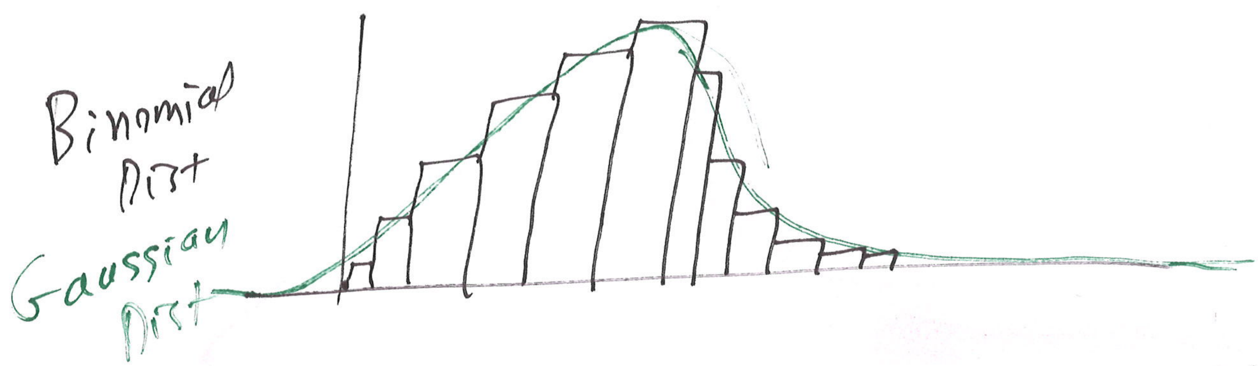
For Large N Binomial → Gaussian

↑discrete ↑continuous

Gaussian is paramotrized by two numbers (cumulants) Mean, Standard Deviation

Mean: $\mu = pN$ $\sigma_n = \sqrt{Npq}$



$$P_B(n) = B(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n} \Rightarrow \sum_{n=0}^{N} B(n) = 1$$

$$P_G(n) = G(n) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(n-\mu)^2}{2\sigma^2}\right]$$

n is continuous $-\infty < n < +\infty$

$$\int_{-\infty}^{+\infty} G(n)\, dn = 1$$

3 (of the many) problems with 19th Century Physics

① Black-body radiation & the UV catastrophe
(Rayleigh - Jeans catastrophe)

② C_v for a diatomic gas

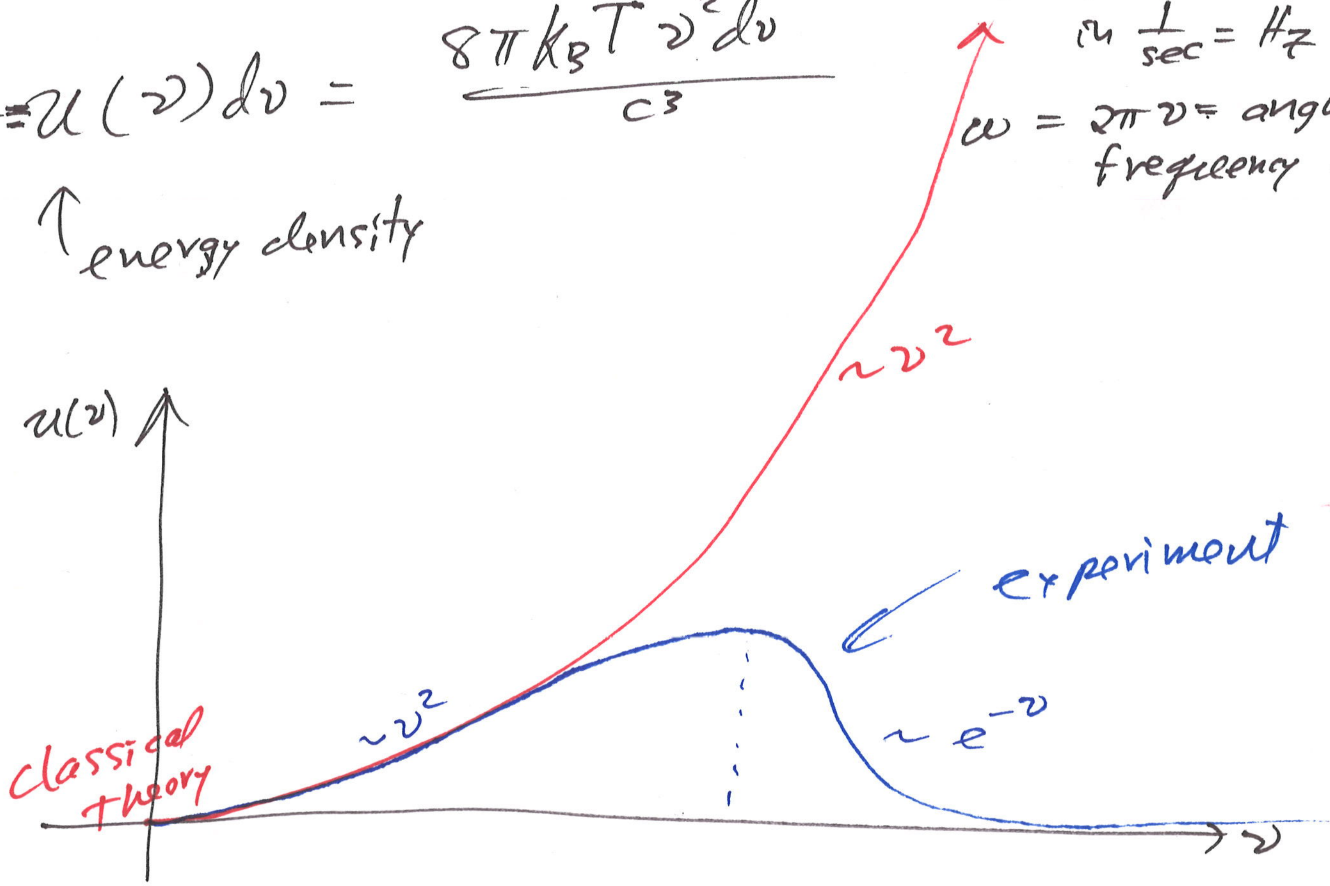
③ Specific heat of solids, especially metals

$\nu = f =$ frequency in $\frac{1}{\sec} = Hz$

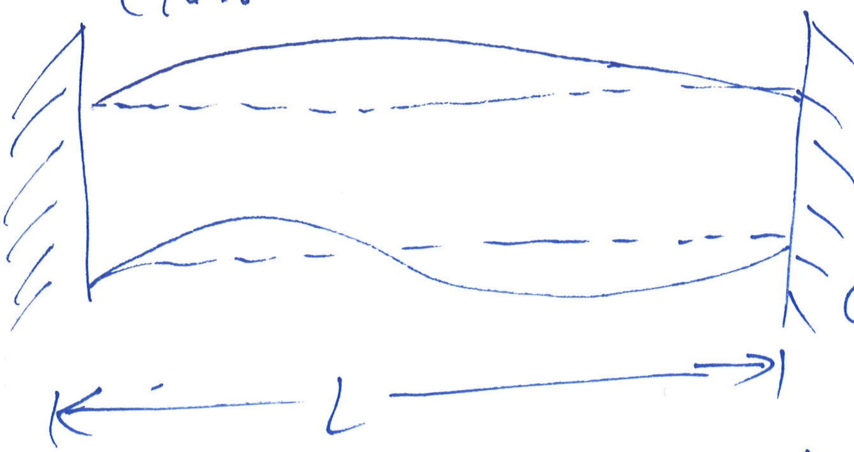
$$\frac{U}{V} = u(\nu)\,d\nu = \frac{8\pi k_B T \nu^2 d\nu}{c^3}$$

↑ energy density

$\omega = 2\pi\nu =$ angular frequency $\frac{rad}{sec}$



Classical SHO



fundamental $\lambda = 2L$

1st overtone
(2nd harmonic) $\lambda = L$

λ and ν are quantized.

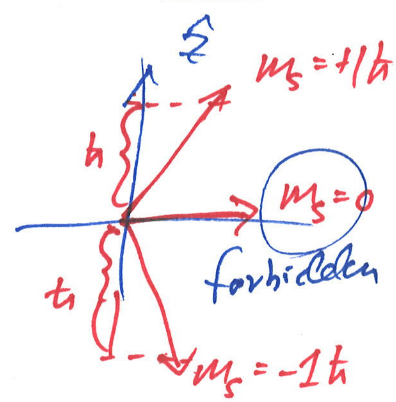
Energy E is not fixed.

Quantum Mech.

$= (n+\frac{1}{2})h\nu$

Energy IS quantized! $E_n = (n+\frac{1}{2})\hbar\omega$, $n = 0,1,2\ldots$

Black-body Spectrum



photon - massless, spin-1 particle
↳ force $\propto \frac{1}{r^2}$

$$|\vec{S}| = \sqrt{\ell(\ell+1)}\,\hbar = \sqrt{2}\,\hbar$$

photon has 2 polarizations (not 3)
↳ transverse

longitudinal polar. is missing.

$$E_j = (j+\tfrac{1}{2})\hbar\omega$$

Partition Function

$$Z_1 = \sum_{j=0}^{\infty} e^{-\beta E_j} = \sum_{j=0}^{\infty} e^{-\beta(j+\frac{1}{2})\hbar\omega}$$

$$= e^{-\beta\frac{\hbar\omega}{2}}\left[1 + e^{-\beta\hbar\omega} + e^{-\beta 2\hbar\omega} + \cdots\right]$$

$$\underbrace{r^0 + r^1 + r^2 + r^3 + \cdots}_{\text{Geometric Series}}$$

$$\sum_{j=0}^{\infty} r^j = \frac{1}{1-r}$$

$$Z_1 = e^{-\beta\frac{\hbar\omega}{2}}\left[\frac{1}{1-e^{-\beta\hbar\omega}}\right]$$

$$\ln(Z_1) = \ln\left(e^{-\beta\frac{\hbar\omega}{2}}\right) + \ln\left(\frac{1}{1-e^{-\beta\hbar\omega}}\right)$$

$$= -\frac{1}{2}\beta\hbar\omega - \ln\left(1-e^{-\beta\hbar\omega}\right)$$

$$U_1 = \langle E_1 \rangle = -\frac{\partial \ln(Z)}{\partial\beta} = \frac{\hbar\omega}{2} + \frac{+e^{-\beta\hbar\omega}\hbar\omega}{1-e^{-\beta\hbar\omega}}$$

multiply $\frac{e^{+\beta\hbar\omega}}{e^{+\beta\hbar\omega}} = 1$

one

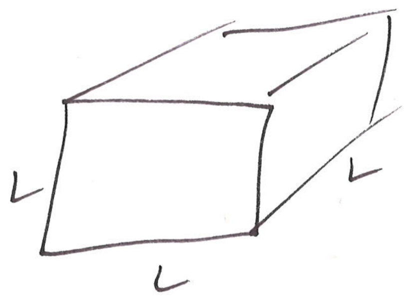
$$U_1 = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$ for single mode.

"Dispersion" Relation $\qquad \omega(k) = ck$ $\quad$ no dispersion

$k \equiv |\vec{k}|$, $\vec{k}$ wave vector

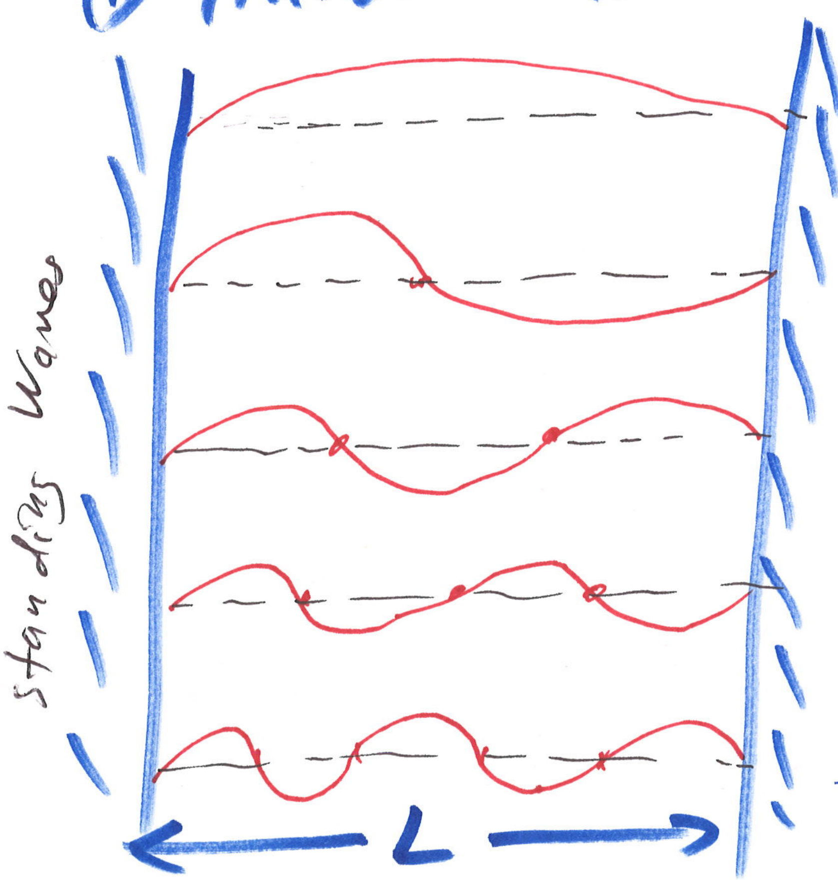
$$k = \frac{2\pi}{\lambda}$$

Put the gas of photons in a 3-dim box, cube



Boundary Conditions

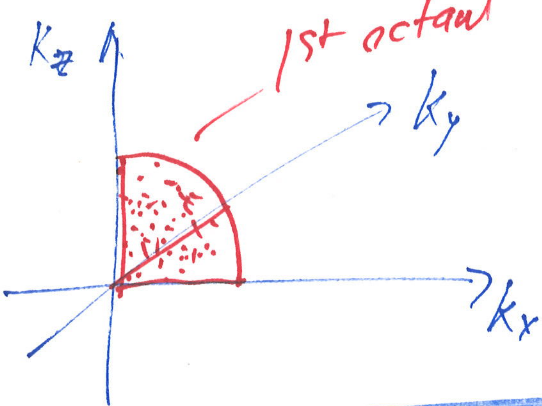
① Hard wall B.C.



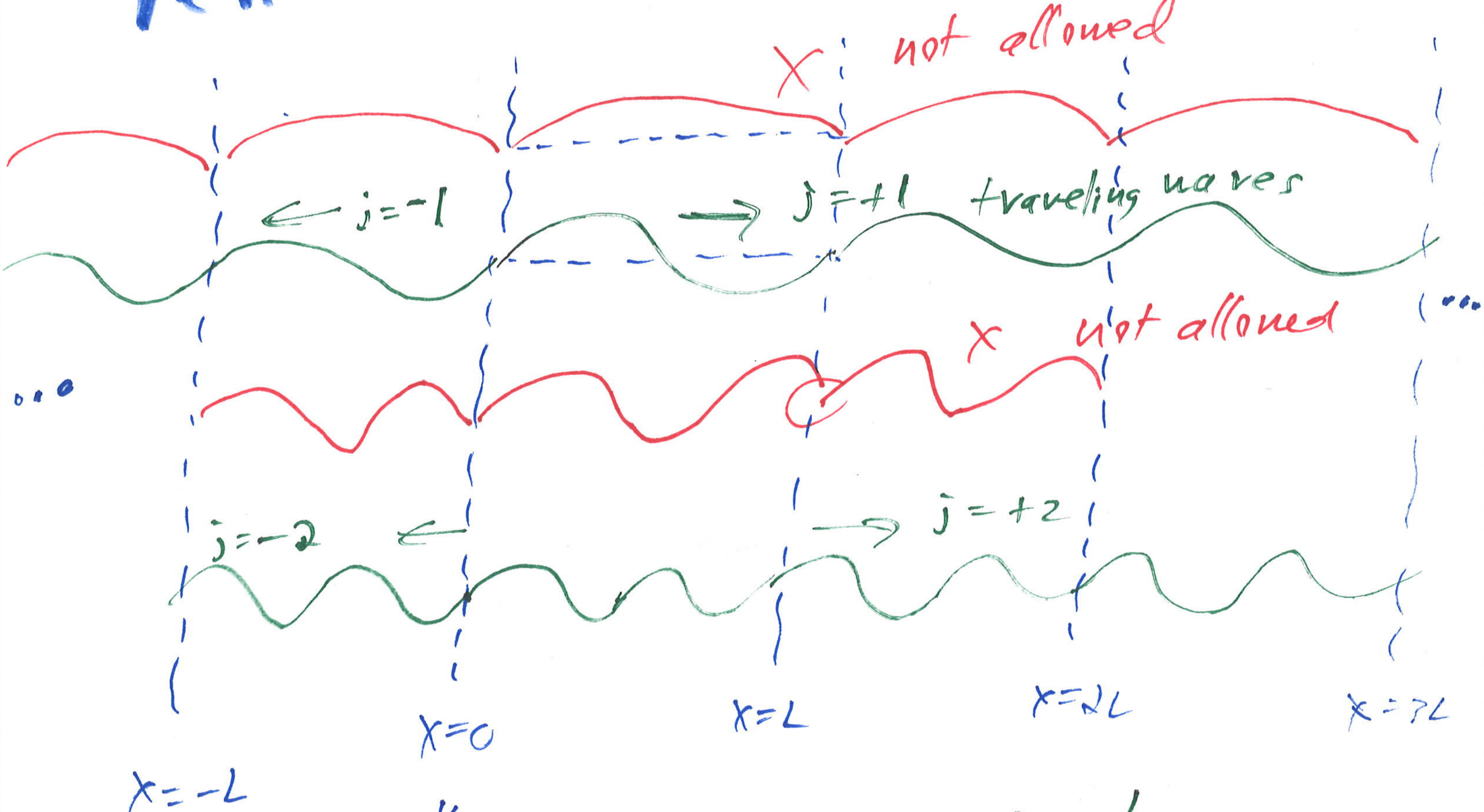
mode n	$\lambda_n = \frac{2L}{n}$	$k_{nx} = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L}$
1	$\frac{2L}{1}$	$\frac{\pi}{L}$
2	$\frac{2L}{2}$	$\frac{2\pi}{L}$
3	$\frac{2L}{3}$	$\frac{3\pi}{L}$
4	$\frac{2L}{4}$	$\frac{4\pi}{L}$
5	$\frac{2L}{5}$	$\frac{5\pi}{L}$

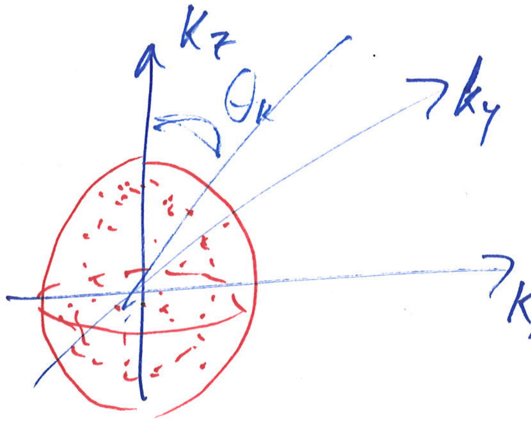
$k > 0$

k-space



Periodic B.C.





$$\lambda = \frac{L}{j}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi j}{L}$$

$$-\infty < j < \infty$$

$$-\infty < k < \infty$$

$$U(T) = \sum_{j_x = -\infty}^{+\infty} \sum_{j_y} \sum_{j_z} \sum_{\epsilon=1}^{2} \hbar\omega \left[\frac{1}{2} + \left(\frac{1}{e^{\beta\hbar\omega} - 1} \right) \right]$$

periodic B.C.

polarizations

Planck distribution

Dispersion relation $\omega = ck$ (no dispersion)

$$\sum_{j_x} \rightarrow \int_{j_x} dj_x \quad \text{continuum limit}$$

$$j_x = \frac{k_x L}{2\pi} \quad \text{(periodic Boundary Conditions)}$$

$$dj_x = dk_x \left(\frac{L}{2\pi} \right)$$

$$\int_{j_x} dj_x \int_{j_y} dj_y \int_{j_z} dj_z$$

$V = L^3$

polarizations

$$U(T) = \Big[\underbrace{\cancel{u_0}}_{\infty} + 2 \int_{k_x}\int_{k_y}\int_{k_z} \left(\frac{(\hbar c k \beta)\, k_B T}{e^{\beta\hbar c k} - 1} \right) \frac{dk_x\, dk_y\, dk_z\, L^3}{2\pi \cdot 2\pi \cdot 2\pi} \Big]$$

Cartesian $k_x, k_y\, k_z \longrightarrow$ Spherical Polar

$$k_r\,,\ \theta_k\,,\ \varphi_k$$

$$U(T) = k_B T V \frac{2}{8\pi^3} \underbrace{\int_{\theta_k=0}^{\pi} \sin\theta_k \, d\theta_k \int_{\varphi_k=0}^{2\pi} d\varphi_k}_{4\pi} \int_{k_r=0}^{\infty} k_r^2 \frac{\hbar c k_r \beta}{e^{\hbar c k_r \beta} - 1} dk_r$$

Define $x = \beta \hbar c k_r$, $dx = \beta \hbar c \, dk_r$

dimensionless $\quad k = |\vec{k}| = k_r$

$$U(T) = V \frac{k_B T}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \underbrace{\int_{x=0}^{\infty} \frac{x^3 \, dx}{e^x - 1}}_{\frac{\pi^4}{15}}$$

Energy Density

$$u = \frac{U(T)}{V} = \underbrace{\left(\frac{8\pi^5 k_B^4}{15 h^3 c^3}\right)}_{a} T^4 = a T^4$$

Radiation Constant

$a = \frac{4}{c}\sigma$ Stefan-Boltzmann constant

Heat Capacity

$$C_V(T) = \left(\frac{\partial U}{\partial T}\right)_V = \frac{32\pi^5 k_B^4 T^3 V}{15\, h^3 c^3}$$

$$\lim_{T\to 0} C_V = 0$$ third law

Entropy

$$dQ = T\,dS \Rightarrow dS = \frac{dQ}{T} = \frac{C_V\,dT}{T}$$

$$S = \int_0^T \frac{C_V(T')}{T'}\,dT' = \frac{32}{45}\frac{\pi^5 k_B^4 V}{h^3 c^3}\,T^3$$

Pressure

$$dF = -S\,dT - p\,dV + \mu\,dN$$

$$F = U - TS$$

$$dF = \left(\frac{\partial F}{\partial T}\right)_{VN} dT + \left(\frac{\partial F}{\partial V}\right)_{TN} dV + \left(\frac{\partial F}{\partial N}\right)_{TV} dN$$

$$F = -k_B T \ln(Z)$$

$$p = -\left(\frac{\partial F}{\partial V}\right)_{TN}$$

$$F = U - TS = aVT^4 - \frac{4}{3}aVT^4 = -\frac{1}{3}aVT^4$$

$$P = \frac{1}{3}aT^4 = \frac{1}{3}\frac{U}{V} = \frac{1}{3}u$$

characteristic of relativistic

Avg. Number of Photons

$$N(T) = \sum_{j_x\, j_y\, j_z = -\infty}^{+\infty} \sum_{\epsilon=1}^{2} \frac{1}{e^{\beta\hbar\omega}-1}$$

$$= 16\pi V \frac{T^3 k_B^3}{h^3 c^3} \underbrace{\frac{1}{2}\int_{x=0}^{\infty} \frac{x^2}{e^x-1}\,dx}_{\zeta(3) \;\simeq 1.20}$$

$\zeta(3)$ Riemann Zeta function

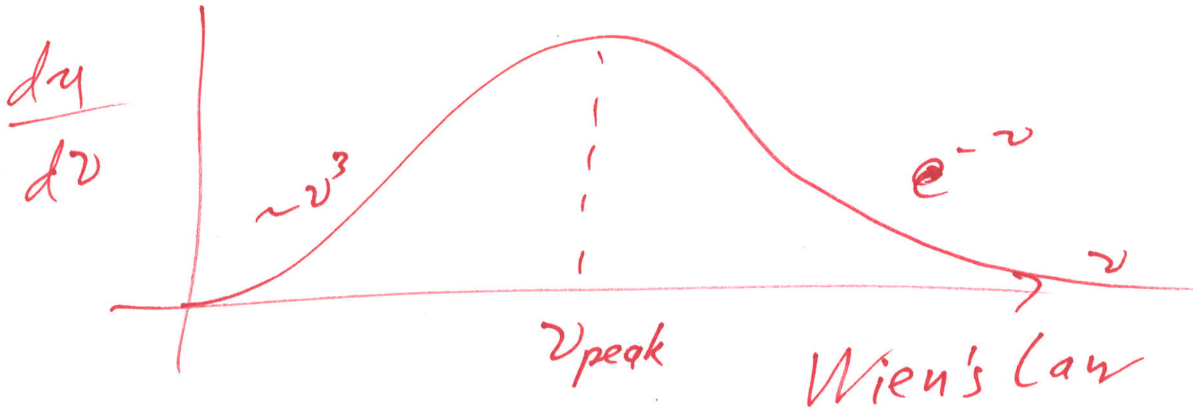
$$u = \frac{U}{V} = \frac{1}{\pi^2}\int_{k=0}^{\infty} dk \boxed{\frac{k^2 \hbar c k}{e^{\beta\hbar c k}-1}} \quad \frac{du}{dk}$$

$$\hbar c k = \hbar\omega = h\nu \leftarrow \nu \text{ is frequency}$$

$$\hbar c\, dk = h\, d\nu$$

$$\frac{du}{d\nu} = \frac{1}{\pi^2 c^2}\frac{h\nu^3}{e^{\beta h\nu}-1}$$

$$\frac{du}{d\lambda} = ?$$



Wien's Law

$\nu \propto T$

$$\lambda_{peak} \neq \frac{c}{\nu_{peak}}$$

What if we don't throw out the zero point energy?

$$Z = e^{-\frac{\beta\hbar\omega}{2}} \frac{1}{1 - e^{-\beta\hbar\omega}} \cdot \left(\frac{e^{+\frac{\beta\hbar\omega}{2}}}{e^{+\frac{\beta\hbar\omega}{2}}} \right)$$

$$= \frac{1}{e^{+\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}} = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$$

$$= \frac{1}{2} \operatorname{csch}\left(\frac{\beta\hbar\omega}{2}\right)$$

$$U = -\frac{\partial \ln(Z)}{\partial \beta} = \frac{\hbar\omega}{2} \coth\left(\frac{\beta\hbar\omega}{2}\right)$$

$$= \frac{\hbar c k}{2} \coth\left(\frac{\beta\hbar c k}{2}\right)$$

$$\int dV\, u \propto \int_{k=0}^{\infty} k^3 \coth(\ldots k) \to \infty$$