Now for Radiation – Photon gas

$$2mv_x \longrightarrow 2p_x \quad \text{for massless particles}$$

$$P' = \frac{N}{V} 2m v_x^2 \rightarrow \frac{N}{V} 2 p_x v_x$$

Average Pressure

assume N = constant

$$P = \frac{N}{V} \langle p_x v_x \rangle$$

$$\langle p_x v_x \rangle = \langle p_y v_y \rangle = \langle p_z v_z \rangle$$

$$\langle p_x v_x \rangle = \frac{1}{3} \langle p_x v_x + p_y v_y + p_z v_z \rangle = \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle$$

$$P = \frac{N}{V} \frac{1}{3} \langle \vec{p} \cdot \vec{v} \rangle$$

For photons $\vec{p} \parallel \vec{v}$ and $v = c$

$$\vec{p} \cdot \vec{v} = pc = E = \frac{U}{N}$$

$$PV = \frac{1}{3} U \equiv (\gamma - 1) U \Rightarrow \gamma = \frac{4}{3} \neq \frac{c_p}{c_V}$$

Adiabatic Process $\Rightarrow Q = 0, \Delta S = 0$
(Slow, Reversible)

$$PV^{\gamma} = \text{constant} \Rightarrow PV^{4/3} = \text{const}$$

Isentropic Compressibility: $$K_s = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_s = \frac{1}{\gamma P}$$

Lagrange Multiplier Refresher

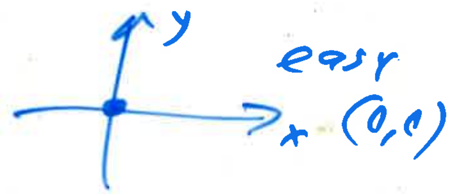
max
min
inflection

"Easy" way to find stationary solutions
subject to constraints.

Mechanics - minimize Action : S

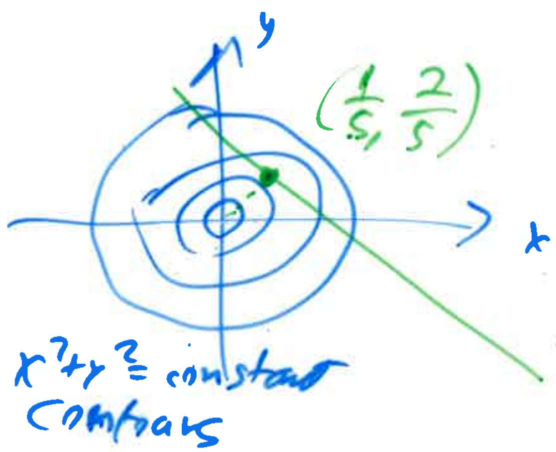
Thermodynamics - maximize Entropy : S

e.g. Math : minimize $g = x^2 + y^2$



but subject to the constraint

$f = x + 2y - 1 = 0$ $\qquad y = -\frac{x}{2} + \frac{1}{2}$



"Hard" way $\quad x = 1 - 2y$

$$g = x^2 + y^2 = (1-2y)^2 + y^2$$

$$g = 1 + 5y^2 - 4y = g(y)$$

$$\left.\frac{dg(y)}{dy}\right|_{y=y_0} = 0 = \left.(10y - 4)\right|_{y=y_0} = 0 \rightarrow y_0 = +\frac{2}{5} \checkmark$$

$$x_0 = 1 - 2y_0 = 1 - \frac{4}{5} = +\frac{1}{5} \checkmark$$

"Easy" way with Lagrange multiplier β

Minimize $x^2+y^2+\beta(x+2y-1)$ with respect to x, y

$$\frac{\partial}{\partial x}\left[x^2+y^2+\beta(x+2y-1)\right]=0 \Rightarrow 2x+\beta=0 \Rightarrow x=-\frac{\beta}{2}$$

$$\frac{\partial}{\partial y}\left[x^2+y^2+\beta(x+2y-1)\right]=0 \Rightarrow 2y+2\beta=0 \Rightarrow y=-\beta$$

Choose β to satisfy the constraint

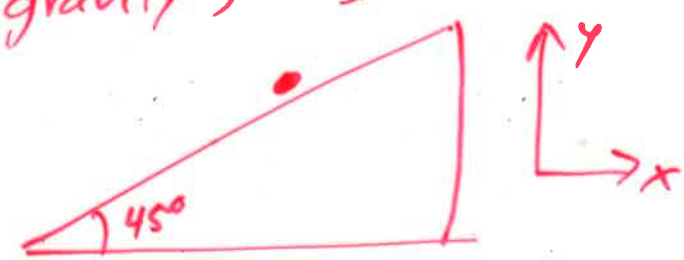
$$x+2y-1=0 \Rightarrow -\frac{\beta}{2}-2\beta-1=0 \Rightarrow \beta=-\frac{2}{5}$$

Find x_0, y_0

$$x_0=-\frac{\beta}{2}=+\frac{1}{5} \qquad \Big/ \qquad y_0=-\beta=+\frac{2}{5}$$

Example from Classical Mechanics

Particle of mass m in 2 dimensions with gravity, subject the constraint $y=x$.



Kinetic Energy

$$T=\frac{m(\dot{x}^2+\dot{y}^2)}{2}$$

Potential Energy $V = mgy + \cancel{\text{constant}}^{\,0}$

Constraint: $f=y-x=0$

Lagrangian: $L = T - V + \lambda f$

Minimize Action: $S = \int_{t_1}^{t_2} L\, dt$

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2}\dot{x}^2 + \frac{m}{2}\dot{y}^2 - mgy + \lambda(y-x)\right] dt$$

Euler - Lagrange Equations

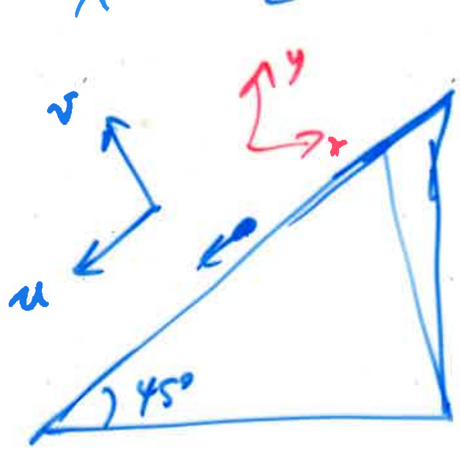
$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \Longrightarrow -\lambda - m\ddot{x} = 0$$

$$\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = 0 \Longrightarrow -mg + \lambda - m\ddot{y} = 0$$

Constraint $y = x$, $\dot{y} = \dot{x}$, $\ddot{y} = \ddot{x}$

$$m\ddot{x} = m\ddot{y} \longrightarrow -\lambda = -mg + \lambda \Longrightarrow \lambda = +\frac{mg}{2}$$

$\ddot{x} = -\frac{g}{2}$ ✓ $\ddot{y} = -\frac{g}{2}$ ✓



$a_z = 0$

$a_u = g\sin\theta = \frac{g}{\sqrt{2}}$

$a_x = \frac{-g}{\sqrt{2}} \cos 45° = -\frac{g}{2} = -g \sin\alpha \cos\alpha$

$a_y = \frac{-g}{\sqrt{2}} \sin 45° = \frac{-g}{2} = -g\sin^2\alpha$