Suppose $y = \operatorname{arctanh}(x)$

$$X = \tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \cdot \frac{e^{y}}{e^{y}}$$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow (e^{2y} + 1)x = e^{2y} - 1$$

$$e^{2y}x + x = e^{2y} - 1$$

$$e^{2y}(x-1) = -x - 1 \Rightarrow e^{2y}(1-x) = 1 + x$$

$$e^{2y} = \frac{1+x}{1-x} \Rightarrow 2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\left(\frac{1+x}{1-x}\right)^{1/2} = \ln\sqrt{\frac{1+x}{1-x}}$$

$$\cosh(y) = \cosh[\operatorname{artanh}(x)]$$

$$\hookrightarrow \frac{e^{y} + e^{-y}}{2} = \frac{1}{2}\left(\sqrt{\frac{1+x}{1-x}} + \sqrt{\frac{1-x}{1+x}}\right)$$

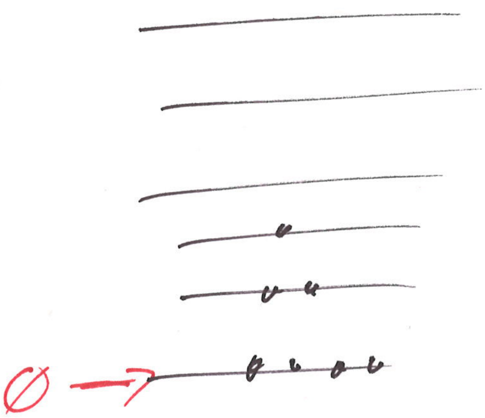
$$= \frac{1}{2}\,\frac{1+x+1-x}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

Marbles on a staircase

Last time, energy fixed $E = 3mgh \Rightarrow$ micro-canonical

Now put system in contact with a reservoir at temp T. $\Rightarrow$ canonical ensemble, states will be Boltzmann weighted. $\sim e^{-\frac{E}{k_B T}}$

How many microstates are there? No

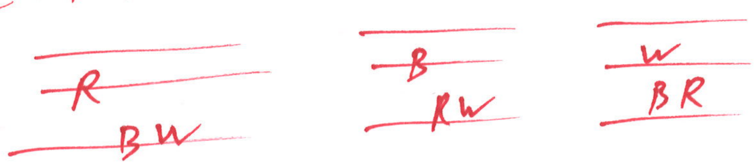


~~Hard way~~ $N = 3$ marbles

$\underline{E = 0}$ Macrostate, # microstates

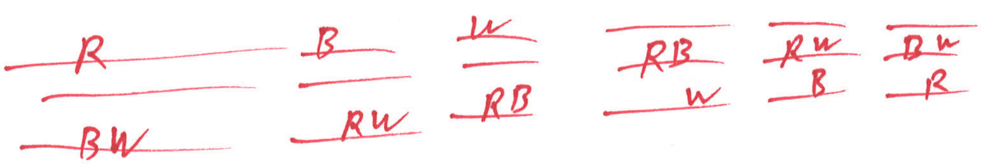
1 microstate

$E = 1\,mgh$ Macrostate



3 microstates

$E = 2mgh$ Macrostates # microstates = 6



$\binom{N}{2} + \binom{N}{1}$

$$Z_t = \sum_{\substack{i = \text{micro} \\ \text{states}}} e^{-\frac{E_i}{k_B T}} = 1 e^{-\beta 0} + 3 e^{-\beta mgh} + 6 e^{-\beta 2mgh} + 10 e^{-\beta 3mgh} + \cdots$$

$$= \sum_{\substack{\text{macro} \\ \text{states}}} g_i e^{-\frac{E_i}{k_B T}}$$

Easier Way

z_1 = partition function for one marble. $= \sum_{n=0}^{\infty} e^{-\beta mgh n}$ = geometric series

$$\sum_k r^k = \frac{1}{1-r} \Rightarrow z_1 = \frac{1}{1 - e^{-\beta mgh}}$$

$$Z_t = z_1^3 = \left(\frac{1}{1 - e^{-\beta mgh}}\right)^3$$

Probability all 3 marbles will be on step 1.

$$P_{111} = \frac{e^{-\beta mgh 3}}{Z_t} = e^{-\beta mgh 3}\left(1 - e^{-\beta mgh}\right)^3$$

$$P_{npq} = \frac{e^{-\beta mgh [n+p+q]}}{Z_t}$$

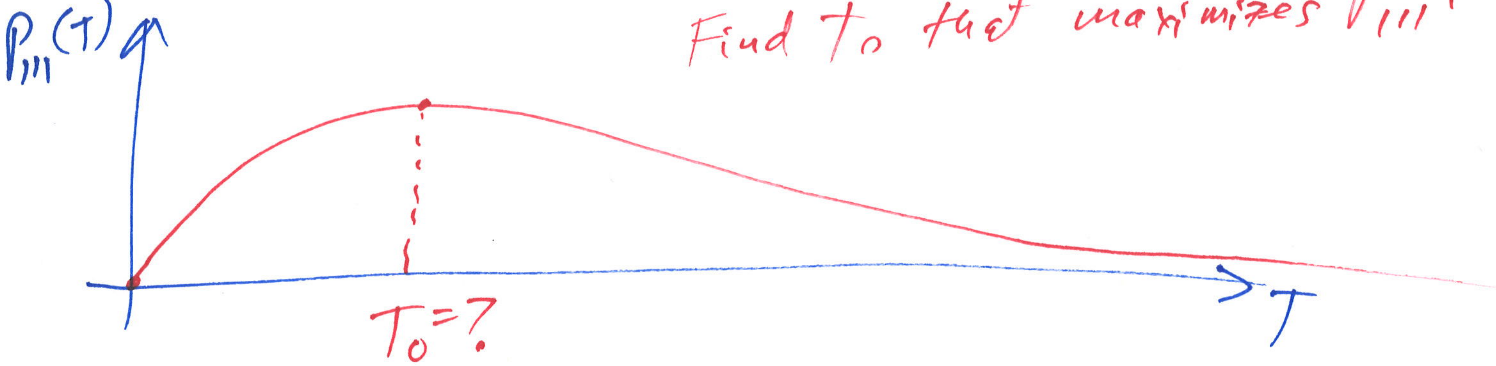
What is P_{111} at low T (high β)

$$P_{111} = \left[e^{-\beta mgh}\left(1 - e^{-\beta mgh}\right)\right]^3 \longrightarrow 0$$

What is P_{111} at high T (low β)

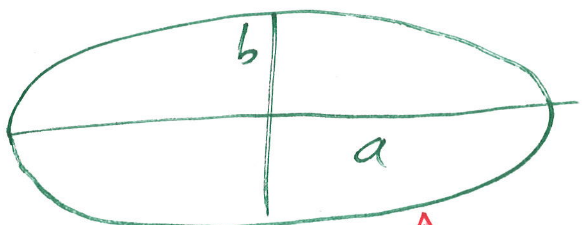
$$P_{111} \longrightarrow 0$$

Find T_0 that maximizes P_{111}



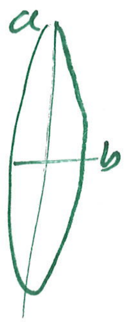
One-dimensional chain with massless links

Tension in chain? $\tau = Mg$



↑ lower energy

At temperature T, what is L, the length of the chain



← higher energy.

Lower energy

$-\tau a$

$\cancel{E_1} = -\tau a$

H_1

higher energy state

$\tau(a-b)$

$-\tau a$

$\cancel{E_2} = -\tau b$

H_2

All links independent

$$z = e^{-(-\tau a)\beta} + e^{-(-\tau b)\beta} = e^{\beta\tau a} + e^{\beta\tau b}$$

$$Z = z^N = \left(e^{\beta\tau a} + e^{\beta\tau b}\right)^N$$

$$\ln(Z) = N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)$$

$$U = -\frac{\partial \ln(z)}{\partial \beta}, \quad F = -\frac{\ln(z)}{\beta}$$

$$G = -\frac{\ln(\mathcal{Z})}{\beta}$$

$$\text{Expect } \langle L \rangle = \pm \left.\frac{\partial(\text{Potential})}{\partial(\text{variable})}\right|_{x,y}$$

$$dF = -S\,dT + \tau\,dL + \mu\,\cancel{dN}^{\,0}$$

$-p\,dV$ for a gas

mechanical work

$$dF = \left(\frac{\partial F}{\partial T}\right)_{LN} dT + \left(\frac{\partial F}{\partial L}\right)_{TN} dL + \left(\frac{\partial F}{\partial N}\right)_{TL} dN$$

Could you this if I wanted τ.

Legendre transformation

switch τ, L and change the sign.

$$dG = -S\,dT - L\,d\tau + \mu\,dN$$

$$dG = \left(\frac{\partial G}{\partial T}\right)_{\tau N} dT + \left(\frac{\partial G}{\partial \tau}\right)_{TN} d\tau + \left(\frac{\partial G}{\partial N}\right)_{T\tau} dN$$

$$\langle L \rangle = -\left(\frac{\partial G}{\partial \tau}\right)_{TN}$$

$$\mathcal{Z}_G = \sum_{\substack{\text{micro}\\\text{state}}} e^{-\frac{H_i}{k_B T}}$$ Gibbs partition function

$$= \left(e^{+\beta\tau a} + e^{+\beta\tau b}\right)^N$$

$$G = -k_B T \ln(\mathcal{Z}_G) = -k_B T N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)$$

$$\langle L \rangle = -\left(\frac{\partial G}{\partial \tau}\right)_{TN} = -\frac{\partial}{\partial \tau}\left[-k_B T N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)\right]$$

$$\langle L \rangle = \frac{\cancel{k_B}\cancel{T} N \left[\cancel{\beta} a e^{\beta\tau a} + \cancel{\beta} b e^{\beta\tau b} \right]}{e^{\beta\tau a} + e^{\beta\tau b}}$$

$$\langle L \rangle = \frac{N \left[a e^{\beta\tau a} + b e^{\beta\tau b} \right]}{e^{\beta\tau a} + e^{\beta\tau b}}$$

low T, high β : $\langle L \rangle = Na$

high T, low β : $\langle L \rangle = N \frac{a+b}{2}$