Random Walk (Drunkard's Walk)

1-dimension

N total steps. Probability p of taking one step to the right. Prob $q = (1-p)$ of stepping to the left. Start at origin.

Prob. that we end up N steps to the right?

$$P(N) = ? = p^N = \underbrace{R \text{ and } R \text{ and } R \ldots R}_{N \text{ times}}.$$

Prob that we end up N steps to the left?

= Prob " " " 0 steps to the right.

$$P(0) = q^N = (1-p)^N$$

Prob that we take exactly one step to the right.

$$P(1) = N p^1 q^{N-1}$$

$$\binom{N}{1} \rightarrow = N p (1-p)^{N-1}$$

$$\left.\begin{matrix} R\,L\,L\,L \ldots L \\ L\,R\,L\,L \ldots L \\ L\,L\,R\,L \ldots L \\ \vdots \\ L\,L\,L \ldots L\,R \end{matrix}\right\} \begin{matrix} N \text{ ways} \\ \text{= } N \text{ microstates} \end{matrix}$$

Prob. that we take n steps to the right. ($N-n$ to the left.)

$$P(n) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}$$

Binomial Distribution

$(p+q)^0 = 1$

$(p+q)^1 = 1p + 1q$

$(p+q)^2 = 1p^2 + 2pq + 1q^2$

$(p+q)^3 = 1p^3 + 3p^2q + 3pq^2 + 1q^3$

$$\begin{array}{ccccc} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{array}$$

$\binom{4}{1}$

$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{4\cdot3\cdot2\cdot1}{2\cdot2} = 6$

Polling relative error $\pm 3\%$ means $N = 1000$

absolute error $\sqrt{N} = \sqrt{1000}$

~~fraction~~ relative error
fractional error. $= \frac{\sqrt{N}}{N} = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1000}}$

$\approx \frac{1}{33} \approx \frac{3}{100} = 3\%$

Remember:

$$(p+q)^N = \underset{\binom{N}{N}}{1}p^N q^0 + \underset{\binom{N}{N-1}}{N}p^{N-1}q^1 + \underset{\binom{N}{N-2}}{\frac{N(N-1)}{2}}p^{N-2}q^2 + \cdots + \underset{\binom{N}{1}}{N}p^1 q^{N-1} + \underset{\binom{N}{0}}{1}p^0 q^N$$

Is $P(n)$ Normalized?

$$\sum_{n=0}^{N} P(n) = P(0) + P(1) + \cdots + P(N)$$

$$= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n \underbrace{(1-p)}_{q}{}^{N-n}$$

$$= (p+q)^N = (1)^N = 1 \quad \checkmark$$

Mean number of steps to the right

$$\langle n_R \rangle = \langle n \rangle = \sum_{n=0}^{N} P(n)\, n$$

$$= \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n\, p^n q^{N-n}$$

p, q independent

$$n\, p^n = p \left(\frac{\partial}{\partial p} p^n \right)_q$$

$$\langle n \rangle = p \frac{\partial}{\partial p} \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n} = p \frac{\partial}{\partial p} \left[(p+q)^N \right]_q$$

$$\langle n\rangle = pN(p+q)^{N-1} \qquad \text{use } p+q=1$$

$$\boxed{\langle n\rangle = pN = \langle n_R\rangle}$$

$$\langle n_L\rangle = qN = (1-p)N$$

$$\langle n_R + n_L\rangle = \langle n_R\rangle + \langle n_L\rangle = N$$

displacement $d = n_R - n_L = 2n_R - N = 2n - N$

$$\langle d\rangle = \langle n_R\rangle - \langle n_L\rangle = pN - (1-p)N = (2p-1)N$$

$$\langle n^2\rangle = \sum_{n=0}^{N} n^2 P_{(n)} = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} n^2 p^n q^{N-n}$$

$$n^2 p^n = \left(p\frac{\partial}{\partial p}\right)^2 p^n \Big|_q$$

$$\langle n^2\rangle = \left(p\frac{\partial}{\partial p}\right)^2 \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n} = \left(p\frac{\partial}{\partial p}\right)^2 (p+q)^N$$

$$= p\frac{\partial}{\partial p}\left[pN(p+q)^{N-1}\right]$$

$$= p\left[N(p+q)^{(N-1)} + pN(N-1)(p+q)^{N-2}\right] \qquad p+q=1$$

$$= p\left[N + pN(N-1)\right] = pN + p^2N^2 - p^2N$$

$$\langle n^2\rangle = \langle n\rangle^2 + pN(1-p) = \langle n\rangle^2 + Npq$$

Standard Deviation d - steps to the right

$$\boxed{\sigma_n = \sqrt{\langle n^2\rangle - \langle n\rangle^2} = \sqrt{Npq}}$$

Displacement $d = n_R - n_L$

$$\langle d\rangle = 2\langle n\rangle - N = (2p-1)N$$

$$\sigma_d = \sqrt{\langle d^2\rangle - \langle d\rangle^2} = 2\sqrt{Npq}$$

If $p = \frac{1}{2} = q \Longrightarrow \boxed{\sigma_d = \sqrt{N}}$ root mean square

Large N, Binomial distribution $\rightarrow$ Gaussian distribution

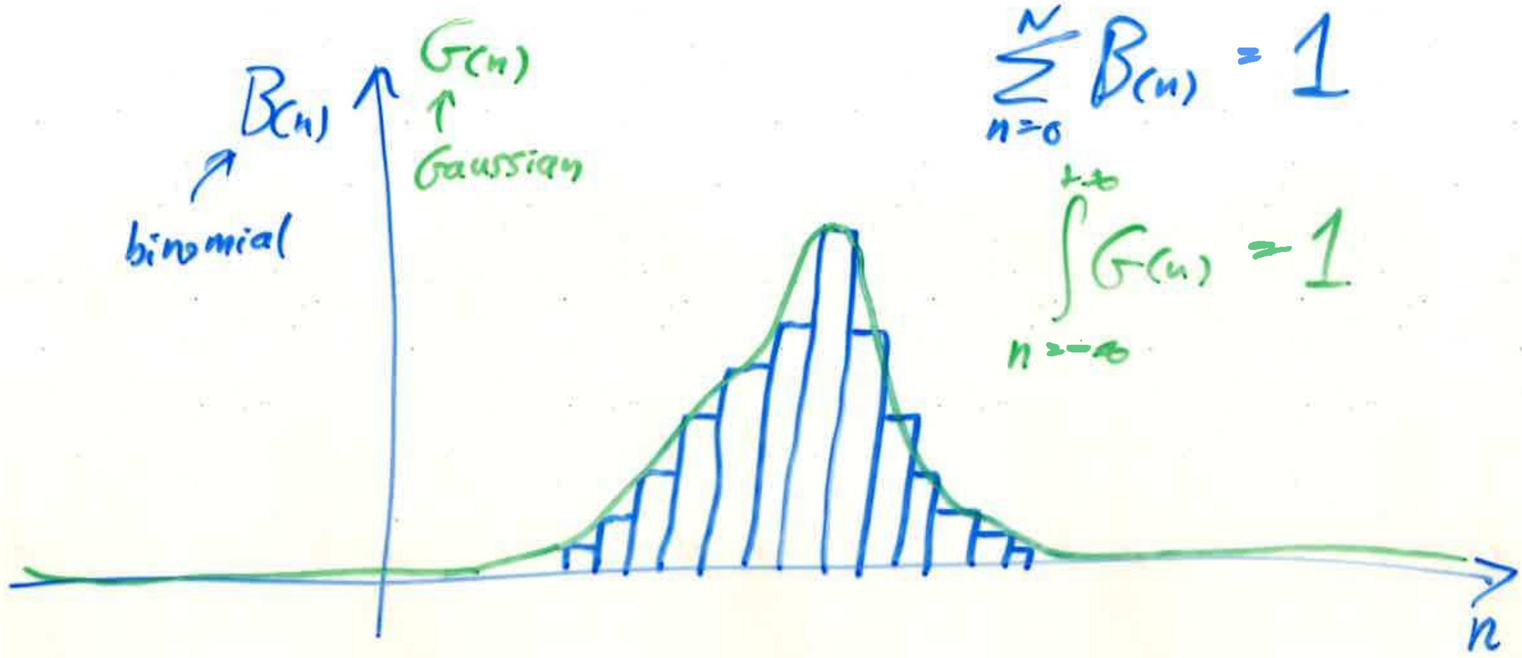
discrete continuous

Mean $\mu = Np$

$$\sigma_n = \sigma = \sqrt{Npq} = \sqrt{Np(1-p)}$$

$$P_G(n) \rightarrow \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(n-\mu)^2}{2\sigma^2}\right]$$

Central Limit Theorem.



$$\sum_{n=0}^{N} B(n) = 1$$

$$\int_{n=-\infty}^{+\infty} G(n) = 1$$