Approximations

(1) Lower limit $-\mu\beta \to -\infty$ since $e^{-|x|}$ in numerator gives a negligible contribution.

(2) Expand $\epsilon^{3/2}$ about $x=0$ or $\epsilon=\mu$.

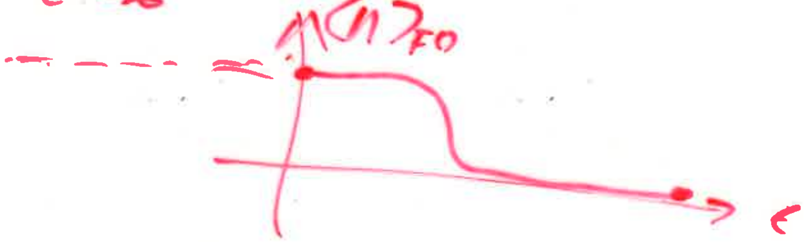
$$\epsilon^{3/2} = \mu^{3/2} + (\epsilon-\mu)\left.\frac{d(\epsilon^{3/2})}{d\epsilon}\right|_{\epsilon=\mu} + \frac{1}{2!}(\epsilon-\mu)^2\left.\frac{d^2(\epsilon^{3/2})}{d\epsilon^2}\right|_{\epsilon=\mu} + \dots$$

$\frac{1}{2!}\,\frac{3}{2}\,\frac{1}{2}$

$$= \mu^{3/2} + \frac{3}{2}(\epsilon-\mu)\mu^{1/2} + \frac{3}{8}(\epsilon-\mu)^2\frac{1}{\mu^{1/2}} + \dots$$

$$N \simeq \frac{2}{3}g_0 \underbrace{\int_{-\infty}^{+\infty}\frac{e^x}{(e^x+1)^2}}_{1}\Big[\underbrace{\mu^{3/2}}_{(1)} + \underbrace{\frac{3}{2}(\epsilon-\mu)\mu^{1/2}}_{(2)} + \underbrace{\frac{3}{8}(\epsilon-\mu)^2\mu^{-1/2}}_{(3)} + \dots\Big]dx$$

(1) $$\frac{2}{3}g_0\mu^{3/2}\int_{\epsilon=-\infty}^{+\infty}\left(-\frac{d\langle n\rangle_{FD}}{d\epsilon}\right)d\epsilon = \frac{2}{3}g_0\mu^{3/2}\Big[\overbrace{\langle n\rangle_{FD}\big|_{\epsilon=-\infty}}^{1} - \underbrace{\langle n\rangle_{FD}\big|_{\epsilon=+\infty}}_{0}\Big]$$



(1) $$\frac{2}{3}g_0\mu^{3/2} = \frac{2}{3}\left(\frac{3N}{2\epsilon_F^{3/2}}\right)\mu^{3/2} = N\left(\frac{\mu}{\epsilon_F}\right)^{3/2}$$

$$② \propto \int_{x=-\infty}^{+\infty} \frac{e^{x} x}{(e^{x}+1)^2}\,dx \frac{e^{-x}}{e^{-x}} = \int_{x=-\infty}^{+\infty} \frac{x\,dx}{(e^{x}+1)(1+e^{-x})} = 0$$

$$③ \propto \int_{-x=-\infty}^{+\infty} \frac{e^{x} x^2}{(e^{x}+1)^2}\,dx = \frac{\pi^2}{3}$$

$$g_0 = \frac{3N}{2\epsilon_F^{3/2}}$$

$$N \simeq N\left(\frac{\mu}{\epsilon_F}\right)^{3/2} + \cancel{0} + \frac{1}{4} g_0 (k_B T)^2 \mu^{-1/2} \frac{\pi^2}{3}$$

$$\cancel{N} \simeq \cancel{N}\underbrace{\left(\frac{\mu}{\epsilon_F}\right)^{3/2}}_{\text{solve for this } \mu} + \underbrace{\cancel{N}\frac{\pi^2}{8} \frac{(k_B T)^2}{\epsilon_F^{3/2} \mu^{1/2}}}_{\text{small}} + \cdots$$

↑ set $\mu = \epsilon_F + \cdots$

$$\left(\frac{\mu}{\epsilon_F}\right)^{3/2} = 1 - \frac{\pi^2}{8}(k_B T)^2 \frac{1}{\epsilon_F^2} + \cdots$$

$$\frac{\mu}{\epsilon_F} = \Big[1 - \underbrace{\frac{\pi^2}{8}(k_B T)^2 \frac{1}{\epsilon_F^2}}_{\text{small}}\Big]^{2/3}$$

Binomial expansion

$$\frac{\mu}{\epsilon_F} = 1 - \frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2$$

Same Approximations, but for U instead of N

$$U = \int_{\epsilon=0}^{\infty} \epsilon\, g(\epsilon) \langle n \rangle_{FD}\, d\epsilon = g_0 \int_{\epsilon=0}^{\infty} \underbrace{\epsilon^{3/2}}_{du} \overset{v}{\langle n \rangle_{FD}}\, \underbrace{d\epsilon}_{du}$$

integrate by parts - surface term vanishes

$$U = \frac{2}{5} g_0\, \epsilon^{5/2} \langle n \rangle_{FD} \Big|_{\epsilon=0}^{\infty} - \underbrace{\frac{2}{5} g_0 \int_{\epsilon=0}^{\infty} \epsilon^{5/2}}_{u} \underbrace{\left(\frac{d\langle n \rangle_{FD}}{d\epsilon}\right)}_{dv} d\epsilon$$

$\epsilon^{5/2} \to 0$ as $\epsilon \to 0$

$\langle n \rangle_{FD} \to 0$ as $\epsilon \to \infty$

Approx.

$$U = \frac{2}{5} g_0 \int_{x=-\infty}^{+\infty} \frac{e^x\, \epsilon^{5/2}}{(e^x+1)^2}\, dx$$

Taylor expand $\epsilon^{5/2}$ about $\epsilon = \mu$

$$\epsilon^{5/2} = \mu^{\frac{5}{2}} + (\epsilon-\mu) \frac{d(\epsilon^{5/2})}{d\epsilon}\Big|_{\epsilon=\mu} + \frac{1}{2!}(\epsilon-\mu)^2 \frac{d^2(\epsilon^{5/2})}{d\epsilon^2}\Big|_{\epsilon=\mu} + \cdots$$

$$= \mu^{\frac{5}{2}} + \frac{5}{2}(\epsilon-\mu)\mu^{3/2} + \frac{15}{8}(\epsilon-\mu)^2 \mu^{1/2} + \cdots$$

$$U = \frac{2}{5} g_0 \int_{x=-\infty}^{+\infty} \frac{e^x}{(e^x+1)^2} \Big[\underbrace{\mu^{\frac{5}{2}}}_{1} + \underbrace{\frac{5}{2}(\epsilon-\mu)\mu^{3/2}}_{2} + \underbrace{\frac{15}{8}(\epsilon-\mu)^2 \mu^{1/2}}_{3} + \cdots \Big] dx$$

$$① \to \frac{2}{5} g_0 \, 1 \, \mu^{\frac{5}{2}} = \frac{2}{5}\left(\frac{3N}{2\epsilon_F^{3/2}}\right)\mu^{\frac{5}{2}} = \frac{3}{5} N \frac{\mu^{\frac{5}{2}}}{\epsilon_F^{3/2}}$$

$$② \propto \int_{x=-\infty}^{+\infty} \frac{e^x x}{(e^x+1)^2} dx = 0$$