Degenerate Fermi Gas

↑ odd - relative to what?

e.g. electrons in white dwarf, neutrons in a neutron star.
electrons in metal at room temp.

When is the gas degenerate?

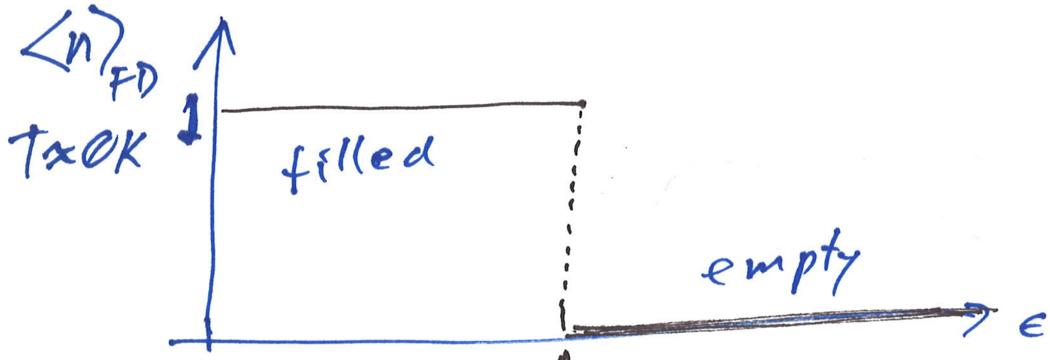
$$\text{Test: } \frac{1}{n} = v = \frac{V}{N} \ll v_Q = \lambda_Q^3 = \left(\frac{h}{\sqrt{\pi\, 2 m k_B T}}\right)^3$$

↑ electron mass

conduction electrons in Cu at 300K

$$v = (0.2\,\text{nm})^3 \ll v_Q = (4.3\,\text{nm})^3$$

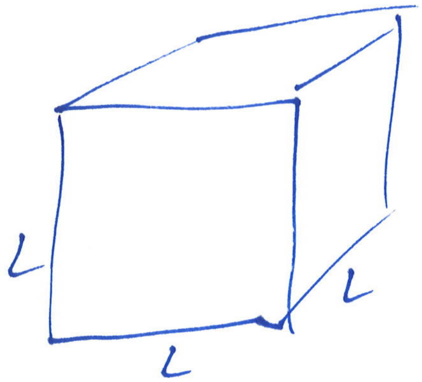
Good Approximation for metals: $\Rightarrow T \approx 0\,K$



ϵ_F Fermi energy
$\parallel$
μ - chemical potential } only at $T = 0K$

Drude model - fill a box with free electrons
treat Quantum mechanically. (waves).

$j_i > 0$



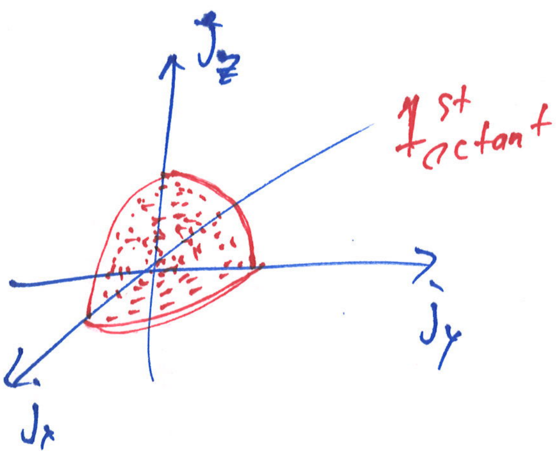
Hard-wall Boundary Conditions

$$\lambda_x = \frac{2L}{j_x} \Longrightarrow P_x = \frac{h}{\lambda_x} = \hbar k_x = \frac{h j_x}{2L}$$

same for y, z : $P_y = \dfrac{h j_y}{2L}$, $P_z = \dfrac{h j_z}{2L}$

Energy

$$\epsilon_{\vec{j}} = \frac{\vec{P}^2}{2m} = \frac{P_x^2 + P_y^2 + P_z^2}{2m} = \frac{h^2}{8mL^2}\underbrace{\left(j_x^2 + j_y^2 + j_z^2\right)}_{\vec{j}^2 = \vec{j}\cdot\vec{j} = j^2}$$

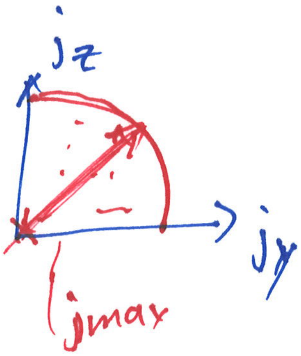


Number of electrons = # dots

$$N \equiv \underset{\uparrow}{2} \cdot \frac{1}{8} \cdot \frac{4\pi}{3}\left(j_{max}\right)^3 = \frac{\pi}{3} j_{max}^3$$

polarization
up or down

solve for j_{max}



$$\epsilon_{Fermi} \equiv \frac{h^2}{8mL^2}\left(j_{max}\right)^2$$ substitute

$$\epsilon_{Fermi} = \frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3N}{\pi V}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3n}{\pi}\right)^{2/3}$$

intensive

n = number density = $m^{-3} = \dfrac{\rho}{m}$

intensive

Average Energy for system

$$U = 2 \sum_{j_x}\sum_{j_y}\sum_{j_z}^{j_{max}} \epsilon_j = 2 \sum\sum\sum_{1}^{j_{max}} \frac{h^2}{8mL^2}(j_x^2 + j_y^2 + j_z^2)$$

↑ spins $= 1$

points are numerous, close together ⟹ integrals

Cartesian ⟹ Polar

$$U = 2 \underbrace{\int_0^{\pi/2} \sin\theta_j \, d\theta_j \int_0^{\pi/2} d\varphi_j}_{\frac{1}{8}(4\pi)} \int_0^{j_{max}} \underbrace{j^2}_{V_j} dj \, \frac{h^2}{8mL^2} j^2$$

$$U = \frac{\pi h^2}{8mL^2} \underbrace{\int_{j=0}^{j_{max}} j^4 dj}_{\frac{(j_{max})^5}{5}} = \frac{\pi h^2 (j_{max})^5}{40 m L^2}$$

dimension dependent $d=3$

$$U = \frac{\pi h^2}{40 m L^2}\left(\frac{3N}{\pi}\right)^{5/3} = \cdots = \frac{3}{5} N \epsilon_{Fermi}$$

U ← extensive; N ← extensive; ϵ_{Fermi} ← intensive

For conduction electrons in Cu at room temperature

$T = 300K$, $T_{Fermi} = \frac{\epsilon_{fermi}}{k_B} = 11{,}000K$

⟹ degenerate Fermi gas

Degeneracy Pressure   holds up a white dwarf.

$$P = -\left(\frac{\partial U}{\partial V}\right)_{SN} = -\frac{\partial}{\partial V}\left[\frac{3}{5} N \frac{h^2}{8m_e}\left(\frac{3N}{\pi}\right)^{2/3} V^{-\frac{2}{3}}\right]$$

$$= +\frac{2}{3}\left[\frac{3}{5} N \frac{h^2}{8m_e}\left(\frac{3N}{\pi}\right)^{2/3} V^{-\frac{5}{3}}\right] = \cdots$$

$$= \frac{2}{5}\frac{N}{V}\epsilon_{fermi} = \frac{2}{3}\frac{U}{V} = \frac{2}{3}u$$

energy density

$$U = \frac{3}{5}N\epsilon_{fermi}$$

$\frac{2}{3} \Rightarrow$ non-relativistic, ideal gas

$\frac{1}{3} \Rightarrow$ relativistic (photon gas)

$$N = 2 \sum_{j_x}^{j_{max}} \sum_{j_y} \sum_{j_z} 1$$

$$\to 2 \iiint 1 \, d^3_j$$

$$U = 2 \sum\sum\sum_{\vec{j}} \epsilon_{\vec{j}} \quad \epsilon(j_x, j_y j_z)$$

$$\to 2 \iiint \epsilon_{\vec{j}} \, d^3_j$$

change variables $\epsilon_{j} = \dfrac{h^2}{8mL^2} j^2$

$$j = \sqrt{\frac{8mL^2}{h^2}} \sqrt{\epsilon} \implies dj = \sqrt{\frac{8mL^2}{h^2}} \frac{1}{2\sqrt{\epsilon}} d\epsilon$$

$$N = \int_{\epsilon=0}^{\epsilon_{fermi}} 1 g(\epsilon) \, d\epsilon$$

$$U = \int_{\epsilon=0}^{\epsilon_{fermi}} \epsilon \, g(\epsilon) \, d\epsilon$$

$$N = \int_\epsilon \frac{dN}{d\epsilon} d\epsilon = \int dN$$

$g(\epsilon)$ - degeneracy, multiplicity, density of states, density of orbitals.

$$g(\epsilon) = \frac{\pi}{2} \frac{(8m)^{3/2}}{h^3} \sqrt{\epsilon} = \frac{3}{2} \frac{N}{\epsilon_{Fermi}^{3/2}} \sqrt{\epsilon} = \frac{dN}{d\epsilon}$$

$$N = \int g(\epsilon) d\epsilon = \int \left(\frac{dN}{d\epsilon}\right) d\epsilon = \int dN$$

$$N = \int_{\epsilon=0}^{\epsilon_{fermi}} 1 \, g(\epsilon) \langle n \rangle_{FD}^{T=0} \, d\epsilon$$

step function $= \begin{cases} 1, & \epsilon < \epsilon_F \\ 0, & \epsilon > \epsilon_F \end{cases}$

$$U = \int_{\epsilon=0}^{\epsilon_{fermi}} \epsilon \, g(\epsilon) \langle n_{FD} \rangle^{T=0} \, d\epsilon$$

$$N = \int_{\epsilon=0}^{\infty} g(\epsilon) \langle n_{FD} \rangle^{T=0} \, d\epsilon$$

$$U = \int_{\epsilon=0}^{\infty} \epsilon \, g(\epsilon) \langle n_{FD} \rangle^{T=0} \, d\epsilon$$

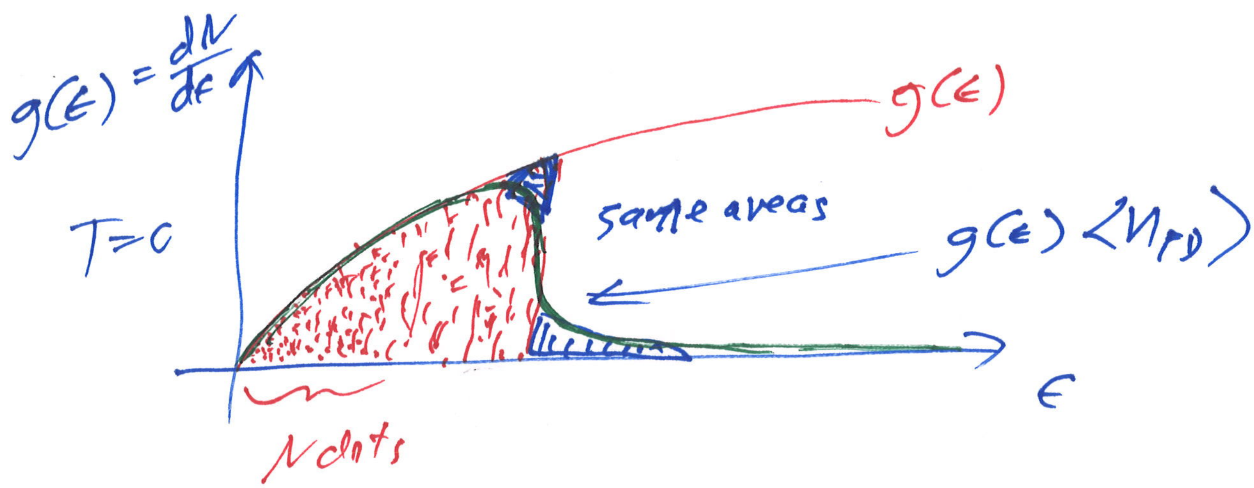
$T \neq 0$ small

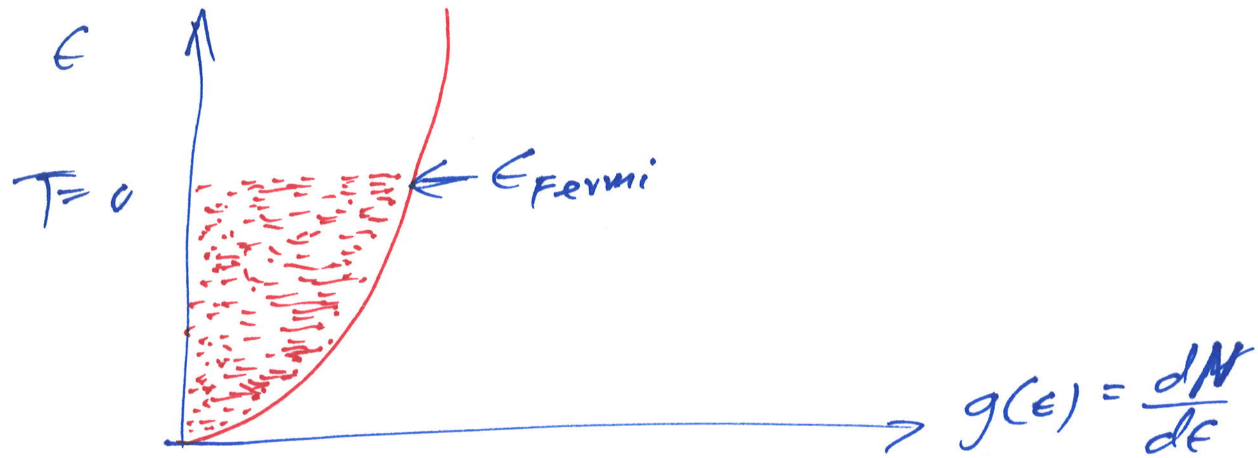
$$N = \int_{\epsilon=0}^{\infty} g(\epsilon) \langle n_{FD} \rangle \, d\epsilon$$

$$U = \int_{\epsilon=0}^{\infty} \epsilon \, g(\epsilon) \langle n_{FD} \rangle \, d\epsilon$$

$$N = \int_{\epsilon=0}^{\infty} \frac{1 \, g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \, d\epsilon$$

$$U = \int_{\epsilon=0}^{\infty} \frac{\epsilon \, g(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \, d\epsilon$$





Plan: get μ from the N integral. Plug μ into the U integral and find

$$U(N,T) \Rightarrow C_V = \left(\frac{\partial U}{\partial T}\right)_{V,N} \propto T$$