTRAL LIMIT THM" → ALMOST ANY PROB FOR INDIV. s_i GIVES THIS FORM)

RANGE OF VALIDITY:

- GOOD APPROX FOR $\ln W$ IF

$$\left|\frac{1}{3!} B_3 \eta^3\right| << \left|\frac{1}{2} B_2 \eta^2\right|$$

ie fails for large enough η

any - WILL SEE: WHEN " ~ "

W NEGLIGIBLE

$\Rightarrow$ APPROX GOOD FOR ALL η

<u>ex</u> (REIF):

$$f(y) = \frac{1}{(1+y)^N} = e^{-N \ln(1+y)}$$

{ powers, exponentials ~same

→ CHANGES SHARPLY IF N LARGE } <u>see picture</u>
→ $\ln f(y)$ MUCH SMOOTHER }

APPROX NEAR $y = 0$

<u>DIRECTLY</u>: $f(y) \sim 1 - Ny + \frac{1}{2}N(N+1)y^2 + \cdots$

N large → NOT SENSIBLE UNLESS $|Ny| \lesssim 1$
OR $|y| \lesssim 1/N$

<u>LOG</u>: $\ln f(y) \sim -N(y - \frac{1}{2}y^2 + \cdots)$

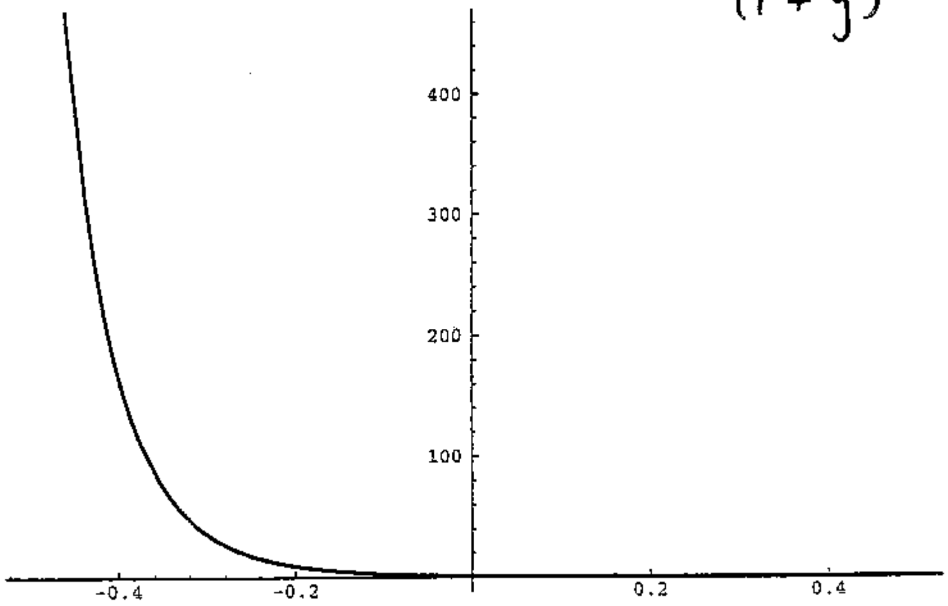
FAILS IF $|y^2| \approx |y|$ OR $|y| \approx 1$
(indep of N)

$f(y) \sim e^{-N(y - \frac{1}{2}y^2 + \cdots)}$

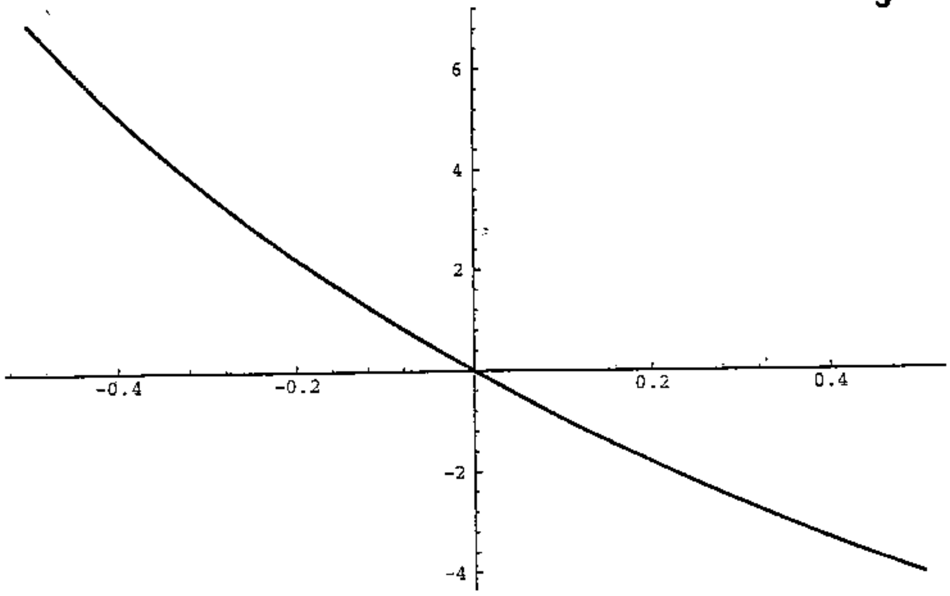
⇒ picture:

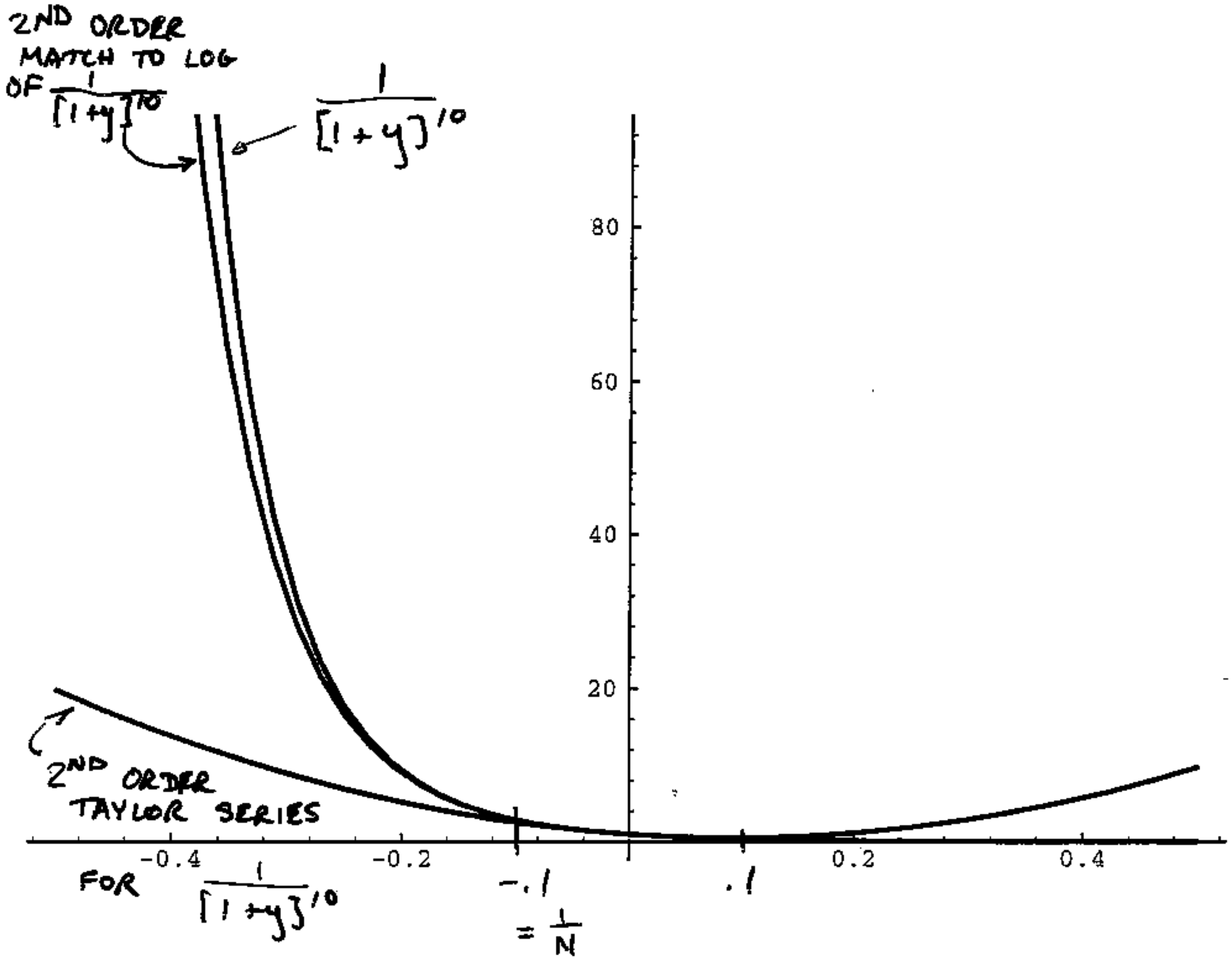
$N = 10$
1ST pict dies at $y \sim \pm \frac{1}{10} = \pm .1$

$$\frac{1}{(1+y)^{10}}$$



$$\ln\left[\frac{1}{(1+y)^{10}}\right]$$




2ND ORDER
MATCH TO LOG
OF $\frac{1}{[1+y]^{10}}$
$\frac{1}{[1+y]^{10}}$
80
60
40
20
2ND ORDER
TAYLOR SERIES
FOR $\frac{1}{[1+y]^{10}}$
-0.4
-0.2
-.1
$= \frac{1}{N}$
.1
0.2
0.4

APPROX $\ln W(n_R)$: (cf REIF FOR DETAILS or better, try yourself)

FIND $\tilde{n}_R$ AT MAX:

(a) $$\frac{d \ln W(n_R)}{d n_R} = 0 \approx \ln W(n_R+1) - \ln W(n_R)$$

SOLN $\boxed{\tilde{n}_R = Np}$

(b) EXPAND AROUND $\tilde{n}_R$ $(\eta = n_R - \tilde{n}_R)$

COEFFICIENTS :

$$B_1 = 0$$

$$\boxed{B_2 = -\frac{1}{Npq}} \quad < 0, \text{ AS EXPECTED}$$

$$B_3 = \frac{q^2 - p^2}{N^2 p^2 q^2} \quad \text{ETC}$$

$\Rightarrow$ OBSERVE:
EACH NEW TERM $B_k \eta^k$ HAS ADDITIONAL FACTOR $\frac{\eta}{Npq}$ TIMES PREVIOUS SERIES OK IF $\frac{\eta}{Npq} \ll 1$ simp; [illegible] note

note (N BIG; p, q NOT TOO SMALL; CF PROB 1.9)
(and when $\eta \sim Npq$, $W \sim 0$)

(c) $$W(\tilde{n}_R) \cong \frac{1}{(2\pi Npq)^{1/2}}$$ STIRLING FORMULA) (cf text or NOTES
(could also use normal curve)

COMBINE: $$W(n_R) = W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$$

$$= \frac{1}{(2\pi Npq)^{1/2}}\, e^{-\frac{(n_R - Np)^2}{2Npq}} + \left[\begin{array}{c}\text{CORRECTIONS}\\ \text{WHICH VANISH}\\ \text{AS } N \to \infty\end{array}\right]$$

APPROX $\ln W(n_R)$: (cf REIF FOR DETAILS (or try yourself))

PROBLEM: NEED DERIVS

ONLY DEF'D AT INTEGER n_R'S

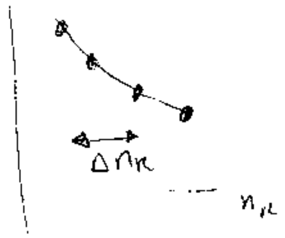
BEST APPROX:

$$\frac{d \ln W(n_R)}{d n_R} \approx \frac{\Delta \ln W(n_R)}{\Delta n_R}$$

SMALLEST $\Delta n_R = 1$

$$= \ln W(n_R+1) - \ln W(n_R)$$

OK IF $\ln W(n_R)$ SMOOTH:



VS

(saw in plot that W becomes smoother as N incr.)

(a) MAX: $\frac{d \ln W}{d n_R} = 0$ *

SOLN: $\boxed{\tilde{n}_R = N_p}$ (note: same as $\overline{n_R}$)

---

* NEED THINGS LIKE (for n large)

$$\frac{\Delta \ln n!}{\Delta n} \approx \frac{\ln (n+1)! - \ln n!}{1} = \ln \frac{(n+1)!}{n!} \Big| = \ln(n+1) \sim \ln n$$

$\{$ or $\ln(n+1) + \ln n + \ln(n-1) + \ldots$
$- \ln(n) - \ln(n-1) + \ldots \;]$

so $n \gg 1 \rightarrow \frac{d \ln n!}{dn} \sim \ln n$ ETC

{ IN DETAIL:

$W(\tilde{n}_R)$ using Stirling's formula:

$$W(n_R) = \frac{N!}{(N-n_R)!\,n_R!}\, p^{n_R} q^{N-n_R}$$

$$\tilde{n}_R = Np$$

$$W(\tilde{n}_R) = \frac{N!}{(N-Np)!\,(Np)!}\, p^{Np} q^{N-Np}$$

$$= \frac{N!}{(Nq)!\,(Np)!}\, p^{Np} q^{Nq}$$

$$\ln W(\tilde{n}_R) = \ln N! - \ln(Nq)! - \ln(Np)! + Np\ln p + Nq\ln q$$

Stirling:

$$\ln n! \underset{n\ \text{large}}{\sim} n\ln n - n + \frac{1}{2}\ln(2\pi n) + \cdots$$

$$\begin{aligned}\ln W(\tilde{n}_R) \sim\ & N\ln N - N + \frac{1}{2}\ln(2\pi N)\\ & -Nq\ln Nq + Nq - \frac{1}{2}\ln(2\pi Nq)\\ & -Np\ln Np + Np - \frac{1}{2}\ln(2\pi Np)\\ & +Np\ln p + Nq\ln q\end{aligned}$$

from $p+q=1 \rightarrow -Nq\ln Nq - Np\ln Np = \underbrace{-Nq\ln N - Np\ln N}_{-N\ln N} - Nq\ln q - Np\ln p$

everything cancels but

$$\frac{1}{2}\left[\ln(2\pi N) - \ln(2\pi Nq) - \ln(2\pi Np)\right] = \frac{1}{2}\ln\frac{2\pi N}{(2\pi Nq)(2\pi Np)}$$

$$= \frac{1}{2}\ln\frac{1}{2\pi Npq}$$

$$\Rightarrow\ W(\tilde{n}_R) = \left(\frac{1}{2\pi Npq}\right)^{\frac{1}{2}}$$

}

THEN $W(n_R) \approx W(\tilde{n}_R)\, e^{-\frac{1}{2}|B_2|\eta^2}$

USE $W(\tilde{n}_R) \hat{=} \frac{1}{(2\pi Npq)^{1/2}}$ (STIRLING'S FORMULA)

$$|B_2| = \frac{1}{Npq}$$

AND $\eta \equiv n_R - \tilde{n}_R = n_R - Np$

$$W(n_R) = \frac{1}{(2\pi Npq)^{\frac{1}{2}}} e^{-\frac{(n_R - Np)^2}{2Npq}} + \underset{\sim \frac{1}{N}}{\text{CORRECTIONS}}$$

CONVERT TO $\mathcal{P}(x) = \frac{P(m)}{2l}$

USE $n_R = \frac{1}{2}(N+m)$, $x = ml$

ALSO
$$\boxed{\begin{aligned} \mu &\equiv (p-q)Nl \\ \sigma^2 &\equiv 4Npql^2 \end{aligned}}$$

(put on side)

{ micro information encapsulated in these 2 parameters

THEN
$$\boxed{\mathcal{P}(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} e^{-(x-\mu)^2/2\sigma^2}}$$

GAUSSIAN DISTRIBUTION

cf PLOT

slide

→ ALMOST ALWAYS GET THIS
WHEN ADD LARGE # RAND. VARS.
(CENTRAL LIMIT THM — cf * SECTIONS FOR PROOF)

DISCUSS: μ, σ JUST DEFNS, BUT HAVE SIMPLE INTERP:

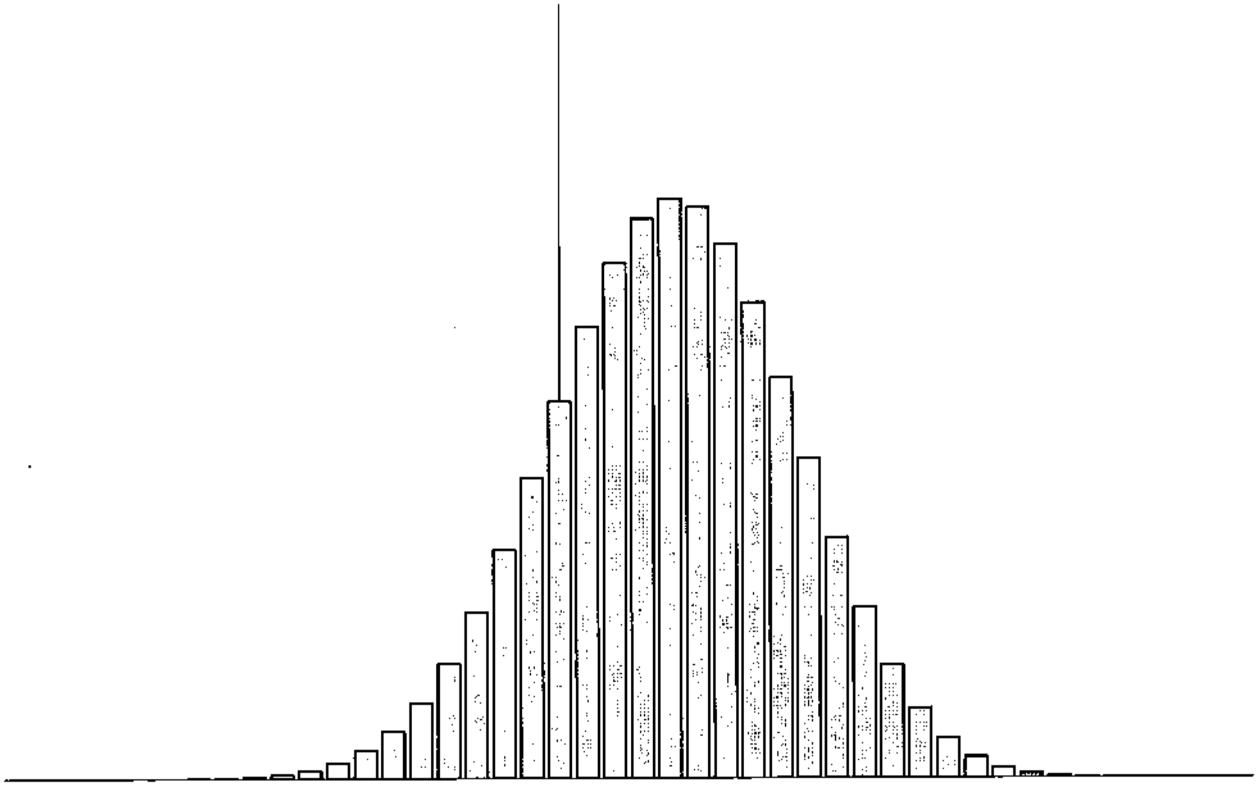
PEAKED AT $x = \mu$

FALLS OFF FOR $|x-\mu| \gtrsim \sigma \Rightarrow \sigma \sim$ WIDTH

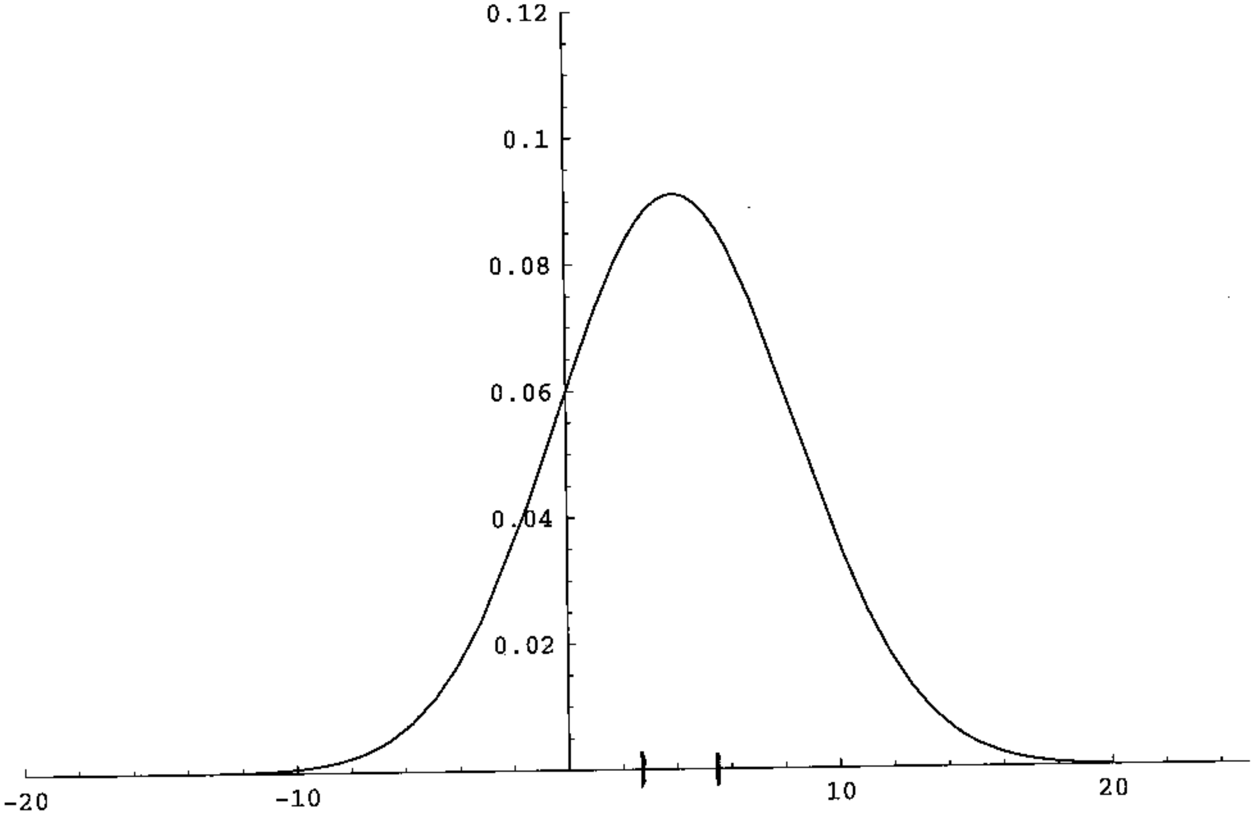
$N = 20 \quad p = 0.6 \quad \ell = 1$

GAUSSIAN vs BINOMIAL

DISTRIBUTIONS



1.23-B
OVERLAY



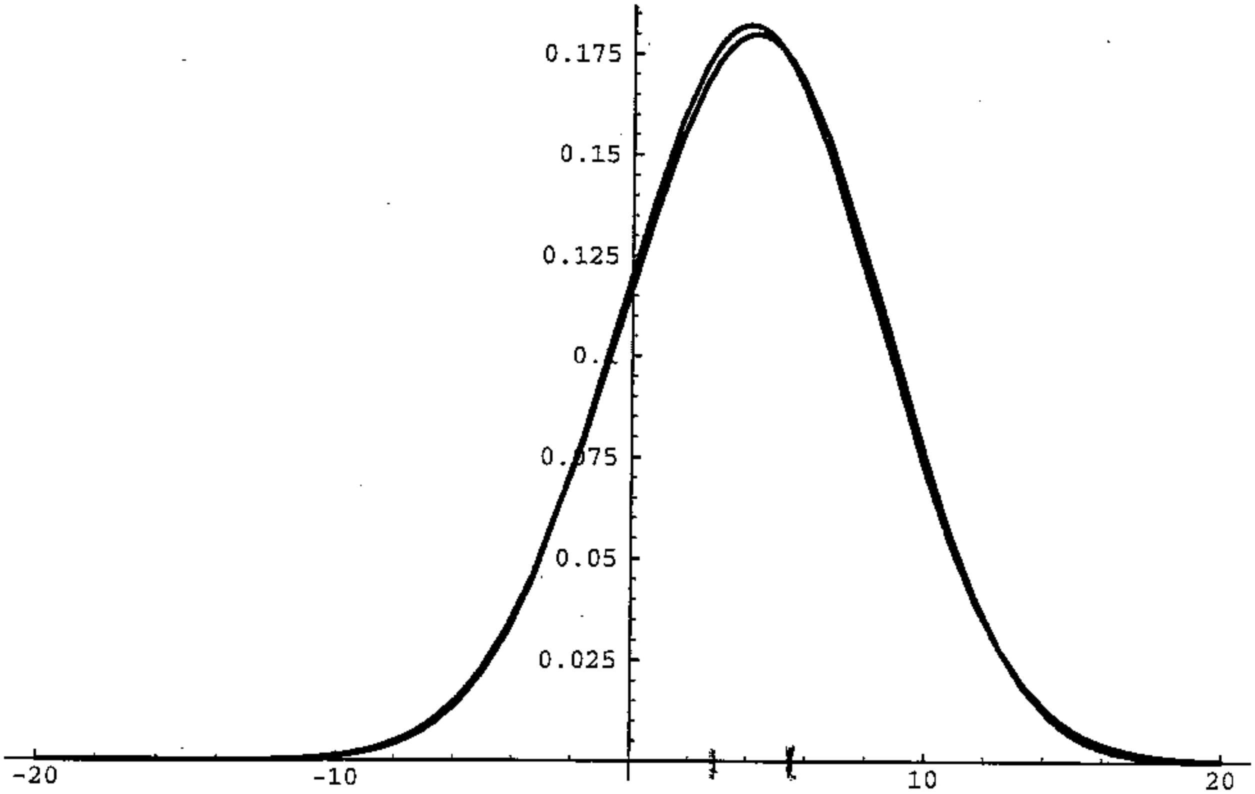
RANGE OF TAYLOR SERIES

$P(x)$, $\lambda=1$

note even this approx fails for some x; looks good for all x here; why?
{⇒ at x where fails, $P \sim 0$}

N=20 p=.6

GAUSSIAN VS BINOMIAL
DISTRIBUTIONS
(PLOTTED AS
CONTINUOUS FNS)



RANGE OF TAYLOR SERIES