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A Deeper Look at the Sadly Cannot Thermodynamic Cycle

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The "Sadly Cannot" (SC) thermodynamic cycle was introduced by Willis and Kirwan in a paper that appeared in the January 1980 issue of this journal.[1] Comprising but two steps, it appears remarkably simple. First, an ideal monatomic gas is expanded from initial state (P_0, V_0) to final state (P_1, V_1) along a path that is a straight line of negative slope in the PV plane, but with (P_0, V_0) and (P_1, V_1) chosen to be connected by an adiabat: $P_0V_0^{\gamma} = P_1V_1^{\gamma}$, where we set $\gamma = 5/3$. The return path is the adiabat, along which there can be no heat energy exchange. The cycle is sketched (not to scale) in Fig. 1. As with all such cycles, all processes are assumed to be quasistatic.

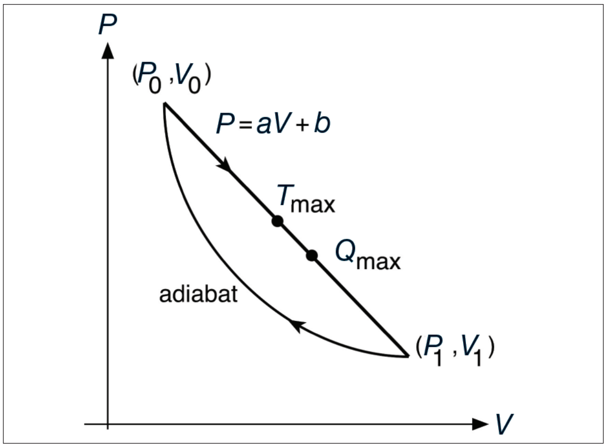

Fig. 1. Sketch of the Sadly Cannot cycle (not to scale; no particular compression ratio is assumed). The points labeled T_{max} and Q_{max} schematically indicate volumes of maximum temperature and heat input, respectively. The precise volumes at which these occur depend on the compression ratio, but T_{max}/Q_{max} = 0.8 for all compression ratios greater than ~1.5, as discussed in the text.

Despite its simplicity, this cycle possesses some pedagogically instructive features. Perhaps the most interesting is that while heat energy initially enters the system, there comes a volume at which this reverses, after which heat leaves the system until (P_1, V_1) is reached. If this behavior is not accounted for, it can be easy to misapply the second law of thermodynamics and the definition of efficiency to come to the conclusion that the efficiency of the cycle is 100%. Beyond this, while the temperature also initially increases (a statement qualified later), there comes a volume of maximum temperature. This volume is always *less* than that at which the heat flow reverses; specifically, it is always at 0.8 times the volume at which the heat flow reverses, as shown via Eqs. (10) and (14) below. This means that there is a range of volume over which the heat capacity is *negative* in the sense that the temperature decreases even while heat continues to enter the system. At the maximum temperature, the straight-line trajectory runs tangent to the isotherm of this temperature. With only two steps, this cycle is simple enough to be taken up in a single class period and its nuances discussed.

Soon after its introduction, the SC cycle appeared as a problem in Zemansky and Dittman's venerable *Heat and Thermodynamics* text.[2] Students should see this problem worked in class or as a homework exercise in order to help them appreciate the above behaviors. The cycle was apparently rediscovered some years later by Dickerson and Mottmann, who undertook a more detailed examination of cycles with straight-line processes.[3] In particular, they examined textbook exercise problems involving triangular cycles where the possibility of the heat in/heat out reversal was not made clear and the answers given for efficiencies could have greater or smaller errors depending on the relative extents of the heat in/heat out volume ranges; they did not cite Willis and Kirwan.

Subsequently, many papers featuring unusual cycles and methods for calculating their efficiencies have appeared, as have more general treatments of the behavior of heat capacities; see, for example, the recent papers of Könye and Cserti[4] and Pacheco et al.[5] However, but for a brief paper from 1991 by Mills and Huston wherein it was pointed out that the straight-line trajectory is tangential to an adiabat at the point of heat-flow reversal,[6] I am not aware of any further treatments of the SC cycle.

While recently describing the SC cycle to my students, I wondered if, like the classic Otto cycle, its efficiency could be made to approach 100% if the compression ratio

$$r = \frac{V_1}{V_0} \tag{1}$$

was allowed to become arbitrarily large. In their original paper, Willis and Kirwan worked with $r = 8$, which gave an efficiency $\eta \approx 0.52$. Investigating this behavior led me to two apparently new revelations regarding the SC cycle: (1) that its efficiency has an absolute maximum value of $\eta = 16/25 = 0.64$ for $r \rightarrow \infty$, and (2) that the temperature can initially decline if the compression ratio is small enough ($r < \sim1.5$), although the heat in/heat out behavior holds for all values of r. The purpose of this paper is to draw these conclusions to the attention of the thermodynamics teaching community.

Work, heat, efficiency, and temperature in the Sadly Cannot cycle

If the straight-line path in Fig. 1 is described by

$$P = aV + b, \tag{2}$$

the slope and intercept are given by

$$a = \left(\frac{P_0}{V_0}\right)\alpha(r) \tag{3}$$

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and

$$b = P_0[1 - \alpha(r)], \tag{4}$$

where

$$\alpha(r) = \frac{r^{-5/3} - 1}{r - 1}. \tag{5}$$

α can be thought of as a dimensionless measure of the slope; in a plot of P/P_0 vs. V/V_0, the slope would be α and the intercept $(1 - \alpha)$. With $r > 1$, $\alpha(r)$ is always negative, varying from $-5/3$ at $r = 1$ to $\alpha(r) \to -1/r$ for large r. For Willis and Kirwan's choice of $r = 8$, $\alpha = -31/224 \approx -0.138$. (The value of α for $r = 1$ is not obvious; to prove it, set $r = 1 + \epsilon$, perform a series expansion on $r^{-5/3}$, and then take $\epsilon \to 0$.)

The work accomplished around the entire cycle is computed in the usual way:

$$W = \int_{V_0}^{V_1} P dV + W_{\text{adiabat}} = \int_{V_0}^{V_1} (aV + b) dV + \frac{3}{2}(P_1 V_1 - P_0 V_0). \tag{6}$$

This reduces, after some algebra, to a fairly simple function of r:

$$W = \frac{P_0 V_0}{2}(r + 4r^{-2/3} - r^{-5/3} - 4). \tag{7}$$

To compute the *net* heat that has entered the system to a given volume, imagine moving along the straight-line path from V_0 to some arbitrary V, where $V_0 \le V \le V_1$. Write the first law of thermodynamics as $dQ = dU + PdV$, set $dU = C_V dT = (3nR/2)dT$ for an ideal gas, and integrate:

$$Q(V) = \frac{3nR}{2}[T(V) - T_0] + \int_{V_0}^{V} P dV. \tag{8}$$

This emerges as

$$Q(V) = 2\left(\frac{P_0}{V_0}\right)\alpha V^2 + \frac{5}{2}P_0(1-\alpha)V - \left(\frac{P_0 V_0}{2}\right)(5 - \alpha). \tag{9}$$

Even if the compression ratio r approaches infinity, the net heat exchange will always be positive; that is, net heat will always enter the system. The volume at which the direction of heat flow reverses is given by

$$\frac{dQ}{dV} = 0 \Rightarrow V = -\left[\frac{5(1-\alpha)}{8\alpha}\right]V_0. \tag{10}$$

This expression is equivalent to Willis and Kirwan's $-5b/8a$ result. Back-substituting this into Eq. (9) gives the heat *input* to the system:

$$Q_{\text{in}} = -\left(\frac{P_0 V_0}{32\alpha}\right)(9\alpha^2 + 30\alpha + 25). \tag{11}$$

The efficiency is given by the ratio of Eqs. (7) and (11), $\eta = W/Q_{\text{in}}$. This is entirely independent of the initial state (P_0, V_0), and for large values of r behaves as

$$\eta_{r \to \infty} \to \frac{16}{25} = 0.64. \tag{12}$$

That there is a such a well-defined limiting efficiency is rather striking; this traces to the fact that both W and $Q(V)$ are proportional to r when it becomes large. Figure 2 shows

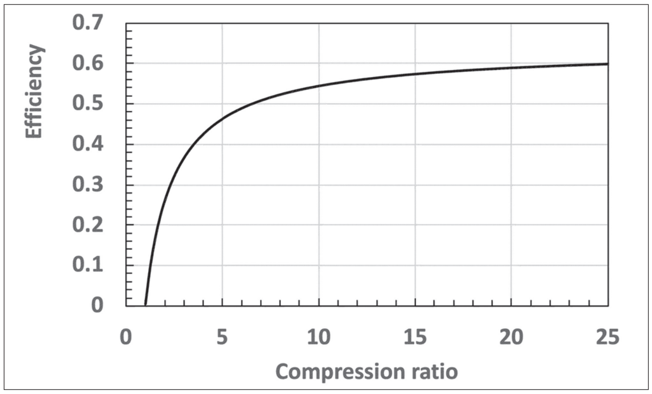

Fig. 2. Efficiency of the Sadly Cannot cycle as a function of compression ratio.

the run of efficiency as a function of r. With their compression ratio of 8, Willis and Kirwan realized just over 80% of the efficiency achievable with this cycle. A worthwhile student exercise might be to investigate the behavior of this limit as a function of the ratio of heat capacities γ.

The temperature evolution of the straight-line path behaves as

$$T(V) = \frac{P_0}{nR}\left[\left(\frac{1}{V_0}\right)\alpha V^2 + (1-\alpha)V\right], \tag{13}$$

This has a maximum at

$$\frac{dT}{dV} = 0 \Rightarrow V = -\left[\frac{(1-\alpha)}{2\alpha}\right]V_0, \tag{14}$$

which must always be less than the volume at which the heat flow reverses, as Willis and Kirwan pointed out.

Now consider the rates of change of T and Q at V_0:

$$\left.\frac{dT}{dV}\right|_{V_0} = \frac{P_0}{nR}(1+\alpha) \tag{15}$$

and

$$\left.\frac{dQ}{dV}\right|_{V_0} = \frac{P_0}{2}(5+3\alpha). \tag{16}$$

Now, α is always negative. Equation (15) then indicates that the temperature will *decline* as the system is expanded from V_0 if $|\alpha| > 1$. This occurs for $r < \sim 1.479$. In such cases, Eq. (14) gives a temperature extremum at a volume less than V_0, which is obviously impossible; that is, we must conclude that T will always remain $< T_0$ for this range of α. Physically, what is happening here is that the slope of the straight-line part of the path is more negative than that of the isotherm that passes through (P_0, V_0). However, Eq. (16) shows that all such cycles will begin with positive heat flow because α is always greater than $-5/3$. To be sure, $r < 1.5$ is a small compression ratio; with their compression ratio of 8, Willis and Kirwan did not uncover this behavior.

Willis and Kirwan also examined the efficiency of a Carnot cycle operating between the maximum and minimum temperatures of the SC cycle. From Eqs. (13) and (14), the maximum operating temperature is, provided r satisfies the

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>1.5 threshold discussed above,

$$T_{\max} = -\frac{T_0}{4\alpha}(1-\alpha)^2. \tag{17}$$

The minimum temperature occurs at $T_1 = T_0/r^{2/3}$, giving the Carnot efficiency as

$$\eta_{\text{Car}} = 1 - \frac{T_1}{T_{\max}} = 1 + \frac{4\alpha}{r^{2/3}(1-\alpha)^2}. \tag{18}$$

For their $r = 8$, this evaluates to $\eta_{\text{Car}} \approx 0.89$.

Concluding remarks

Covering all of the algebraic steps here might be too much for a single class period; students should be encouraged to fill in any gaps. Another challenge for them would be to think about how—in principle at least—such a cycle might be experimentally approximated. One suggestion by Dr. Carl Mungan is to trap gas in a diathermic cylinder and arrange to slowly extract a piston while the temperature is controlled by a surrounding water bath in such a way as to achieve the correct pressure. When the final volume is reached, the cylinder is withdrawn from the bath, wrapped in an insulating blanket, and the gas slowly compressed back to the initial volume. These idealized cycles are never easy to realize in practice!

Revisiting an old problem can sometimes reveal new insights and unexpected behavior. The SC cycle is still a valuable teaching tool for alerting students to the potential pitfalls of blind assumptions and calculations.

Acknowledgments

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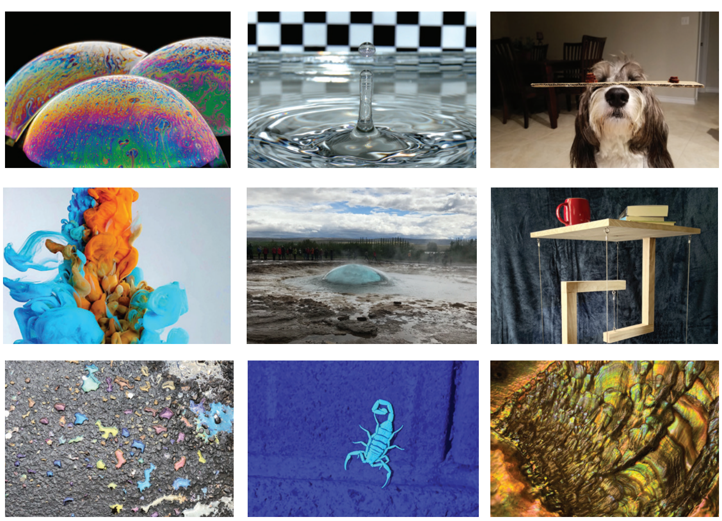

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