# All About Polytropic Processes ⓢⒸⒾ

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# All About Polytropic Processes

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O f all the conceivable ideal-gas processes, almost all introductory physics textbooks as well as more advanced texts on thermodynamics emphasize only four: isochoric, isobaric, isothermal, and adiabatic (isentropic). These are processes in which a state variable—volume, pressure, temperature, or entropy— remains constant. It turns out that these four processes are examples of a more general ideal-gas process, called a *polytropic process*, in which the specific heat remains constant. Polytropic processes are frequently discussed in engineering thermodynamics but rarely in physics. The goal of this paper is to bring polytropic processes to the attention of physics instructors. These processes have interesting properties (such as, in some cases, negative specific heat), they are easily accessible at the introductory physics level, and they expand the range of examples and problems that can be explored in thermodynamics.

## Processes with constant specific heat

A key finding of elementary thermodynamics is that the heat needed to change the temperature of $n$ moles of an ideal gas is $Q = nC_V\Delta T$ for a constant-volume process and $Q = nC_P\Delta T$ for a constant-pressure process. $C_V$ and $C_P$, the molar specific heats at constant volume and constant pressure, are constants.

Consider a general ideal-gas process that we'll label $\beta$ for which $Q = nC_\beta\Delta T$ with a constant molar specific heat $C_\beta$. Isochoric and isobaric processes are two specific members of this class of processes. The small amount of heat $\delta Q$ required for an infinitesimal temperature change $dT$ is

$$\delta Q = nC_\beta dT. \tag{1}$$

We also know that for any ideal-gas process the thermal energy changes by

$$dE_{\text{th}} = nC_V dT. \tag{2}$$

Some textbooks refer to the left-hand side of Eq. (2) as the change in internal energy $dU$.

Now we can use the first law of thermodynamics with the convention that $W$ is the work done *on* the gas[1] to write

$$\delta W = dE_{\text{th}} - \delta Q = n(C_V - C_\beta)dT. \tag{3}$$

The small amount of work done on the gas as the volume changes by $dV$ is $\delta W = -pdV$, so

$$-pdV = n(C_V - C_\beta)dT. \tag{4}$$

If we substitute $nRdT = pdV + Vdp$, from the ideal-gas law, we get an expression involving only $p$ and $V$ that is easily integrated. The result—the details are left to the reader—is that a polytropic process, a process that has a constant molar specific heat $C_\beta$, satisfies the pressure–volume relationship

$$pV^\beta = \text{constant}, \tag{5}$$

where $\beta$, called the polytropic index, is

$$\beta = 1 + \frac{R}{C_V - C_\beta}. \tag{6}$$

Some books[2,3] and papers give Eq. (5) as the definition of a polytropic process, but that would be like defining an isothermal process as one for which $pV$ = constant rather than one for which $T$ = constant. The better definition is a process for which $C_\beta$ = constant, then Eq. (5) follows as a result.

Equation (5) looks very much like the familiar equation $pV^\gamma$ = constant for an adiabatic process, where $\gamma = C_P/C_V$ is the specific heat ratio. Indeed, an adiabatic process is a polytropic process for which $\beta = \gamma$. In fact, as Table I shows, all of the familiar processes are members of the more general class of polytropic processes. The infinite polytropic index of an isochoric process is understood by writing $V = \text{constant}/p^{1/\beta}$, which becomes $V = \text{constant}/p^0 = \text{constant}$ when $\beta = \infty$.

The curve of a polytropic process on a $pV$ diagram is called a polytrope. Figure 1 shows several examples. The blue curves are the four familiar processes; the red curves are examples of other polytropes, including one with a negative value of $\beta$. An ideal gas moving along any of these curves undergoes a process that has a constant specific heat.
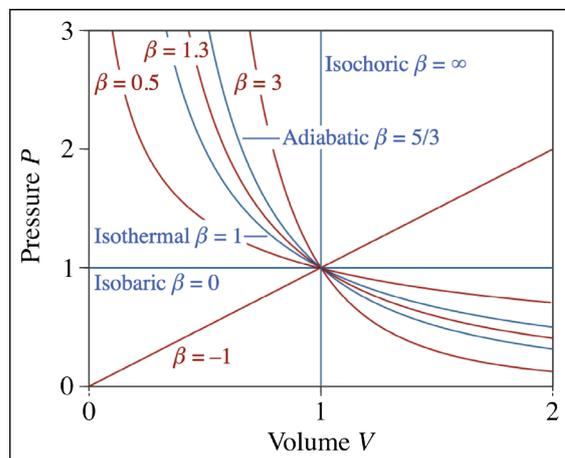


Fig. 1. Polytropic processes can be shown on a *pV* diagram. Each curve is called a polytrope. The adiabatic curve is for a monatomic gas.

Equation (1) assumes that the specific heat $C_\beta$ is constant, and that leads to Eq. (6) for the polytropic index in terms of $C_\beta$. If we solve Eq. (6) for $C_\beta$ in terms of $\beta$ and use $C_V = R/(\gamma - 1)$, which follows from using $C_P = C_V + R$ in $\gamma = C_P/C_V$, we find that the constant molar specific heat of a polytropic process with index $\beta$ is

$$C_\beta = \left(\frac{\beta - \gamma}{\beta - 1}\right)C_V = \left(\frac{1}{\gamma - 1} - \frac{1}{\beta - 1}\right)R. \tag{7}$$

The four familiar cases—isobaric ($\beta = 0$), isothermal ($\beta = 1$), adiabatic ($\beta = \gamma$), and isochoric ($\beta = \infty$)—give, respectively,

**Table I. The four familiar ideal-gas processes are polytropic processes.**

| Process | Constant | Polytropic Index | Specific Heat |
|---|---|---|---|
| Isobaric | Pressure | $\beta = 0$ | $C_P$ |
| Isothermal | Temperature | $\beta = 1$ | $\infty$ |
| Adiabatic | Entropy | $\beta = \gamma$ | 0 |
| Isochoric | Volume | $\beta = \infty$ | $C_V$ |

$C_P$, $\infty$, 0, and $C_V$. We don't normally think of isothermal and adiabatic processes in terms of specific heats, but an isothermal process with $Q \neq 0$ while $\Delta T = 0$ must have $C_1 = \infty$, and an adiabatic process with $Q = 0$ while $\Delta T \neq 0$ must have $C_\gamma = 0$.

What about $C_\beta$ along other polytropes? For example, it's straightforward to calculate that the rising linear polytrope in Fig. 1 with $\beta = -1$ has $C_{-1} = 2R$ for a monatomic gas with $\gamma = 5/3$, a value that falls between $C_V = (3/2)R$ and $C_P = (5/2)R$.

## Negative specific heat

By definition, $C_V > 0$ and $\gamma > 1$. If we use Eq. (7) to examine the sign of $C_\beta$, we find that the specific heat is negative ($C_\beta < 0$) if $1 < \beta < \gamma$; that is, $C_\beta$ is negative for a polytrope that falls between an isotherm and an adiabat. Otherwise the specific heat is positive.

The idea of a negative specific heat certainly seems odd. A process with a negative specific heat is one in which the temperature of the gas increases ($\Delta T > 0$) while heat is being removed ($Q < 0$) or vice versa.
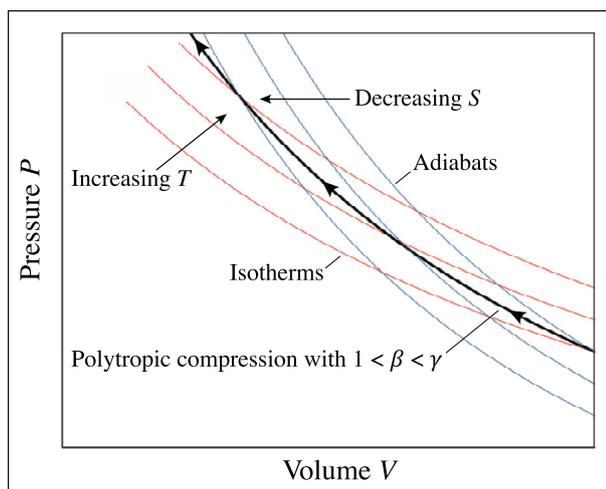


**Fig. 2. A polytropic compression slices across both isotherms and adiabats.**

Figure 2 provides some insight. Consider a polytropic compression with $1 < \beta < \gamma$ that, starting from the right edge of the figure, decreases the volume and increases the pressure. The figure shows three red isotherms (lines of constant temperature) and three blue adiabats (lines of constant entropy). We see that the rising polytrope slices across the isotherms in the direction of increasing temperature; that is, compressing the gas increases its temperature, just as in an adiabatic com-

pression. At the same time, the polytrope slices across the adiabats in the direction of decreasing entropy. Because $dS = \delta Q/T$, a decreasing entropy means that heat is removed. Thus a polytropic compression that falls between an isotherm and an adiabat is a process in which the temperature increases as heat is removed, so the specific heat is negative. For $\beta < 1$ (less steep than an isotherm) or $\beta > \gamma$ (steeper than an adiabat), the entropy and the temperature change in the same direction and the specific heat is positive.

Physically, an adiabatic compression increases the temperature with $Q = 0$ because work is done on the gas. Work is also done on the gas during an isothermal compression, but in that case $\Delta T = 0$ because an equal amount of heat is removed from the gas. The polytropic compression of Fig. 2 is one in which some heat is removed from the gas while work is done to compress it, but not enough to prevent the temperature from increasing.

This might seem interesting but rather arcane, but the polytropic processes of interest to engineers[2,3] are precisely those with $1 < \beta < \gamma$. In fact, these are the only polytropic processes mentioned in various engineering thermodynamics textbooks consulted by the author. In physics, we model a rapid compression as an adiabatic process because, we say, it happens so quickly that there's no time for heat exchange. But a real compression—and the compression and expansion of gases is ubiquitous in industrial applications—is not ideal. There is inevitably some heat loss to the surroundings during the compression as the gas temperature rises above the ambient temperature, so the $pV$ curve is not an adiabat but a slightly less steep polytrope with $\beta < \gamma$. That is, a real compression is a polytropic process with negative specific heat because heat is transferred out of the gas as its temperature increases.

Similarly, any real expansion of a gas is not an ideal adiabatic process because there's some heat flow into the gas from the surroundings as the gas temperature falls. This is the situation described by the polytrope of Fig. 2 when the process runs in the opposite direction as an expansion. The value of the polytropic index has to be determined experimentally for any particular situation.

Interestingly, while engineering thermodynamics texts usually do give the same physical explanation as above—heat is removed as work is done, but not enough to prevent the temperature from increasing—none of the texts consulted by the author mention specific heat or that these processes have negative specific heat.

The idea of a negative specific heat is certainly not new. Fairly recent papers[4,5] have noted that the heat capacity $C = \delta Q/dT$ can be negative during some portions of an ideal-gas process, but these papers did not consider processes for which the specific heat is both negative and constant. In addition, it's long been known that stars have a negative heat capacity in the sense that a star's temperature goes up as it loses energy, and many models of stellar structure are based on a polytropic equation of state $p \propto \rho^\beta$ between pressure and density.[6,7] But it has rarely been called out that some ideal-gas processes, processes important in applications, have a constant negative specific heat throughout the process.

## Thermodynamics with polytropic processes

Standard thermodynamic quantities are easily calculated for polytropic processes. For example, $pV^\beta = p_1V_1{}^\beta = p_2V_2{}^\beta$ can be used to show that the work done on the gas in a polytropic expansion or compression is

$$W = -\int_{V_1}^{V_2} pdV = \frac{p_2V_2 - p_1V_1}{\beta - 1} \quad \beta \neq 1. \tag{8}$$

This agrees with well-known expressions for $W$ in isobaric ($\beta = 0$), adiabatic ($\beta = \gamma$), and isochoric ($\beta = \infty$) processes. Thus polytropes are easily incorporated into heat-engine cycles. One recent paper[8] uses polytropic processes in an alternative cycle for a Stirling engine.

Similarly, Eq. (1) for $\delta Q$ can be used in $dS = \delta Q/T$ to find, by integrating, that the entropy change in a polytropic process is

$$\Delta S = nC_\beta \ln(T_2/T_1). \tag{9}$$

Equation (9) parallels similar expressions[1] for the entropy change during an isochoric or isobaric process, and it confirms that entropy decreases as $T$ increases during a process in which $C_\beta$ is negative.

## An example

We'll conclude with an example that is very much physics rather than engineering. Excluding localized situations near the ground, the temperature of the atmosphere decreases with increasing elevation. The rate of decrease is called the lapse rate $\Gamma$. Parcels of dry air rise and fall adiabatically (to a good approximation), with no heat exchange with their surroundings, so they obey the adiabatic pressure–temperature relationship $p = \text{constant} \times T^{\gamma/(\gamma-1)}$. Each parcel of air also obeys the hydrostatic–equilibrium equation $dp/dz = -g\rho$, where $\rho$ is the density of the air. We can combine these and the ideal-gas law to eliminate $p$ and find that the lapse rate—the temperature gradient—is

$$\Gamma = \frac{dT}{dz} = -\frac{\gamma-1}{\gamma}\frac{Mg}{R}, \tag{10}$$

where $M = 0.029$ kg/mol is the average molar mass of air. Using $\gamma = 1.4$ for the diatomic molecules of the atmosphere gives $\Gamma = -9.8$°C/km as the dry-air lapse rate, a value in good agreement with observations.

However, air is not dry except in extreme desert conditions. When moist air rises, the temperature drops and the air soon saturates at the dew point. Further cooling, as the air continues to rise, causes water vapor to condense, and the condensing water vapor releases the latent heat of vaporization. The parcel of air does not exchange heat with its surroundings, but now it has an internal source of heat and so the temperature of rising moist air does not decrease as rapidly as the temperature of dry air. This is another situation of negative specific heat.

A careful derivation of the moist-air lapse rate is quite complex, but we can model[9] the ascent of a moist-air parcel as a polytropic process rather than as a dry-air adiabatic process. The moist-air lapse rate is typically $\Gamma = -6.5$°C/km, and radio-

sonde measurements show that the rate stays fairly constant to near the top of the troposphere at a height of roughly 10 km. If we replace $\gamma$ in Eq. (10) with $\beta$, for a polytropic process, and use the measured lapse rate, we find that $\beta \approx 1.25$. That is, the rising of a parcel of moist air is well modeled as a polytropic process with $\beta \approx 1.25$ and thus, from Eq. (7), $C_\beta \approx -0.6C_V \approx -12.5$ J/mol·K. The specific heat is negative because the temperature decreases even as heat, from the condensation, is added.

A closely related but more down-to-earth example is the rapid expansion of compressed moist air when a water rocket is launched. One laboratory measurement[10] found that the expansion is well described as a polytropic process with $\beta \approx 1.2$, and an accompanying analysis derives a theoretical expression for $\beta$ in terms of the temperature, the vapor pressure of the water, and the heat of vaporization of water.

In summary, polytropic processes provide many interesting possibilities for expanding instruction in thermodynamics beyond the standard four processes encountered in textbooks.

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