Black body Spectrum

photons – massless, spin-1 particles

quantum number : s (angular momentum)

z-projection $m_s = \pm 1\hbar$ ($0\hbar$ missing)

length of spin vector $|\vec{s}| = \sqrt{s(s+1)}\,\hbar = \sqrt{2}\,\hbar$

$\omega = 2\pi\nu$

QM Stat. Mech.

Energy states $E_j = (j+\frac{1}{2})\hbar\omega = (j+\frac{1}{2})h\nu$

Ground state energy $E_0 = \frac{1}{2}\hbar\omega$

j = # of photons, mode number, occupation #

Partition Function

$$Z = \sum_{j=0}^{\infty} e^{-\beta E_j} = \sum_{j=0}^{\infty} e^{-\beta(j+\frac{1}{2})\hbar\omega}$$

$$= e^{-\beta\frac{\hbar\omega}{2}} + e^{-\beta\frac{\hbar\omega}{2} - \beta\hbar\omega} + e^{-\beta\frac{\hbar\omega}{2} - 2\beta\hbar\omega} + \ldots$$

$$= e^{-\beta\frac{\hbar\omega}{2}} \underbrace{\left(1 + e^{-\beta\hbar\omega} + e^{-2\beta\hbar\omega} + \ldots\right)}_{1 + r + r^2 + \ldots}$$

$$\sum_j r^j = \frac{1}{1-r}$$

$$Z = e^{-\frac{\beta\hbar\omega}{2}} \sum_{j=0}^{\infty} e^{-\beta\hbar\omega j} = e^{-\frac{\beta\hbar\omega}{2}} \left(\frac{1}{1-e^{-\beta\hbar\omega}} \right)$$

$$\ln(Z) = \ln\left(e^{-\frac{\beta\hbar\omega}{2}}\right) - \ln\left(1-e^{-\beta\hbar\omega}\right)$$

$$= -\frac{1}{2}\beta\hbar\omega - \ln\left(1-e^{-\beta\hbar\omega}\right)$$

$$U_1 = \langle E_1 \rangle = -\frac{\partial \ln(Z)}{\partial \beta} = \frac{\hbar\omega}{2} + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}}$$

multiply the last term by

$$\left(\frac{e^{+\beta\hbar\omega}}{e^{+\beta\hbar\omega}}\right) = 1$$

$$U_1 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{+\beta\hbar\omega} - 1}$$ single mode

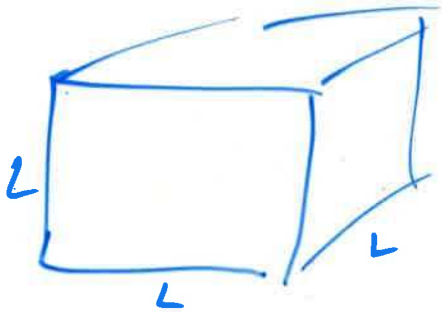
Wavenumber $k = \frac{2\pi}{\lambda}$

Dispersion Relation

$$\omega(|\vec{k}|) = c|\vec{k}|$$

even though no dispersion here.

Now put the photon gas in a box



$$k_x = \frac{2\pi}{\lambda} = \frac{n_x \pi}{L}$$

$$k_y = \frac{n_y \pi}{L}$$

$$k_z = \frac{n_z \pi}{L} \rightarrow n_z = \frac{k_z L}{\pi}$$

Sum over all modes in cavity:

$$U(T) = \sum_{k_x} \sum_{k_y} \sum_{k_z} \underset{\underset{\text{polarizations}}{\uparrow}}{\sum_{\epsilon=1}^{2}} \left[\hbar\omega(|\vec{k}|) \left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega(|\vec{k}|)} - 1} \right) \right]$$

$$\boxed{\frac{1}{e^{\beta\hbar\omega} - 1} \quad \text{Planck Distribution}}$$

$$U(T) = \underbrace{V}_{L^3} \left[\underset{\underset{\infty}{\uparrow}}{u_0} + \overset{\overset{\text{polarizations}}{\downarrow}}{2} \iiint \left(\frac{\hbar c |\vec{k}| \cancel{\beta}}{e^{\beta\hbar c|\vec{k}|} - 1} \right) k_B T \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \frac{dk_z}{2\pi} \right]$$

Cartesian $k_x, k_y, k_z \rightarrow$ Spherical Polar

k_r, θ_k, φ_k

Introduce dimensionless $x = \beta\hbar c|\vec{k}| = \beta\hbar c k_r$ $\qquad dx = \beta\hbar c\, dk_r$

$$U = V\left[u_0 + 2\underbrace{\int_{\theta_k=0}^{\pi} \sin\theta_k\, d\theta_k \int_{\varphi_k=0}^{2\pi} d\varphi_k}_{4\pi} \int_{k_r=0}^{\infty} \frac{k_r^2}{(8\pi^3)} \frac{\beta\hbar c k_r (k_B T)}{e^{\beta\hbar c k_r}-1}\, dk_r\right]$$

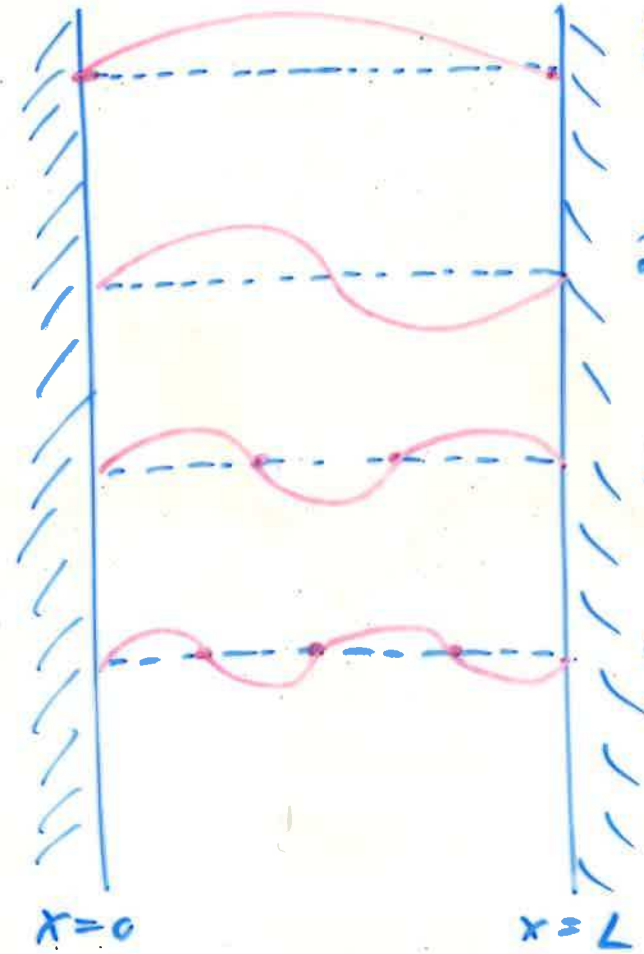
$$U_{(T)} = V\left[u_0 + \frac{k_B T}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3 \underbrace{\int_{x=0}^{\infty} \frac{x^3}{e^x-1}\, dx}_{\frac{\pi^4}{15}}\right]$$

$$\frac{U_{(T)}}{V} = u_0 + \underbrace{\frac{8\pi^5 k_B^4}{15 h^3 c^3}}_{\frac{4}{c}\sigma \leftarrow \text{Stefan-Boltzmann}} T^4$$

Boundary Conditions

① Hard-Wall B.C.

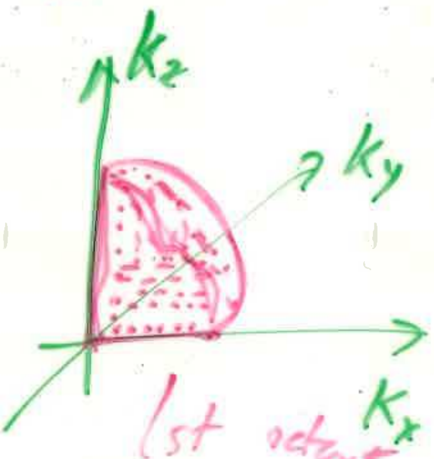
normal modes are standing waves (like a guitar string)



n_x	$\lambda_{n_x} = \frac{2L}{n}$	$k_{n_x} = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$
1	$\frac{2L}{1}$	$\frac{\pi}{L}$
2	$\frac{2L}{2}$	$\frac{2\pi}{L}$
3	$\frac{2L}{3}$	$\frac{3\pi}{L}$
4	$\frac{2L}{4}$	$\frac{4\pi}{L}$

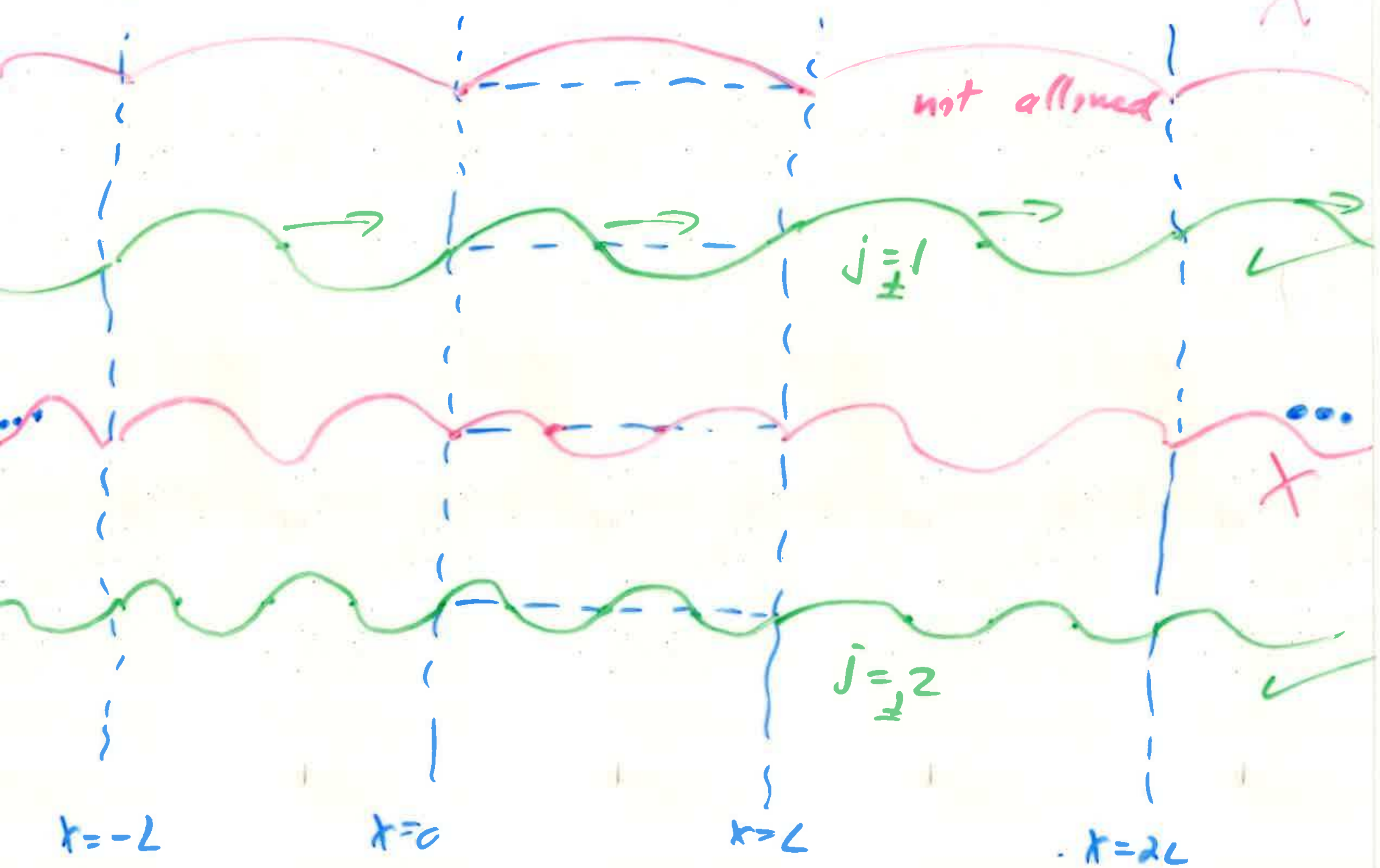
$k > 0$

k- space



$$e^{i\omega_n t}\sin(k_n x) = e^{i\omega t}\sin\left(\frac{n\pi x}{L}\right)$$

Periodic B.C.

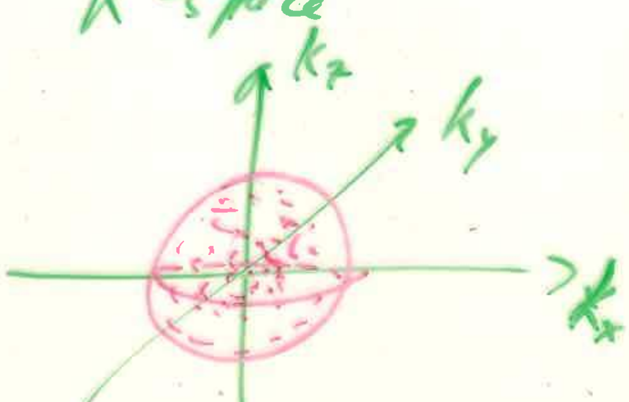


traveling waves : right-moving $k > 0$

left-moving $k < 0$

$$e^{i(\omega t - kx)} \rightarrow \; , \quad e^{i(\omega t + kx)} \leftarrow$$

k-space



$$\lambda_j = \frac{L}{j}$$

$$k_j = \frac{2\pi}{\lambda_j} = \frac{2j\pi}{L}$$

$$U(T) = \sum_{n_x} \sum_{n_y} \sum_{n_z} \sum_{\epsilon=1}^{2} \frac{\hbar\omega(|\vec{k}|)}{e^{+\beta\hbar\omega(|\vec{k}|)} - 1}$$

polarizations

If Hard-wall B.C. $\sum_{n_x=1}^{+\infty}$ $\qquad n_x = \frac{k_x L}{2\pi}$

If Periodic B.C. $\sum_{n_x=-\infty}^{+\infty} \longrightarrow \int_{n_x=-\infty}^{+\infty} dn_x = \left(\frac{L}{2\pi}\right)\int_{-\infty}^{+\infty} dk_x$

$$U(T) = L^3 \cdot 2 \iiint_{k_x\, k_y\, k_z = -\infty}^{+\infty} \frac{\hbar c|\vec{k}|}{e^{+\beta\hbar c|\vec{k}|} - 1} \frac{dk_x}{2\pi}\frac{dk_y}{2\pi}\frac{dk_z}{2\pi}$$

polarizations