Hypercubes + hypercylinders

\[ \int_{-\infty}^{\infty} e^{-x_1^2} \, dx_1 \cdot \int_{-\infty}^{\infty} e^{-x_2^2} \, dx_2 \cdots \int_{-\infty}^{\infty} e^{-x_n^2} \, dx_n = \prod_{k=1}^{n} \int_{-\infty}^{\infty} e^{-x_k^2} \, dx_k \]

\[ = (\sqrt{\pi})^n = \int \, d\Omega \int e^{-r^2} \, r^{n-1} \, dr \]

[\text{solid angle}] \quad r = 0 \quad \frac{1}{2} \Gamma\left(\frac{n}{2}\right)

\[ r^2 = x_1^2 + x_2^2 + \cdots + x_n^2 \]

\[ \Omega_{n-1} = (n-1) \text{ dimensional solid angle} = \frac{2 \pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} = \frac{2 \pi^{n/2}}{(n-1)!} \]

Surface area of an \((n-1)\) dimensional sphere.

Sphere: all points a distance \(R\) from a center.

\[ S_{n-1} = \Omega_{n-1} R^{n-1} \quad \Rightarrow \quad S_n = \Omega_n R^n = \frac{2 \pi^{n/2} R^n}{(n-1)!} \]

Ball: all points a distance \(\leq R\) from a center.
\[ V_n = \sum_{r=0}^{R} S_{n-1}(r) \, dr = \frac{S_{n-1} \, R}{n} = \frac{2 \pi^{n/2} \, R^n}{n(n^{1/2}-1)!} \]

<table>
<thead>
<tr>
<th>n</th>
<th>( S_n )</th>
<th>( V_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( 2 \pi R )</td>
<td>( 2 \pi R )</td>
</tr>
<tr>
<td>2</td>
<td>( 4 \pi R^2 )</td>
<td>( \pi R^2 )</td>
</tr>
<tr>
<td>3</td>
<td>( 2 \pi^2 R^3 )</td>
<td>( \frac{4}{3} \pi R^3 )</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{\pi^2 R^4}{2} )</td>
<td>( \frac{\pi^2 R^4}{2} )</td>
</tr>
</tbody>
</table>
Kinetic Theory of Gases

Atoms + Newton's Laws + Maxwell's Eqs. (not Quantum Mech.)

\[ P = \frac{F}{A} \]

\[ \text{Volume} = \frac{A}{x} \]

Perfect reflection from piston

Collisions are elastic

\( P_x \rightarrow -P_x \)
\( P_y \rightarrow P_y \)
\( P_z \rightarrow P_z \)

\( \rightarrow \) some kinetic energy before \( \rightarrow \) after collision

\( \rightarrow \) some speed before \( \rightarrow \) after

Total momentum delivered to the piston

\[ 2m_v x = 2P_x \]
Now watch the box for a time \( dt \). Only atoms within distance \( v_x \cdot dt \) will hit the piston.

Number of atoms that hit the piston in time \( dt \)

\[
dN = \frac{N}{V} v_x \, dt \, A
\]

Rate at which the piston is hit

\[
\frac{dv}{dt} = \frac{N}{V} v_x \, A
\]

Force applied to the piston

\[
F = \left( \frac{N}{V} v_x A \right) \left( 2m v_x \right) = \frac{dp}{dt}
\]

\[
P = \frac{F}{A} = \frac{N}{V} v_x^2 \, 2m \quad \text{all } v_x \text{'s different} \quad \Rightarrow \text{take average}
\]

Average Pressure \( P = \frac{Nm \langle v_x^2 \rangle}{V} \)

What happened to the 2? Only \( \frac{1}{2} \) of the atoms within \( v_x \cdot dt \) were going forward the wall. The other half were moving away.
\[
\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle \quad \text{isotropic}
\]
\[
\langle v_x^2 \rangle = \frac{1}{3} \langle v_x^2 + v_y^2 + v_z^2 \rangle = \frac{1}{3} \langle v^2 \rangle
\]
\[
\Rightarrow P = \frac{N}{V} \frac{m}{3} \langle v^2 \rangle = \frac{2}{3} \frac{N}{V} \langle \frac{1}{2} m v^2 \rangle
\]

average kinetic energy for one molecule.

\[
PV = \frac{2}{3} N \langle KE \rangle = \frac{2}{3} U
\]

If gas is monatomic
\[
U = \frac{3}{2} k_B T N
\]
equipartition theorem

\[
PV = \frac{2}{3} U = (\gamma - 1) U \quad \gamma = \frac{5}{3} = \frac{5}{2} \nu
\]

Adiabatic \( PV^n = \text{constant} \).