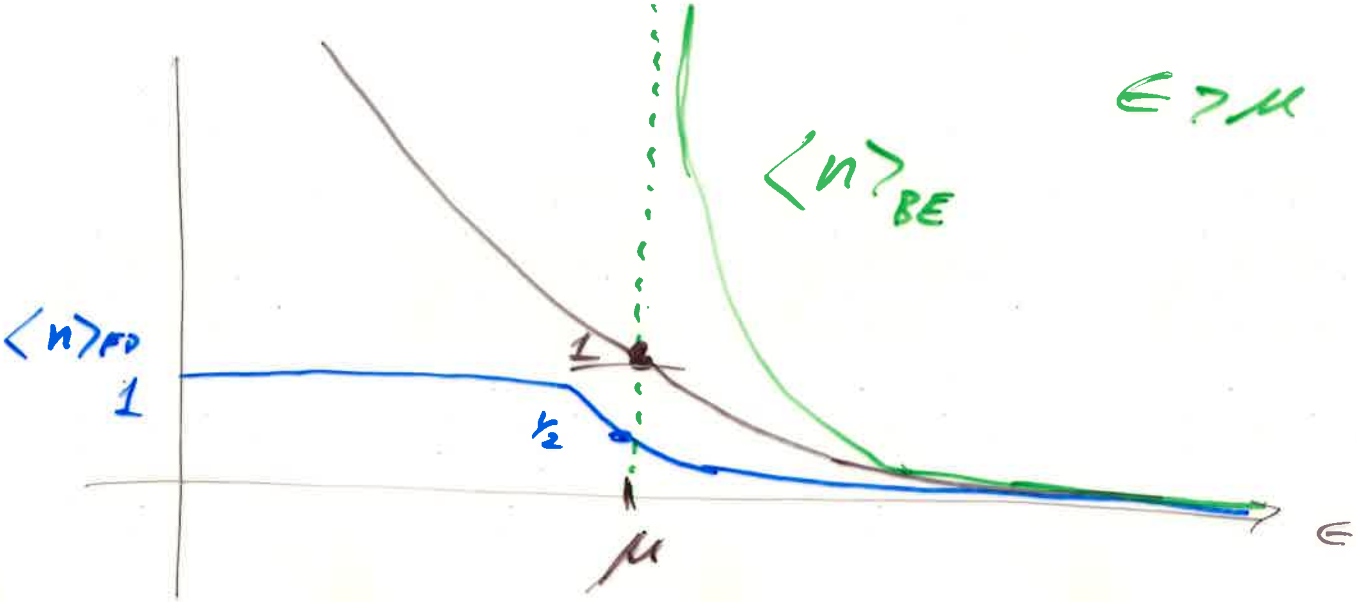


$$\left.\begin{aligned} \langle n_{BE} \rangle &= \frac{1}{e^{+\beta(\epsilon-\mu)} - 1} \\ \langle n \rangle_{FD} &= \frac{1}{e^{+\beta(\epsilon-\mu)} + 1} \end{aligned}\right\} \xrightarrow[\epsilon \gg \mu]{} e^{-\beta(\epsilon-\mu)} = \langle n \rangle_{\text{Boltzmann}}$$

Degenerate Fermi Gas

electrons in white dwarf, metal

neutrons in a neutron star

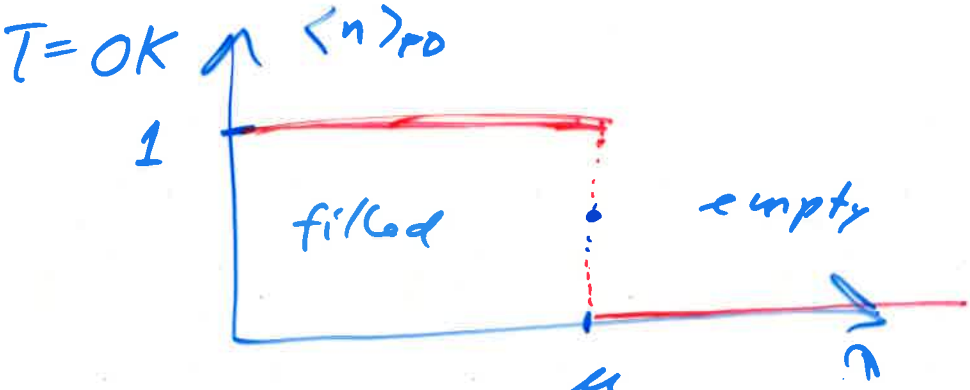
^3He

$$\frac{1}{n} = \frac{V}{N} \ll v_Q = \lambda_Q^3 = \left(\frac{h}{\sqrt{\pi 2 m k_B T}}\right)^3$$

electron in Cu at 300K

$$v_Q = (4.3\,nm)^3 \gg \frac{1}{n} = (0.2\,nm)^3$$

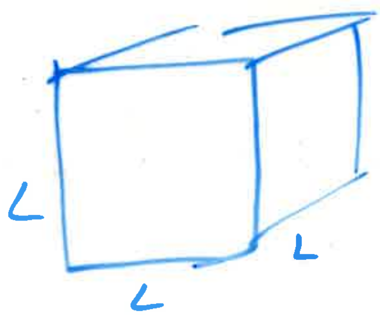
First, look at T=0K



μ

ϵ_F - fermi energy $\Big\}$ only at T = 0K

Fill a box with fermions (electrons)

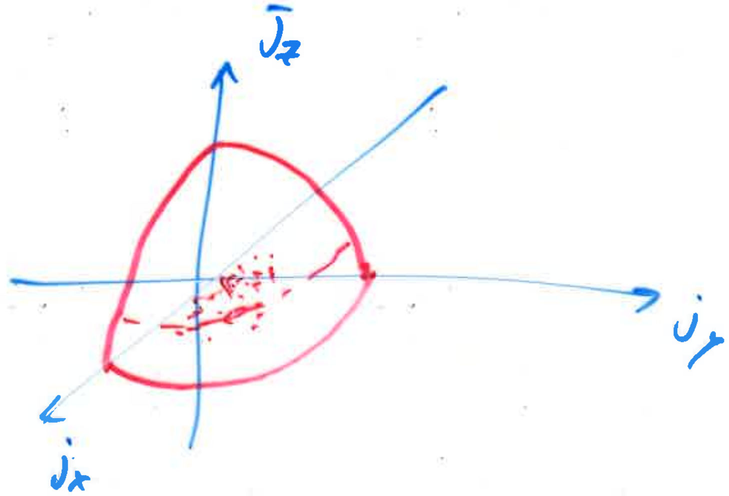


$$\lambda_x = \frac{2L}{j_x} \; ; \; p_x = \frac{h}{\lambda_x} = \hbar k_x = \frac{h j_x}{2L}$$

$$p_y = \frac{h j_y}{2L} \; , \; p_z = \frac{h j_z}{2L}$$

Energy

$$\epsilon_{\vec{j}} = \frac{\vec{p}^{\,2}}{2m} = \frac{p_x^2 + p_y^2 + p_z^2}{2m} = \frac{h^2}{8mL^2}\underbrace{(j_x^2 + j_y^2 + j_z^2)}_{\vec{j}^{\,2}}$$



$j_i > 0$

$$\epsilon_F = \frac{h^2}{8mL^2}\vec{j}_{max}^{\,2}$$

energy of particles on the "surface"

number of particles

$$N = \underset{\substack{\uparrow \\ \text{spins}}}{2} \cdot \frac{1}{8} \cdot \frac{4\pi}{3}|j_{max}|^3 = \frac{\pi}{3}(j_{max})^3$$

$$|j_{max}| = \left(\frac{3N}{\pi}\right)^{1/3} \quad / \quad \epsilon_F = \frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3}$$

intensive

$$= \frac{h^2}{8m}\left(\frac{3N}{\pi L^3}\right)^{2/3} =$$

intensive

$$= \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$$

Average Energy of system

$$U = \underset{\text{spin}}{2}\, \overbrace{\sum_{j_x}\sum_{j_y}\sum_{j_z}}^{j_{max}}_{0} \epsilon_{\vec{j}} = 2\iiint_{0}^{j_{max}} \epsilon_{\vec{j}}\, dj_x\, dj_y\, dj_z$$

Cartesian → Spherical

$$U = 2\underbrace{\int_0^{\pi/2}\sin\theta_j\, d\theta_j \int_0^{\pi/2} d\varphi_j}_{\frac{1}{8}(4\pi) = \frac{\pi}{2}} \int_0^{j_{max}} j^2 dj\, \frac{h^2}{8mL^2} j^2$$

$$j_{max} = \left(\frac{3N}{\pi}\right)^{1/3}$$

$$U = \frac{\pi h^2}{8mL^2}\underbrace{\int_0^{j_{max}} j^4 dj}_{\frac{j_{max}^5}{5}} = \frac{\pi h^2 j_{max}^5}{40\, mL^2}$$

$$U = \frac{\pi h^2}{40\, mL^2}\left(\frac{3N}{\pi}\right)^{5/3} = \underbrace{\left[\frac{h^2\left(\frac{3N}{\pi}\right)^{2/3}}{8mL^2}\right]}_{\epsilon_F}\frac{\cancel{\pi}\, 3N}{5\,\cancel{\pi}} = \frac{3}{5}N\epsilon_F$$

For conduction electrons in a metal

$\epsilon_F \sim eV$

Thermal energy $k_B T \sim \frac{1}{40} eV \ll \epsilon_F$
at room temperature

Fermi temperature $T_F = \frac{\epsilon_F}{k_B} = 11{,}000\,K \gg 300\,K$

also indicates degenerate Fermi gas

Degeneracy Pressure

$$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N} = -\frac{\partial}{\partial V}\left[\frac{3}{5} N \frac{h^2}{8m}\left(\frac{3N}{\pi}\right)^{2/3} V^{-2/3}\right]$$

$$= +\frac{2}{3}\left[\frac{3}{5} N \frac{h^2}{8m}\left(\frac{3N}{\pi}\right)^{2/3} V^{-5/3}\right]$$

$$= \frac{2}{\cancel{3}}\frac{\cancel{3}}{5} N \epsilon_F \frac{1}{V} = \frac{2}{5}\frac{N\epsilon_F}{V} = \frac{2}{3}\frac{U}{V} = \frac{2}{3}u$$

$\boxed{\text{Cf. photons } P = \frac{1}{3}u}$