17 = small number

10^{17} = large number

$10^{(10^{17})}$ = very number

Mass $10^{(10^{17})}$ grams or tons

$10^6 \cdot 10^{(10^{17})} = 10^{[10^{17}+6]}$

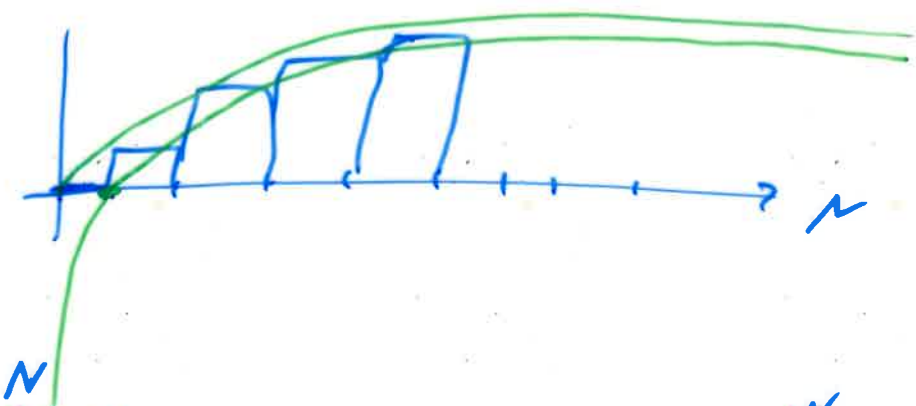
$10^{[100000000000000006]} \sim 10^{17}$

Stirling's Approximation

Need a formula to evaluate $N!$ for large N (e.g. N_A). $N_A!$ is very large.

$$\ln(N!) = \ln[N \cdot (N-1) \cdot (N-2) \cdots 1]$$

$$= \ln(N) + \ln(N-1) + \ln(N-2) \cdots + \xcancel{\ln(1)}^{\,0}$$



$$\ln(N!) \approx \int_{x=0}^{N} \ln(x)\,dx = x\ln(x) - x \Big|_0^N$$

$\lim_{x\to 0} x\ln(x) = 0$

$$\ln(N!) \approx N\ln(N) - N \qquad a\ln(b) = \ln(b^a)$$

$$e^{\ln(N!)} = e^{[N\ln(N) - N]}$$

$$N! \approx e^{N\ln(N)} \cdot e^{-N} = e^{\ln(N^N)} \cdot e^{-N}$$

$$= N^N e^{-N} = \left(\frac{N}{e}\right)^N$$

usually good enough.

First correction — use gaussian approximation to the integrand

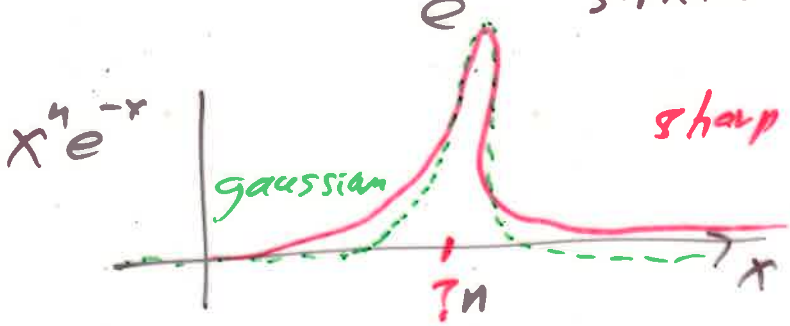
$$n! = \int_{x=0}^{\infty} \underline{x^n e^{-x}}\, dx = \Gamma(n+1)$$

Gamma function

$$0! = 1 \qquad \left(-\tfrac{1}{2}\right)! = \sqrt{\pi} = \Gamma\left(\tfrac{1}{2}\right)$$

Integrand x^n grows very rapidly with x

e^{-x} shrinks even more rapidly with x



sharp peak, height $n^n e^{-n}$

$$\frac{d}{dx}\left[x^n e^{-x}\right]\Big|_{x_0} = 0 = \begin{array}{l} n x_0^{n-1} e^{-x_0} + -x_0^n e^{-x_0} = 0 \\ e^{-x_0} x_0^{n-1}[n - x_0] = 0 \\ x_0 = n \end{array}$$

$$\ln(x^N e^{-x}) = \ln(x^N) + \ln(e^{-x}) = N\ln(x) - x$$

change variables: $y = x - N$ = distance from peak

$dy = dx$

$$N\ln(x) - x \rightarrow N\ln(y+N) - y - N$$

$$= N\ln\left[N\left(1+\frac{y}{N}\right)\right] - y - N$$

$$= N\ln(N) - N + N\ln\left(1+\frac{y}{N}\right) - y$$

Near the peak $x \sim N$, $\frac{y}{N} \ll 1$

Taylor Expansion $\ln\left(1+\frac{y}{N}\right) = \underline{\frac{y}{N} - \frac{1}{2}\left(\frac{y}{N}\right)^2} + \frac{1}{3}\left(\frac{y}{N}\right)^3 - \frac{1}{4}\left(\frac{y}{N}\right)^4$

$$N\ln(x) - x \simeq N\ln(N) - N + -\frac{1}{2}\frac{y^2}{N}$$

$$e^{N\ln(x)-x} = x^N e^{-x} \simeq \exp\left[N\ln(N) - N - \frac{y^2}{2N}\right]$$

$$= N^N \cdot e^{-N} \cdot e^{-\frac{y^2}{2N}}$$

$$N! = \int_{x=0}^{\infty} x^N e^{-x}\,dx \simeq \int_{x=0}^{\infty} N^N e^{-N} e^{-\frac{y^2}{2N}}\,dy \qquad \begin{matrix} x\to\infty \\ y\to\infty \end{matrix}$$

lower limit should be $y = -N \rightarrow -\infty$

$$N! \simeq N^N e^{-N} \int_{y=-\infty}^{\infty} e^{-\frac{y^2}{2N}} dy = N^N e^{-N} \sqrt{2\pi N}$$

always for low

Need even more accuracy?

always for high

$$N! \simeq N^N e^{-N} \sqrt{2\pi N} \left(1 + \frac{1}{12N}\right)$$

Areas of n-dimensional "spheres"

Volumes of n-dimensional balls.

e.g. soap bubble: 2-sphere _ dimension of manifold

~~3-sphere ‒ dimension of space containing the manifold.~~

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circle  1 sphere

disk  1 ball