$$① \rightarrow \frac{2}{5} g_0 \, 1 \, \mu^{\frac{5}{2}} = \frac{2}{5}\left(\frac{3N}{2\epsilon_F^{3/2}}\right)\mu^{\frac{5}{2}} = \frac{3}{5} N \frac{\mu^{\frac{5}{2}}}{\epsilon_F^{3/2}}$$

$$② \propto \int_{x=-\infty}^{+\infty} \frac{e^x x}{(e^x+1)^2} dx = 0$$

$$③ \quad \frac{2}{5} g_0 \frac{15}{8} \mu^{1/2} (k_B T)^2 \underbrace{\int_{-\infty}^{+\infty} \frac{e^x x^2}{(e^x+1)^2} dx}_{\frac{\pi^2}{3}}$$

$$= \frac{\pi^2}{4} g_0 \mu^{1/2} (k_B T)^2$$

$$= \frac{3}{8} \pi^2 N \frac{\mu^{1/2}}{\epsilon_F^{3/2}} (k_B T)^2$$

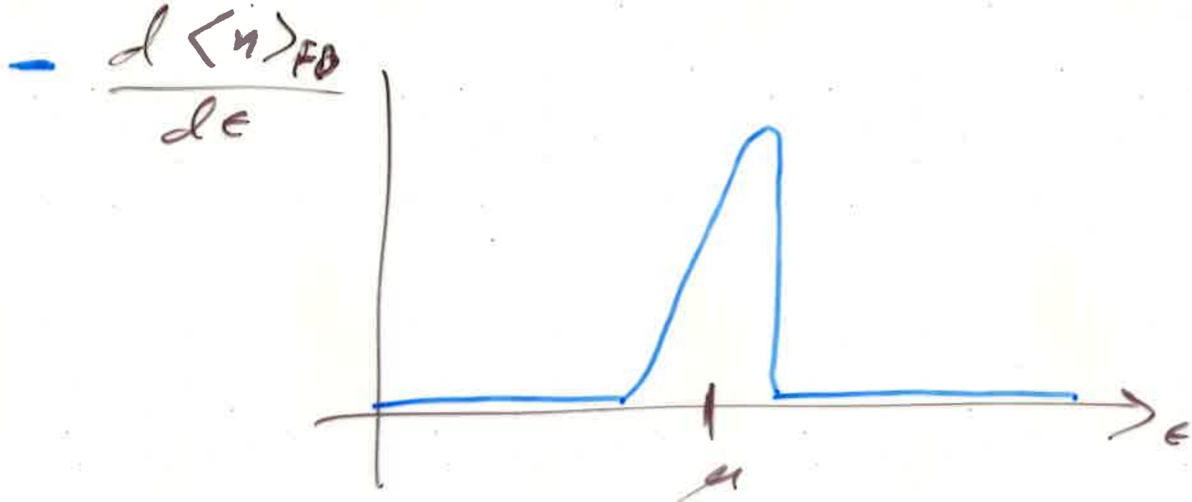
$$U = \frac{3}{5} N \frac{\mu^{5/2}}{\epsilon_F^{3/2}} + 0 + \frac{3\pi^2}{8} N \frac{\mu^{1/2}}{\epsilon_F^{3/2}} (k_B T)^2$$

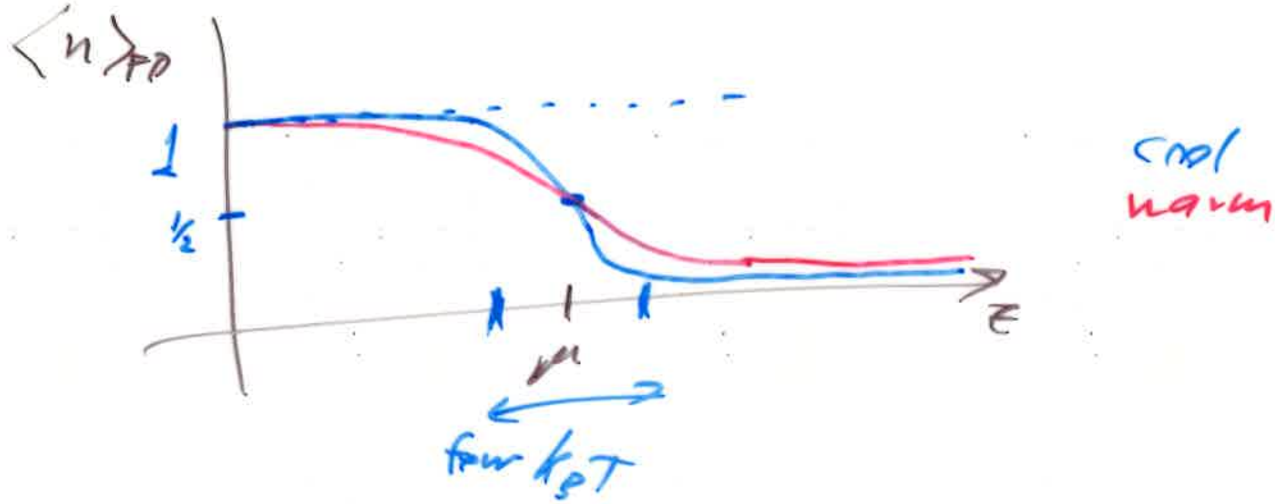
substitute: $\mu = \epsilon_F - \frac{\pi^2}{12} \frac{(k_B T)^2}{\epsilon_F}$

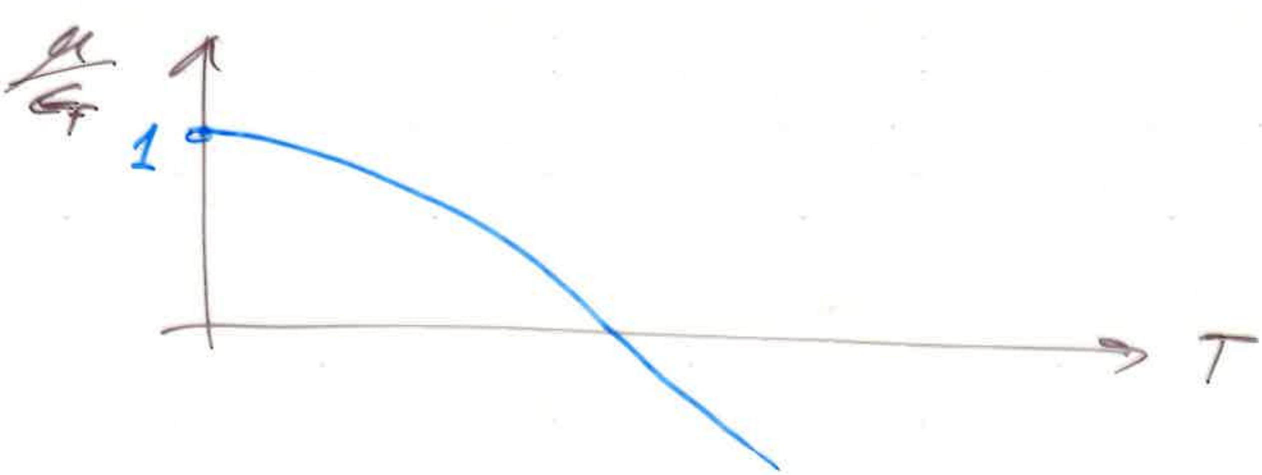
$$U = \underbrace{\frac{3}{5} N \epsilon_F}_{T=0K \text{ value}} + \underbrace{\cancel{\frac{3}{5}} N \cancel{\frac{5}{2}} \left(-\frac{\pi^2}{\cancel{12}_{4}}\right)}_{-\frac{\pi^2}{8}} \frac{(k_B T)^2}{\epsilon_F} + \frac{3\pi^2}{8} N \frac{(k_B T)^2}{\epsilon_F}$$

$$U = \frac{3}{5} N \epsilon_F + \frac{\pi^2}{4} N \frac{(k_B T)^2}{\epsilon_F} + \cdots$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} = \frac{\pi^2}{2} N \frac{k_B^2}{\epsilon_F} T$$ linear in T







Examples of iterative solutions

(1) Inversion of series $\qquad x < 1$

$$y = \sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

want $\quad x = \sum_{k=1}^{\infty} b_k y^k$

To lowest order : $y = a_1 x \longrightarrow x = \frac{y}{a_1} + \mathcal{O}(y^2)$

$$x = \frac{1}{a_1}\left[y - a_2 x^2 - a_3 x^3 - a_4 x^4 - \cdots \right]$$

substitute first-order guess $x = \frac{y}{a_1} + \mathcal{O}(y^2)$

Keep terms of order y^2, ignore the rest $\mathcal{O}(y^3)$

$$x = \frac{1}{a_1}\left[y - a_2 \left(\frac{y}{a_1}\right)^2 - \cdots \right] = \frac{y}{a_1} - \frac{a_2}{a_1^3} y^2 + \mathcal{O}(y^3)$$

2nd-order guess

substitute into above, keep y^3 ignore $\mathcal{O}(y^4)$

$$x = \frac{1}{a_1}\left[y - a_2 \left(\frac{y}{a_1} - \frac{a_2}{a_1^3} y^2 \right)^2 - a_3 \left(\frac{y}{a_1} \cancel{- \frac{a_2}{a_1^3} y^2} \right)^3 + \mathcal{O}(y^4) \right.$$

ignore

$$= \frac{y}{a_1} - \frac{a_2}{a_1^3} y^2 + \frac{2a_2^2}{a_1^5} y^3 - \frac{a_3}{a_1^4} y^3 + \mathcal{O}(y^4)$$

$$= \underbrace{\frac{y}{a_1}}_{b_1} \overbrace{- \frac{a_2}{a_1^3}}^{b_2} y^2 + \overbrace{\left(\frac{2a_2^2}{a_1^5} - \frac{a_3}{a_1^4} \right)}^{b_3} y^3 + \mathcal{O}(y^4)$$

② Solving the Quadratic Equation

E.g. $2x = 10^6(x^2 - 4)$ solve for x

0^{th} order guess $x_0 = 2$

1^{st} order guess: $x_1 = 2(1-\epsilon)$ substitute

$$2\cdot 2(1-\cancelto{\text{ignore}}{\epsilon}) = 10^6[\cancel{4} - 8\epsilon + \cancelto{\text{ignore}}{4\epsilon^2} - \cancel{4}]$$

$$4 = 10^6(-8)\epsilon \Rightarrow \epsilon = -\tfrac{1}{2}\times 10^{-6}$$

$$x_1 = 2 + 10^{-6}$$

2^{nd} order guess $x_2 = 2 + 10^{-6} + \eta$

$$2(2+10^{-6}+\eta) = 10^6([2+10^{-6}+\eta]^2 - 4)$$

$$10^{12}\eta(4-\cancelto{\text{ignore}}{\eta}) = 1 \Rightarrow \eta = \tfrac{1}{4}\times 10^{-12}$$

$$x_2 = 2.00000100000025\ldots$$