Two-state Paramagnet with Partition Function Z

(see Schroeder Sec 3.3 w/o Z)
in a heat bath at temp. T

N of magnetic dipoles – spin $\frac{1}{2}$ particles

$$S = \frac{\hbar}{2}, \quad m_s = \pm\frac{1}{2}, \quad |\vec{S}| = \sqrt{s(s+1)}\,\hbar = \frac{\sqrt{3}}{2}\hbar$$

in a magnetic field $\vec{B}$ define an axis.

$\uparrow \vec{B}$

$\uparrow\uparrow\uparrow\uparrow$ spin "up" – energy $-\mu B$

$\downarrow\downarrow\downarrow$ spin "down" – energy $+\mu B$

μ is the magnetic moment $\mu = \frac{gq}{2m}s$

$$Z = \sum_i e^{-\beta E_i}, \quad \beta \equiv \frac{1}{k_B T}$$

sum is over microstates

N+1 macrostates $\Longleftrightarrow$ N+1 different Energies

2^N microstates

E_N = N spins down $\downarrow\downarrow\downarrow\downarrow$ = $N\mu B$ — 1 state

⋮

E_2 = two spins down $\downarrow\downarrow\uparrow\uparrow\uparrow$ = $-(N-2)\mu B + 2\mu B$ | $\frac{N(N-1)}{2!}$

E_1 = one spin down $\downarrow\uparrow\uparrow\uparrow\cdots$ = $-(N-1)\mu B + \mu B$ | N states

E_0 = zero spins down $\underbrace{\uparrow\uparrow\uparrow\uparrow\cdots}_{N}$ = $-N\mu B$ — 1 state

Hard way

$$Z = \sum_i e^{-\beta E_i} = 1 \exp\left[\frac{+N\mu B}{k_B T}\right] + N \exp\left[\frac{(N-2)\mu B}{k_B T}\right]$$

$$+ \frac{N(N-1)}{2!} \exp\left[\frac{(N-4)\mu B}{k_B T}\right] + \dots 1 \exp\left[\frac{-N\mu B}{k_B T}\right]$$

Easy way

$$(x+y)^N = x^N + N x^{N-1} y^1 + \frac{N(N-1)}{2!} x^{N-2} y^2 \dots + y^N$$

For one dipole: $z_1 = e^{\frac{\mu B}{k_B T}} + e^{-\frac{\mu B}{k_B T}}$

For N non-interacting dipoles: $Z = z_1^N$

$$Z = \left(e^{\frac{\mu B}{k_B T}} + e^{-\frac{\mu B}{k_B T}}\right)^N = \left[2\cosh\left(\frac{\mu B}{k_B T}\right)\right]^N$$

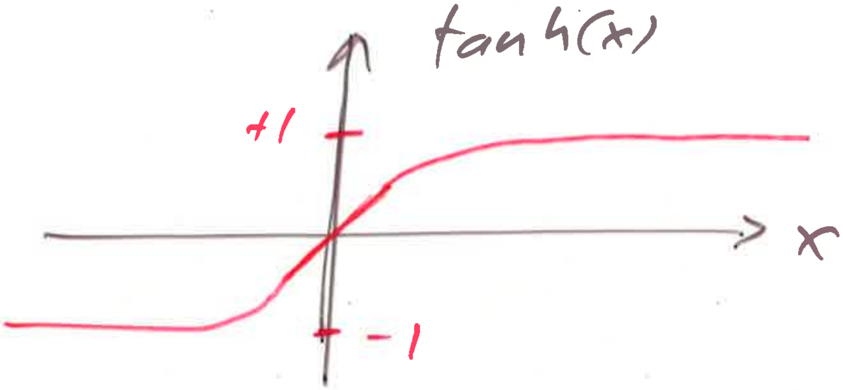
$$\ln(Z) = N \ln\left[2\cosh\left(\frac{\mu B}{k_B T}\right)\right]$$

$$= N \ln\left[2\cosh(\beta\mu B)\right]$$

$$U = -\frac{\partial \ln(Z)}{\partial \beta} = -\frac{\partial}{\partial \beta} N \ln\left[2\cosh(\beta\mu B)\right]$$

$$= -N \frac{\cancel{2}\sinh(\beta\mu B)}{\cancel{2}\cosh(\beta\mu B)} \mu B = -N\mu B \tanh(\beta\mu B)$$

$U(S,V,N)$ $\qquad\qquad$ $F = U - TS$



Look at $x \geq 0$ for now.

— w/ less

— ← more

high T: $k_B T \gg \mu B \leftrightarrow$ small β

$$\tanh(\beta \mu B) \sim \beta \mu B \Rightarrow U = -N(\mu B)^2 \beta \to 0$$

$U = 0 \Rightarrow$ equal # of spins up and down.

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low T: $k_B T \ll \mu B \Leftrightarrow$ large β

$\tanh(\beta \mu B) \to 1$, $U = -N\mu B$ all spins up.

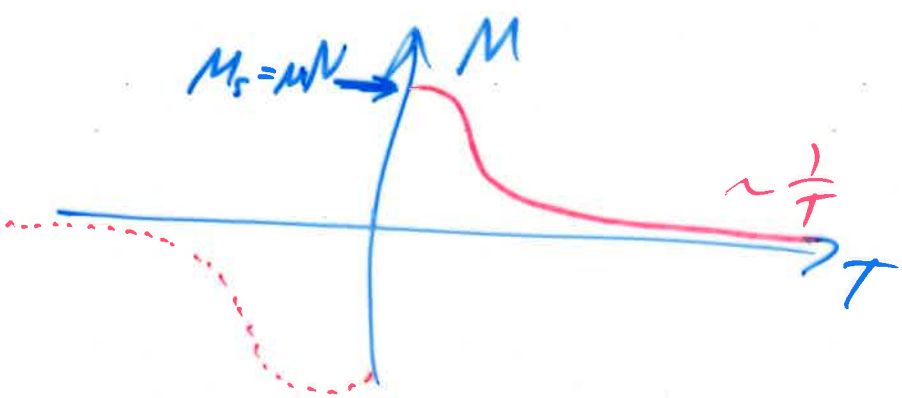
$$U = -\vec{\mu} \cdot \vec{B}$$

<u>Magnetization</u>

$$M = \mu(N_\uparrow - N_\downarrow) = -\frac{U}{B} = N\mu \tanh\left(\frac{\mu B}{k_B T}\right)$$

high T (small β) $M \sim N \frac{\mu^2 B}{k_B T}$ Pierre Curie Law

Low T (large β): $M \sim N\mu$ saturation, all spins up.



Magnetization a different way.

Helmholtz Free Energy: $F = A = -\dfrac{\ln(z)}{\beta}$

$$F = -\frac{N}{\beta}\ln\left[2\cosh(\beta\mu B)\right]$$

$$dF = -S\,dT - \cancel{P\,dV} = -S\,dT - M\,dB$$

$$dF = \left(\frac{\partial F}{\partial T}\right)_B dT + \left(\frac{\partial F}{\partial B}\right)_T dB$$

$$dU = T\,dS - P\,dV + \cancel{\mu\,dN}$$

$$M = -\left(\frac{\partial F}{\partial B}\right)_T = +\frac{N}{\cancel{\beta}}\,\frac{\cancel{2}\sinh(\beta\mu B)}{\cancel{2}\cosh(\beta\mu B)}\,\mu\cancel{\beta}$$

$$= N\mu\tanh(\beta\mu B)$$

Heat Capacity at constant B

$$C_B = \left(\frac{\partial U}{\partial T}\right)_{BN} = \frac{\partial}{\partial T}\left[-N\mu B\tanh\left(\frac{\mu B}{k_B T}\right)\right]$$

$$= -N\mu B\,\mathrm{sech}^2\left(\frac{\mu B}{k_B T}\right)\frac{\mu B}{k_B}\left(-\frac{1}{T^2}\right)$$

$$= N k_B\,\frac{\left(\frac{\mu B}{k_B T}\right)^2}{\cosh^2\left(\frac{\mu B}{k_B T}\right)}$$