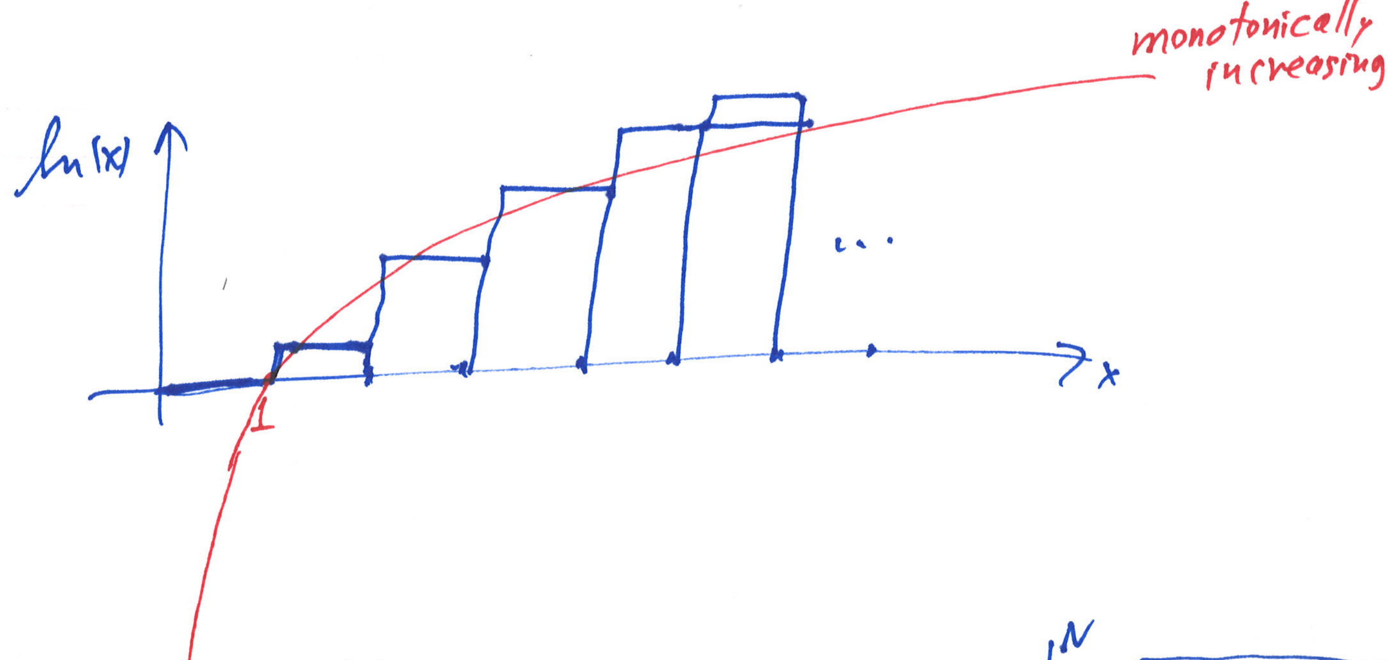
Stirling's Approximation

Need a formula to evaluate $N!$ for large N

e.g. $N = N_A = 10^{23}$, $N_A!$ is very large

$$\ln(N!) = \ln[N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1]$$

$$= \ln(N) + \ln(N-1) + \cdots \ln(2) + \ln(1)$$



① $\ln(N!) \approx \int_{x=0}^{N} \ln(x)\,dx = [x\ln(x) - x]\Big|_0^N = \boxed{N\ln(N) - N}$

$\lim_{x \to 0} x\ln(x) = 0$ check with L'Hôpital

$a\ln(b) = \ln(b^a)$

$\ln(N!) \approx N\ln(N) - N$ exponentiate both sides

$$e^{\ln(N!)} = N! = e^{N\ln(N)-N} = e^{N\ln N} \cdot e^{-N}$$

$$= e^{\ln(N^N)} \cdot e^{-N} = N^N \cdot e^{-N} = \left(\frac{N}{e}\right)^N$$

$$\boxed{N! \approx \left(\frac{N}{e}\right)^N}$$

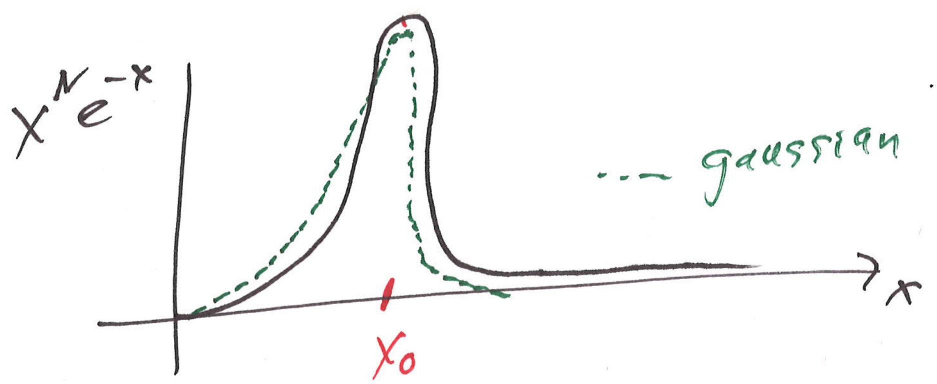
0^{th} approximation usually good enough

First correction – use gaussian approximation to the integral

$$N! = \int_{x=0}^{\infty} \underline{x^N \cdot e^{-x}}\, dx \equiv \Gamma(N+1)$$ ← Gamma function

$0! = 1$, $(-\frac{1}{2})! = \sqrt{\pi} = \Gamma(\frac{1}{2})$

Integrand $x^N e^{-x}$ $\begin{cases} x^N \text{ grows rapidly with } x \\ e^{-x} \text{ shrinks more rapidly with } x \end{cases}$



$$\frac{d}{dx}\left[x^N e^{-x}\right]\Big|_{x=x_0} \stackrel{!}{=} 0 = N x^{N-1} e^{-x} + -x^N e^{-x}\Big|_{x=x_0}$$

$$= e^{-x_0}\, x_0^{N-1}(N - x_0) = 0 \Rightarrow \boxed{x_0 = N}$$

$$\ln\left(x^N e^{-x}\right) = \ln\left(x^N\right) + \ln\left(e^{-x}\right) = N\ln(x) - x$$

change variables $y \equiv x - N$ distance from peak

$$dy = dx$$

$$N\ln(x) - x = N\ln(y+N) - y - N$$

$$= N\ln\left[N\left(1+\frac{y}{N}\right)\right] - y - N$$

$$= N\ln(N) - N + N\ln\left(1+\frac{y}{N}\right) - \underline{y}$$

Near the peak, $x \sim N$, $\frac{y}{N} \ll 1$

Taylor Expansion: $\ln\left(1+\frac{y}{N}\right) = \underline{\frac{y}{N}} - \underline{\frac{1}{2}\left(\frac{y}{N}\right)^2} + \frac{1}{3}\left(\frac{y}{N}\right)^3 + \ldots$

$$N\ln(x) - x = N\ln(N) - N - \frac{1}{2}\frac{y^2}{N} + O\left(\frac{y}{N}\right)^3$$

$$x^N e^{-x} = \exp\left(N\ln(x) - x\right) \approx \exp\left[N\ln(N) - N - \frac{y^2}{2N} + \cdots\right]$$

$$N! = \int_{x=0}^{\infty} x^N e^{-x}\,dx \approx \int_{y=-N}^{\infty} N^N e^{-N} e^{-\frac{y^2}{2N}}\,dy + \cdots$$

Replace $-N$ by $-\infty$

$$N! \approx \int_{-\infty}^{+\infty} N^N e^{-N} e^{-\frac{y^2}{2N}}\,dy = N^N e^{-N} \underbrace{\int_{-\infty}^{+\infty} e^{-\frac{y^2}{2N}}\,dy}_{\substack{\text{Gaussian} \\ \text{integral} \\ = \sqrt{\pi 2N}}}$$

$$N! \approx N^N e^{-N} \sqrt{2\pi N}\left(1 + \frac{1}{12N}\right)$$

Gaussian Integrals

$$I = \int_{x=-\infty}^{+\infty} e^{-x^2}\,dx = \text{number (not a function of } x)$$

$$I^2 = \left(\int_{x=-\infty}^{+\infty} e^{-x^2}\,dx\right)\left(\int_{y=-\infty}^{+\infty} e^{-y^2}\,dy\right)$$

Now consider x + y as Cartesian coordinates.

change variables to polar coordinates

$$I^2 = \int_{x=-\infty}^{+\infty}\int_{y=-\infty}^{+\infty} e^{-(x^2+y^2)}\,dx\,dy$$

$\updownarrow$ Jacobian

$$I^2 = \int_{r=0}^{\infty}\int_{\varphi=0}^{2\pi} e^{-r^2}\,r\,dr\,d\varphi$$

$x = r\cos\varphi$

$y = r\sin\varphi$

$r = \sqrt{x^2+y^2}$

$\varphi = \arctan\left(\frac{y}{x}\right)(+180^\circ)$

maybe

$$= \left(\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\right)\left(\underbrace{\int_{\varphi=0}^{2\pi} d\varphi}_{2\pi}\right)$$

$$I^2 = 2\pi \left[-\frac{e^{-r^2}}{2} \right]_0^\infty = 2\pi \left[0 - \left(-\frac{1}{2}\right) \right]$$

$$\frac{d}{dr}\left(-\frac{e^{-r^2}}{2}\right) = -\frac{1}{2}(-2r) e^{-r^2} = r e^{-r^2}$$

$$I^2 = \pi \quad \Rightarrow \quad I = \sqrt{\pi}$$

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

change variable

$x = \sqrt{a} z$

$x^2 = a z^2$

$dx = \sqrt{a} dz$

$$= \sqrt{a} \int_{-\infty}^{\infty} e^{-a z^2} dz = \sqrt{\pi}$$

Rename $z \to x$

$$\int_{-\infty}^{+\infty} e^{-a x^2} dx = \sqrt{\frac{\pi}{a}} \leftarrow \frac{d}{da} \text{ both sides}$$

$$\int_{-\infty}^{+\infty} (-x^2) e^{-a x^2} dx = \sqrt{\pi} \frac{d}{da}\left(a^{-1/2}\right) = \sqrt{\pi}\left(-\frac{1}{2}\right) a^{-3/2}$$

$$\int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2}\left(\frac{\pi}{a^3}\right)^{1/2}$$

lather, rinse, repeat

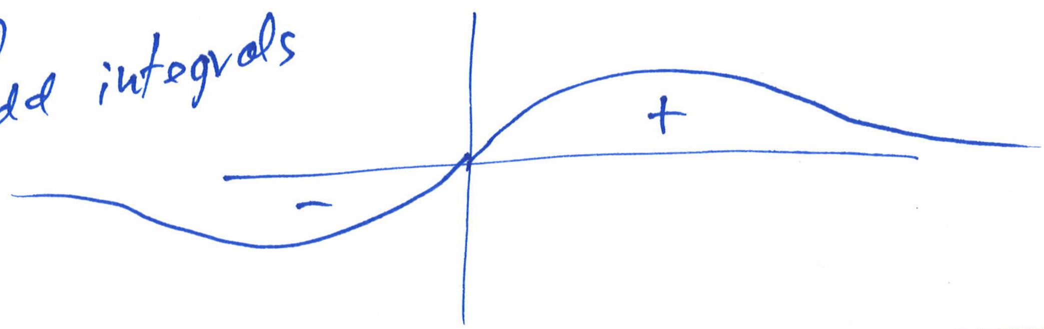
$$\frac{d^n}{da^n}$$

$\vdots$

$$\int_{-\infty}^{+\infty} x^{2n} e^{-ax^2}\,dx = ?$$

$$\int_{-\infty}^{+\infty} x^{2n+1} e^{-ax^2}\,dx = \emptyset$$

odd integrals



$$\int_{0}^{\infty} x e^{-x^2}\,dx = \frac{1}{2}$$

$x \to \sqrt{a}\, z$

$\frac{d^n}{da^n}$ under $\int$ sign.

Hyper area + hyper volumes

$$\underbrace{\int_{x_1=-\infty}^{+\infty} e^{-x_1^2}\,dx_1}_{\sqrt{\pi}} \cdot \underbrace{\int_{x_2=-\infty}^{+\infty} e^{-x_2^2}\,dx_2}_{\sqrt{\pi}} \cdots \int_{x_n=-\infty}^{+\infty} e^{-x_n^2}\,dx_n = \prod_{k=1}^{n} \int_{x_k=-\infty}^{+\infty} e^{-x_k^2}\,dx_k$$

$$= \left(\sqrt{\pi}\right)^n = \pi^{n/2} = \underbrace{\int d\Omega^{n-1}}_{\text{solid angle}} \underbrace{\int_{r=0}^{\infty} e^{-r^2} r^{n-1}\,dr}_{\frac{1}{2}\Gamma\left(\frac{n}{2}\right)}$$

$$r^2 = x_1^2 + x_2^2 + \cdots x_n^2$$

$$\Omega_{n-1} = \text{(n-1)-dimensional solid angle} = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} = \frac{2\pi^{n/2}}{\left(\frac{n}{2}-1\right)!}$$

Surface area of an $(n-1)$-dimensional sphere.

$$S_{n-1} = \Omega_{n-1} R^{n-1} \rightarrow S_m = \Omega_m R^m$$

$$= \frac{2\pi^{\frac{m+1}{2}} R^m}{\left(\frac{m-1}{2}\right)!}$$

n - Sphere = n-dimensional manifold
all points the same distance R from origin

e.g. Soap bubble is 2-sphere.

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n-ball = n-dimensional manifold
all points at or less than distance R from origin.

e.g. Earth is 3-ball

n	S_n	
0	2	 line segment = 1-ball
1	$2\pi R$	 circle = 1-sphere disk = 2-ball
2	$4\pi R^2$	 soap bubble = 2-sphere interior of 2-sphere is 3-ball
3	$2\pi^2 R^3$	

n	V_n	volume of n-ball
0	1	
1	$2R$	 line segment Length
2	πR^2	 disk area
3	$\frac{4}{3}\pi R^3$	 baseball volume
4	$\frac{1}{2}\pi^2 R^4$	hypersphere hypervolume – glome hypervolume

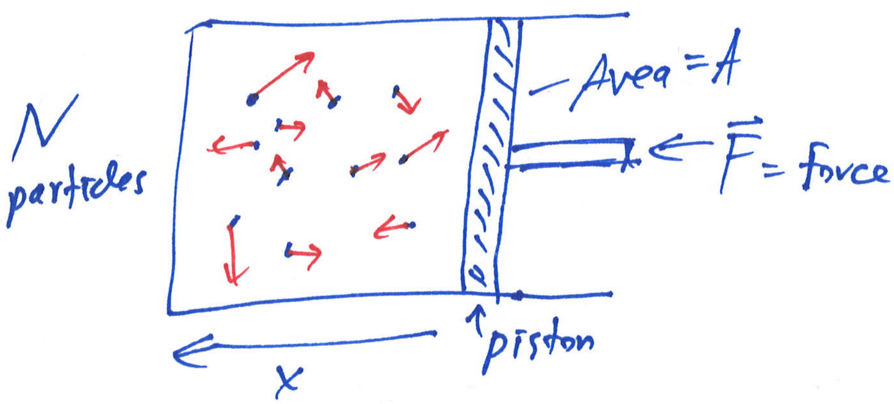
$$V_n = \int_{r=0}^{R} S_{n-1}(r)\,dr$$

$$= \frac{S_{n-1}R}{n} = \frac{2\pi^{n/2}R^n}{n\left(\frac{n}{2}-1\right)!}$$

Kinetic Theory of Gases

Atoms + Newton's Laws + Maxwell's Eqs

Not Quantum Mechanics



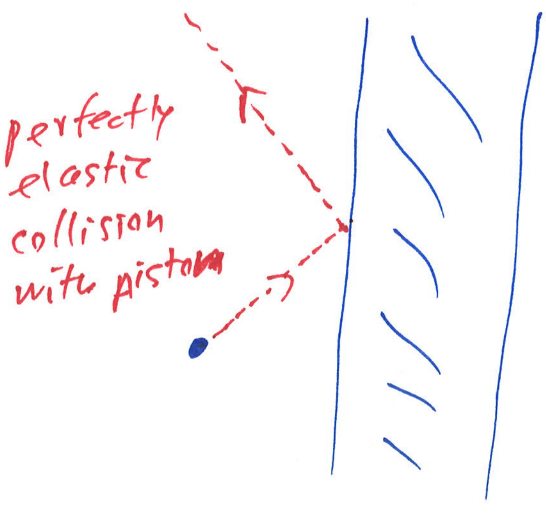
Pressure $P = \frac{F}{A}$

Volume = Ax

Any collision conserves linear Momentum

Elastic collision conserves kinetic energy

$\Rightarrow$ same speed before and after the collision.



$P_y \to P_y$, $P_z \to P_z$, $P_x \to -P_x$

Momentum delivered to piston $= 2P_x$

$= 2mv_x$