There are more cosmic photons than neutrinos.

For bosons, the number of particles at temperature T is

$$N = \underset{\substack{\uparrow \\ \text{degeneracy:} \\ g=2 \text{ for photons} \\ \text{spin } \pm 1.}}{g} \sum_{n_x, n_y, n_z = -\infty}^{+\infty} \underset{\substack{\uparrow \\ \text{Bose-Einstein} \\ \text{distribution}}}{\langle n \rangle_{BE}} = g \sum_{\vec{n}} \frac{1}{e^{\beta E(\vec{n})} - 1}$$

$\beta = \frac{1}{k_B T}$

For periodic B.C.

$k_i = \frac{2\pi}{\lambda_i} = \frac{2\pi L}{n_i}$

$-\infty \le n_i < +\infty$

$dn_i = \left(\frac{L}{2\pi}\right) dk_i$

$$\longrightarrow g \iiint_{\vec{n}} \frac{1}{e^{\beta E} - 1} d^3\vec{n}$$

$$= L^3 g \iiint_{\vec{k}} \frac{1}{e^{\beta \hbar c k} - 1} \frac{d^3\vec{k}}{(2\pi)^3}$$

V = volume

Cartesian $\rightarrow$ Spherical coordinates

Define $x = \beta \hbar c k$

$dx = \beta \hbar c \, dk$

$$= \frac{Vg}{(2\pi)^3} \underset{\substack{\uparrow \\ \text{from} \\ \theta, \varphi \text{ integrals}}}{(4\pi)} \int_{k=0}^{\infty} \frac{k^2 dk}{e^{\beta \hbar c k} - 1}$$

$$= \frac{Vg}{2\pi^2} \underbrace{\int_{x=0}^{\infty} \frac{x^2 dx}{e^x - 1}}_{2\zeta(3) \leftarrow \text{Riemann zeta function}} \frac{1}{(\beta \hbar c)^3} = 8\pi V g \frac{k_B^3 T^3}{h^3 c} \zeta(3)$$

$$n = \frac{N}{V} = 8\pi \overset{\substack{2 \text{ for } \gamma \\ \downarrow}}{g} \zeta(3) \frac{k_B^3 T^3}{h^3 c^3} = 410 \text{ cm}^{-3}$$

$T^3 \leftarrow$ 2.725K from Planck CMB fit where $M_\odot$ is one solar mass. (In comparison, the mass of our galaxy is about $10^{11} M_\odot$.) After recombination, the pressure dropped by a factor 10^9, so the Jeans mass dropped to

$$M_J = (10^{-9})^{3/2} \times 5 \times 10^{18}\, M_\odot = 1.6 \times 10^5 M_\odot$$

It is interesting that this is roughly the mass of the large globular clusters within our galaxy.

Note 6 Neutrino Temperature and Density

As long as thermal equilibrium is preserved, the total value of the quantity known as "entropy" remains fixed. For our purposes, the entropy per unit volume S is given to an adequate approximation at temperature T by

$$S \propto N_T T^3$$

where N_T is the effective number of species of particles in thermal equilibrium whose threshold temperature lies below T. In order to keep the total entropy constant, S must be proportional to the inverse cube of the size of the universe. That is, if R is the separation between any pair of typical particles, then

$$SR^3 \propto N_T T^3 R^3 = \text{constant}$$

Just before the annihilation of electrons and positrons (at about $5 \times 10^9\ ^\circ$K) the neutrinos and antineutrinos had already gone out of thermal equilibrium with the rest of the universe, so the only abundant particles in equilibrium were the electron, positron, and photon. Referring to Table One on page 156, we see the effective total number of particle species before annihilation was

$$N_{\text{before}} = \frac{7}{2} + 2 = \frac{11}{2}$$

On the other hand, after the annihilation of electrons and positrons in the fourth frame, the only remaining abundant particles in equilibrium were the photons. The effective number of particle species then was simply

$$N_{\text{after}} = 2$$

It follows then from the conservation of entropy that

$$\frac{11}{2}(TR)^3_{\text{before}} = 2(TR)^3_{\text{after}}$$

The First Three Minutes
– Steven Weinberg

That is, the heat produced by the annihilation of electrons and positrons increases the quantity TR by a factor

$$\frac{(TR)_{\text{after}}}{(TR)_{\text{before}}} = \left(\frac{11}{4}\right)^{1/3} = 1.401$$

Before the annihilation of electrons and positrons, the neutrino temperature T_ν was the same as the photon temperature T. But from then on, T_ν simply dropped like $1/R$, so for all subsequent times $T_\nu R$ was equal to the value of TR before the annihilation:

$$(T_\nu R)_{\text{after}} = (T_\nu R)_{\text{before}} = (TR)_{\text{before}}$$

We conclude therefore that after the annihilation process is over, the photon temperature is higher than the neutrino temperature by a factor

$$(T/T_\nu)_{\text{after}} = \frac{(TR)_{\text{after}}}{(T_\nu R)_{\text{after}}} = \left(\frac{11}{4}\right)^{1/3} = 1.401$$

Even though out of thermal equilibrium, the neutrinos and antineutrinos make an important contribution to the cosmic energy density. The effective number of species of neutrinos and antineutrinos is 7/2, or 7/4 of the effective number of species of photons. (There are two photon spin states.) On the other hand, the fourth power of the neutrinos' temperature is less than the fourth power of the photon temperature by a factor $(4/11)^{4/3}$. Thus the ratio of the energy density of neutrinos and antineutrinos to that of photons is

$$\frac{u_\nu}{u_\gamma} = \frac{7}{4}\left(\frac{4}{11}\right)^{4/3} = 0.4542$$

The Stefan-Boltzmann law (see Chapter III) tells us that at *photon* temperature T the photon energy density is

$$u_\gamma = 7.5641 \times 10^{-15}\ \text{erg/cm}^3 \times [T(^\circ\text{K})]^4$$

Hence the total energy density after electron-positron annihilation is

$$u = u_\nu + u_\gamma = 1.4542 u_\gamma = 1.100 \times 10^{-14}\ \text{erg/cm}^3 [T(^\circ\text{K})]^4$$

We can convert this to an equivalent mass density by dividing by the square of the speed of light, and find

$$\rho = u/c^2 = 1.22 \times 10^{-35}\ \text{gm/cm}^3 \times [T(^\circ\text{K})]^4$$

$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \leftarrow 2.725\text{K}$

$\uparrow 1.945\text{K}$

(2)

For fermions

$$N = \underset{\substack{\uparrow \\ \text{degeneracy} \\ g=6 \text{ for 3 flavors} \\ \text{of neutrinos +} \\ \text{antineutrinos}}}{g} \sum_{\vec{n}=-\infty}^{+\infty} \underset{\substack{\uparrow \\ \text{Fermi-Dirac} \\ \text{distribution}}}{\langle n \rangle_{FD}} = g \sum_{\vec{n}} \frac{1}{e^{\beta E(\vec{n})} \underset{\substack{\uparrow \\ \text{sign change} \\ \text{from Bose Einstein}}}{+} 1}$$

$$\rightarrow g \iiint_{\vec{n}} \frac{1}{e^{\beta E}+1} d^3\vec{n} = Vg \iiint_{\vec{k}} \frac{1}{e^{\beta\hbar c k}+1} \frac{d^3\vec{k}}{(2\pi)^3}$$

$$= \frac{Vg}{(2\pi)^3}(4\pi) \int_{k=0}^{\infty} \frac{k^2\,dk}{e^{\beta\hbar c k}+1} = \frac{Vg}{2\pi^2} \underbrace{\int_{x=0}^{\infty} \frac{x^2\,dx}{e^x+1}}_{\frac{3}{2}\zeta(3)} \left(\frac{1}{\beta\hbar c}\right)^3$$

$$= 6\pi V g \frac{T_\nu^3 k_B^3}{h^3 c^3} \zeta(3)$$

(This is $\frac{3}{4}$ x the BE result but the g and T are different.)

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with $g=6$ and $T_\nu = 1.945\,K$

$$n_\nu = \frac{N_\nu}{V} = 335\ cm^{-3} < n_\gamma = 410\ cm^{-3}$$

$$n_\nu = \frac{9}{11} n_\gamma$$