Suppose $y = \operatorname{arctanh}(x)$

$$x = \tanh(y) = \frac{\sinh(y)}{\cosh(y)} = \frac{e^{y} - e^{-y}}{e^{y} + e^{-y}} \cdot \frac{e^{y}}{e^{y}}$$

$$x = \frac{e^{2y} - 1}{e^{2y} + 1} \Rightarrow (e^{2y} + 1)x = e^{2y} - 1$$

$$e^{2y}x + x = e^{2y} - 1$$

$$e^{2y}(x - 1) = -x - 1 \Rightarrow e^{2y}(1 - x) = 1 + x$$

$$e^{2y} = \frac{1 + x}{1 - x} \Rightarrow 2y = \ln\left(\frac{1 + x}{1 - x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) = \ln\left(\frac{1 + x}{1 - x}\right)^{1/2} = \ln\sqrt{\frac{1 + x}{1 - x}}$$

$$\cosh(y) = \cosh[\operatorname{artanh}(x)]$$

$$\hookrightarrow \frac{e^{y} + e^{-y}}{2} = \frac{1}{2}\left(\sqrt{\frac{1 + x}{1 - x}} + \sqrt{\frac{1 - x}{1 + x}}\right)$$

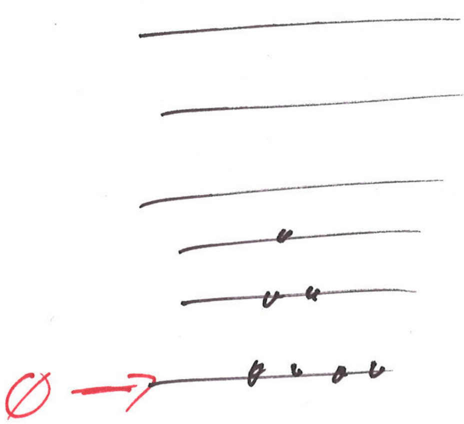
$$= \frac{1}{2}\,\frac{1 + x + 1 - x}{\sqrt{1 - x^{2}}} = \frac{1}{\sqrt{1 - x^{2}}}$$

Marbles on a staircase

last time, energy fixed $E = 3mgh \Rightarrow$ micro-canonical

Now put system in contact with a reservoir at temp T. $\Rightarrow$ canonical ensemble, states will be Boltzmann weighted. $\sim e^{-\frac{E}{k_B T}}$

How many microstates are there? No

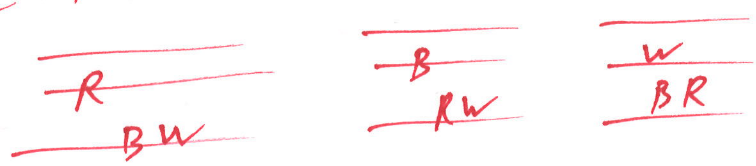


~~X~~ Hard way $N = 3$ marbles

$\underline{E=0}$ Macrostate, # microstates

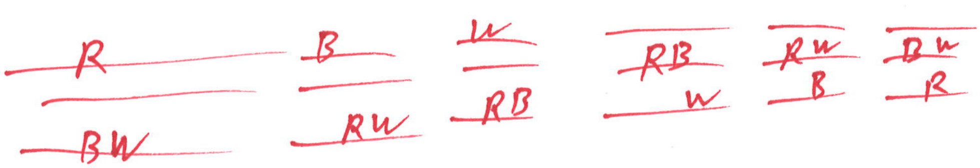
1 microstate

$E = 1^{mgh}$ Macrostate



3 microstates

$E = 2mgh$ Macrostates   # microstates $= 6$



$\binom{N}{2} + \binom{N}{1}$

$$Z_t = \sum_{i = \text{micro states}} e^{-\frac{E_i}{k_B T}} = 1 e^{-\beta 0} + 3 e^{-\beta mgh} + 6 e^{-\beta 2mgh} + 10 e^{-\beta 3mgh} + \cdots$$

$$= \sum_{\text{macro states}} g_i e^{-\frac{E_i}{k_B T}}$$

Easier Way

z = partition function for one marble. $= \sum_{n=0}^{\infty} e^{-\beta mgh n}$ = geometric series

$$\sum_k r^k = \frac{1}{1-r} \Rightarrow z = \frac{1}{1 - e^{-\beta mgh}}$$

$$Z_t = z^3 = \left(\frac{1}{1-e^{-\beta mgh}}\right)^3$$

Probability all 3 marbles will be on step 1.

$$P_{111} = \frac{e^{-\beta mgh 3}}{Z_t} = e^{-\beta mgh 3}\left(1-e^{-\beta mgh}\right)^3$$

$$P_{npq} = \frac{e^{-\beta mgh[n+p+q]}}{Z_t}$$

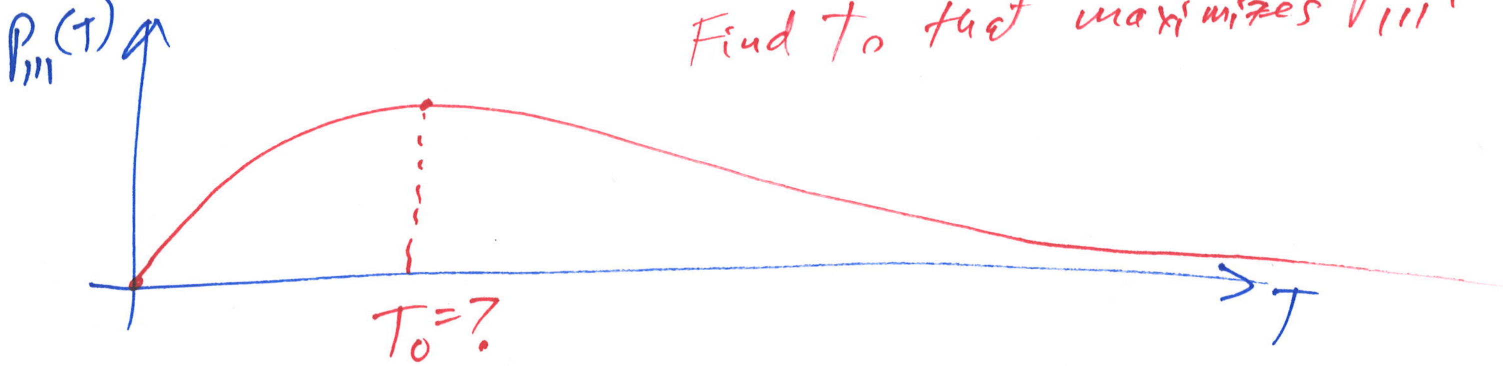
What is P_{111} at low T (high β)

$$P_{111} = \left[e^{-\beta mgh}\left(1-e^{-\beta mgh}\right)\right]^3 \longrightarrow 0$$

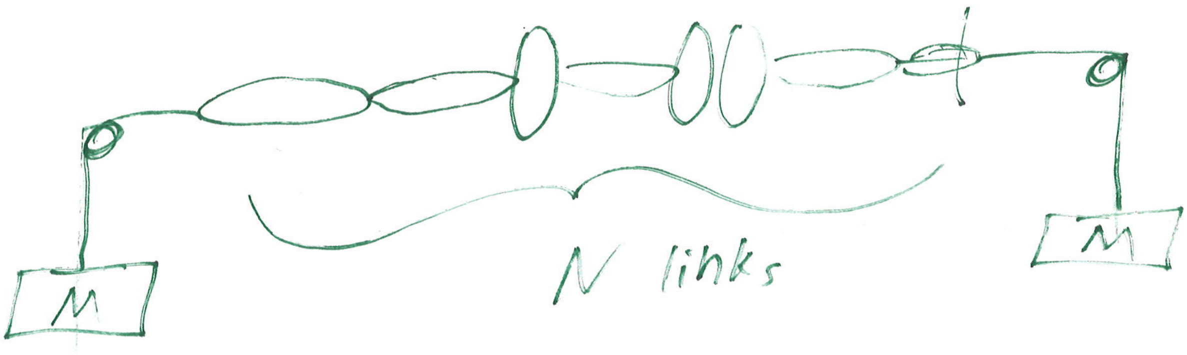
What is P_{111} at hight (low β)

$$P_{111} \longrightarrow 0$$

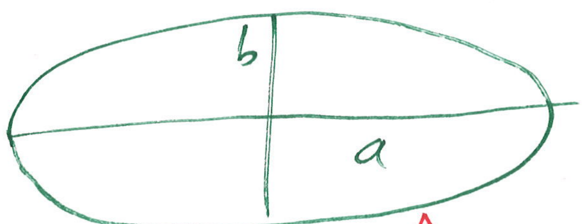
Find T_0 that maximizes P_{111}



One-dimensional chain with massless links



Tension in chain? $\tau = Mg$



At temperature T, what is L, the length of the chain

↑ lower energy

← higher energy.

Lower energy

$-\tau a$

$\cancel{E_1} = -\tau a$

H_1

higher energy state

$\tau(a-b)$

$-\tau a$

$\cancel{E_2} = -\tau b$

H_2

All links independent

$$z = e^{-(-\tau a)\beta} + e^{-(-\tau b)\beta} = e^{\beta\tau a} + e^{\beta\tau b}$$

$$Z = z^N = \left(e^{\beta\tau a} + e^{\beta\tau b}\right)^N$$

$$\ln(Z) = N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)$$

$$U = -\frac{\partial \ln(Z)}{\partial \beta}, \quad F = -\frac{\ln(Z)}{\beta}$$

$$G = -\frac{\ln(\mathcal{Z})}{\beta}$$

$$\text{Expect } \langle L \rangle = \pm \left.\frac{\partial(\text{Potential})}{\partial(\text{variable})}\right|_{x,y}$$

$$dF = -S\,dT + \tau\,dL + \cancel{\mu\,dN}^{\to 0}$$

$-p\,dV$ for a gas
mechanical work

$$dF = \left(\frac{\partial F}{\partial T}\right)_{LN} dT + \left(\frac{\partial F}{\partial L}\right)_{TN} dL + \left(\frac{\partial F}{\partial N}\right)_{TL} dN$$

Could you this if I wanted τ.

Legendre transformation

switch τ, L and change the sign.

$$dG = -S\,dT - L\,d\tau + \mu\, dN$$

$$dG = \left(\frac{\partial G}{\partial T}\right)_{\tau N} dT + \left(\frac{\partial G}{\partial \tau}\right)_{TN} d\tau + \left(\frac{\partial G}{\partial N}\right)_{T\tau} dN$$

$$\langle L \rangle = -\left(\frac{\partial G}{\partial \tau}\right)_{TN}$$

$$Z_G = \sum_{\text{micro state}} e^{-\frac{H_i}{k_B T}}$$ Gibbs partition function

$$= \left(e^{+\beta\tau a} + e^{+\beta\tau b}\right)^N$$

$$G = -k_B T \ln(Z_G) = -k_B T N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)$$

$$\langle L \rangle = -\left(\frac{\partial G}{\partial \tau}\right)_{TN} = -\frac{\partial}{\partial \tau}\left[-k_B T N \ln\left(e^{\beta\tau a} + e^{\beta\tau b}\right)\right]$$

$$\langle L \rangle = \frac{\cancel{k_B T} N \left[\cancel{\beta} a e^{\beta \tau a} + \cancel{\beta} b e^{\beta \tau b} \right]}{e^{\beta \tau a} + e^{\beta \tau b}}$$

$$\langle L \rangle = \frac{N \left[a e^{\beta \tau a} + b e^{\beta \tau b} \right]}{e^{\beta \tau a} + e^{\beta \tau b}}$$

Low T, high β : $\langle L \rangle = Na$

high T, low β : $\langle L \rangle = N \frac{a+b}{2}$