The Spin-Statistics Theorem and Identical Particle Distribution Functions

by Dwight E. Neuenschwander

In the restaurants of some countries, the *maître d′* will seat your party at a table already occupied by strangers if the table has sufficient empty chairs. In other cultures, new diners expect to be seated at empty tables even when occupied tables are surrounded by surplus seats. Elementary particles, and composites made of them, fall analogously into two categories: bosons and fermions. Bosons readily share the same state, analogous to the first group of diners. Fermions prefer solitude, like the second group of diners. This article offers a simple argument for their respective distribution functions that describe how a system of identical particles populates the states available to them.

Since the most stable configuration of a system typically finds it in the state of lowest energy, why don't all the electrons in an atom reside in the 1*s* orbital? If they did, there would be no chemistry, and thus no biology and no life—nor would metals conduct electricity, and white dwarf stars would collapse into black holes. The universe would be very different indeed, and we would not be around to appreciate it! The Pauli exclusion principle formally articulates the hypothesis that electrons, as fermions, do not all collapse into an atom's 1*s* orbital; only two of them can live there, and then only with opposite spins, to not be in the same space-spin state.

The exclusion principle grew out of the realization that the discrete quantum states for electrons could not, by themselves, explain the periodic table of the elements. The crucial clue towards a resolution came in 1924, when Edmund Stoner published a paper noting a correlation between the number of electron states of alkali metals placed in a magnetic field and the number of closed-shell electrons in noble gases.[1,2] From this observation Pauli realized the sequence of closed-shell electron numbers 2, 8, 18,…is equivalent to the rule that no state can hold more than one electron. To implement this rule he had to invent an extra quantum number. It was soon identified with electron spin, when Samuel Goudsmit and George Uhlenbeck proposed it in the autumn of 1925.[3] Pauli's spin-statistics theorem of 1940 generalized the argument about spin and statistics.[4] Its flip side, that bosons can coalesce, finds dramatic illustration in superfluidity and superconductivity, variations of Bose-Einstein condensation.

The word "state" carries three contexts in this discussion: macroscopic thermodynamic states, and microscopic states in two varieties—single-particle and multiparticle microstates. The macroscopic state of a bottle of hydrogen gas in thermal equilibrium is characterized by observables such as pressure, temperature, and volume. Microscopically, each atom has a set of available internal states, such as the atomic orbitals with spin, $\psi_{nlm_l m_s}$ that describe the electron relative to the nucleus. For other purposes, as in the kinetic theory of gases, it may be sufficient to model a hydrogen atom as a solid point mass moving with momentum $\mathbf{p}$ and located at position $\mathbf{r}$. Here the atom's microstate consists of six numbers, the momentum and position coordinates in phase space.

Between macrostates and single-particle microstates are states of a system of N particles. For $N = 2$, with one particle in the single-particle state ψ_a and the other in microstate ψ_b, how simple it would be mathematically if the two-particle state was merely the product of the two one-particle wave functions, $\psi = \psi_a\psi_b$. This works fine for systems of distinguishable particles, such as the deuteron—a proton and a neutron bound together.[5] But the plot thickens when the particles are indistinguishable. If $\psi = \psi_a\psi_b$ was the whole story for identical particles, the universe would be rather sterile. Simplicity is beautiful, but complexity can be essential.

Statistical mechanics aims to understand macroscopic states in terms of microstates. The next section reviews how this game is played.

Statistical Mechanics Review

Consider a system of N molecules, each in one of the many possible microstates available to it ("molecule" here is a generic term which could also denote a nucleus or a neutrino, depending on the problem). Label the various single-particle microstates as state a, state b, and so on. Let n_i denote the number of molecules in state i. If I were to reach into a box of these molecules and pick one at random, the probability P_i of selecting one in state i is n_i/N. Since every molecule must be in one microstate or another,

$$\sum_i P_i = \sum_i \frac{n_i}{N} = 1. \qquad (1)$$

If E_i denotes a molecule's energy when in state i, and U the total energy carried by all the molecules (the internal energy of thermodynamics), then

$$U = \sum_i n_i E_i = N \sum_i E_i P_i = N\langle E\rangle; \qquad (2)$$

$\langle E\rangle$ denotes the average molecular energy.

A primary responsibility of statistical mechanics expresses P_i in terms of E_i and the ambient thermal energy kT in which the molecules find themselves, when in thermal equilibrium with their surroundings at temperature T. Boltzmann's constant $k = 1.381\times10^{-23}$ J/K forms a conversion factor between temperature and energy. A derivation of P_i as a function of T can be approached at least two ways: through the "method of most probable distribution,"[6] or with phase space arguments based on Liouville's theorem.[7] One finds

$$P_i = \frac{1}{Z}\, e^{-E_i/kT}, \qquad (3)$$

where the normalization factor Z, determined by Eq. (1), is the "partition function,"

$$Z = \sum_i e^{-E_i/kT}. \qquad (4)$$

In terms of Z, the computation of U can be expressed succinctly. From Eqs. (2) and (3),

$$U = \frac{N}{Z}\sum_i E_i\, e^{-E_i\beta}, \qquad (5)$$

where $\beta = 1/kT$. Thanks to a nice property of the exponential,

$a\, e^{ax} = \frac{d}{dx} e^{ax}$, Eq. (5) becomes

$$U = -N\frac{\partial}{\partial\beta}\ln Z. \qquad (6)$$

The partition function offers a window from the macroscopic world of thermometers and pressure gauges into the microscopic world of atomic structures and interactions. For example, consider a system of N identical switches, each one either off (with energy $E_1 = 0$) or on (with energy $E_2 = \varepsilon$), so that $Z = 1 + e^{-\varepsilon\beta}$. To see if a real macroscopic system can be well modeled by such a set of switches, we could compare the measured heat capacity $C = dU/dT$ to the prediction of it calculated from Z.

Indistinguishable Particles in Quantum Mechanics

Now let's enlarge our perspective and speak of the states of an N-particle system. As before, each individual particle will be in a single-particle microstate of energy E_i. A possible N-particle state is specified by a list

$$\tilde{n} = (n_1, n_2, n_3, \ldots), \qquad (7)$$

where $\sum_i n_i = N$. Each $\tilde{n}$ labels one possible state of the N-particle system. The energy $E_{\tilde{n}}$ of state $\tilde{n}$ is the sum

$$E_{\tilde{n}} = \sum_i n_i E_i. \qquad (8)$$

In this context the partition function of Eq. (4) becomes the sum over the allowed $\tilde{n}$:

$$Z_N = \sum_{\tilde{n}} e^{-E_{\tilde{n}}\beta} = \sum_{\tilde{n}} e^{-\beta\sum_i n_i E_i}. \qquad (9)$$

For example, suppose $N = 2$ and each particle can be in one of three one-particle microstates. The possible states of the two-particle system are:

$\tilde{1} = (1,1,0)$ $\tilde{2} = (1, 0, 1)$ $\tilde{3} = (0, 1, 1)$

$\tilde{4} = (2, 0, 0)$ $\tilde{5} = (0, 2, 0)$ $\tilde{6} = (0, 0, 2)$

with respective energies

$E_{\tilde{1}} = E_1 + E_2$ $E_{\tilde{2}} = E_1 + E_3$ $E_{\tilde{3}} = E_2 + E_3$

$E_{\tilde{4}} = 2E_1$ $E_{\tilde{5}} = 2E_2$ $E_{\tilde{6}} = 2E_3$.

From Eq. (9) the partition function reads

$$Z_2 = e^{-\beta E_{\tilde{1}}} + e^{-\beta E_{\tilde{2}}} + \cdots + e^{-\beta E_{\tilde{6}}}$$

$$= e^{-\beta E_1}e^{-\beta E_2} + e^{-\beta E_1}e^{-\beta E_3} + \cdots + e^{-\beta 2E_3}$$

$$= x_1x_2 + x_1x_3 + x_2x_3 + x_1^2 + x_2^2 + x_3^2, \qquad (10)$$

where

$$x_i = e^{-\beta E_i}. \qquad (11)$$

With N held fixed some $\tilde{n}$ do not occur, as in our illustration where $N = 2$ excludes (0,0,0) and (1,1,1). To work with fixed N means the set of all allowed $\tilde{n}$ must be known before Z_N can be evaluated. That is feasible when $N = 2$, but statistical mechanics deals with systems that contain on the order of 10^{23} particles. However, if N were *not* fixed, then each n_i in every $\tilde{n}$ could range over 0, 1, 2,…, N_{max}. To determine N_{max} for identical particles, the spin-statistics theorem steps in. Such generality makes Z summable.

"Identical" here means that no method exists *in principle* to distinguish one particle from another. This raises concern about possibly double-counting microscopic states. If particles 1 and 2 in a two-particle system exchange places in their one-particle microstates, should that swapped configuration be counted as the same, or as distinct, from the original? According to the rules of quantum mechanics,[8] one sums over both possibilities, which interfere with one another through the cross terms in $|\psi|^2 = |\psi_{\text{original}} + \psi_{\text{exchanged}}|^2$. The $N = 2$ wave function therefore has two options for including one particle in state a and an identical one in state b:

$$\psi_{ab}(1,2) = c_{\text{orig}}\, \psi_a(1)\psi_b(2) + c_{\text{exch}}\, \psi_a(2)\psi_b(1). \qquad (12)$$

If these exhaust the possibilities for arranging the two particles, then $|c_{\text{orig}}|^2 + |c_{\text{exch}}|^2 = 1$. But $|c_{\text{orig}}| = |c_{\text{exch}}|$ because the particles are identical. Thus $c_{\text{exch}} = c_{\text{orig}}\, e^{i\delta}$, where $i^2 = -1$ with δ a real number. Equation (12) then becomes

$$\psi_{ab}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) + e^{i\delta}\, \psi_a(2)\psi_b(1)]. \qquad (13)$$

Now introduce the exchange operator $\mathcal{E}$ that interchanges particles 1 and 2:

$$\mathcal{E}\psi_{ab}(1,2) = \psi_{ab}(2,1). \qquad (14)$$

If the interparticle potential is invariant under the exchange, then the exchange operator commutes with the Hamiltonian, and $\psi_{ab}(1,2)$ is an eigenstate of both $\mathcal{E}$ and the Hamiltonian. Besides the energy eigenvalue of the latter, there also exists an eigenvalue λ of $\mathcal{E}$, which means that under the operation of $\mathcal{E}$ the eigenstate is merely rescaled but otherwise unchanged:

$$\mathcal{E}\, \psi_{ab}(1,2) = \lambda\, \psi_{ab}(1,2). \qquad (15)$$

Therefore

$$\mathcal{E}^2\, \psi_{ab}(1,2) = \lambda^2\, \psi_{ab}(1,2). \qquad (16)$$

But in addition,

$$\mathcal{E}^2\, \psi_{ab}(1,2) = \mathcal{E}\, \psi_{ab}(2,1) = \psi_{ab}(1,2) \qquad (17)$$

and thus $\lambda^2 = 1$, so that $\lambda = \pm 1$. Operating with $\mathcal{E}$ on the $\psi_{ab}(1,2)$ of Eq. (13) gives

$$\mathcal{E}\, \psi_{ab}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(2)\psi_b(1) + e^{i\delta}\, \psi_a(1)\psi_b(2)]. \qquad (18)$$

By virtue of Eq. (15), this equals

$$\mathcal{E}\, \psi_{ab}(1,2) = \pm\frac{1}{\sqrt{2}}\, [\psi_a(1)\psi_b(2) + e^{i\delta}\, \psi_a(2)\psi_b(1)]. \qquad (19)$$

Comparing Eqs. (18) and (19) shows that $e^{i\delta} = \pm 1$. Therefore, Eq. (13) becomes

$$\psi_{ab}(1,2) = \frac{1}{\sqrt{2}}[\psi_a(1)\psi_b(2) \pm \psi_a(2)\psi_b(1)]. \qquad (20)$$

What property of the particles determines which sign applies? One may define an elementary particle as a state of definite mass and spin. The spin **S**, the intrinsic angular momentum carried by the particle, becomes quantized in units of the reduced Planck's constant $\hbar$. In particular, to say a particle carries spin s means that its spin vector squared has magnitude $\hbar^2 s(s+1)$. One component of **S** can also be simultaneously measured, such as $S_z = m_s\hbar$, where m_s takes on the $2s+1$ possible values $s, s-1, \ldots, -s$. The quantum number s can be one of the possible values $0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$ Particles with integer s are bosons, and particles that carry half-odd-integer s are fermions.

Pauli's 1940 proof of the spin-statistics theorem[4] makes consistent with relativistic quantum field theory the assertion that identical fermions use the minus sign in Eq. (20), and identical bosons use the plus sign. The Pauli exclusion principle emerges as a consequence: If two fermions are identical, they cannot be in the same state because, by Eq. (20), $\psi_{aa}(1,2) = 0$. But for identical bosons, the total wave function is enhanced: $|\psi_{aa}(1,2)|^2 = 2|\psi_a|^2$. Consequently, calculations of the root-mean-square distance separating identical bosons or fermions leads to the (misnomered) "exchange forces," whereby identical bosons congregate closer together than do distinguishable particles, and identical fermions are farther apart.[9]

Although the result of the spin-statistics theorem is simple to state, connecting it to deeper principles is not simple. In the *Feynman Lectures,* Richard Feynman remarked,

> *Why is it that particles with half-integral spin are Fermi particles whose amplitudes add with the minus sign, whereas particles with integer spin are Bose particles whose amplitudes add with the plus sign? We apologize for the fact that we cannot give you an elementary explanation. An explanation has been worked out by Pauli from complicated arguments of quantum field theory and relativity. He has shown that the two must necessarily go together, but we have not been able to find a way of reproducing his arguments on an elementary level…This probably means that we do not have a complete understanding of the fundamental principle involved…"*[10]

Feynman's interpretation of what it means to "understand" a point of physics is extremely suggestive. In a 1994 contribution to a "Question and Answer" column in the *American Journal of Physics,* this Feynman quote was recalled, then followed with the question "Has anyone made any progress towards an 'elementary' argument for the spin-statistics theorem?"[11] This question generated a lively discussion over the next three years,[12] culminating in the book *Pauli and the Spin-Statistics Theorem* by George Sudarshan and Ian Duck. They wrote,

> "*Everyone knows the spin-statistics theorem but no one understands it. The key word of course is 'understand.'…The question is whether physics contains this fact, and if so how does this come about; or whether physics is merely consistent with the spin-statistics theorem and that some deeper explanation exists…*[13]

Whether or not we "understand" the spin-statistics theorem, we can nevertheless apply it to systems of N identical bosons or fermions.

Distributions of Identical Bosons or Fermions

Returning to Z_N, we now allow every n_i to range from 0 to $N_{\max}$ in every $\tilde{n}$. According to the spin-statistics theorem, $N_{\max} = 1$ for identical fermions, and $N_{\max} = \infty$ for identical bosons. In our previous example of particles each having three available microstates, as identical bosons the states of the multiparticle system exhibit these options, grouped by the value of N:

$$\{\tilde{n}\} = \{[\,(0,0,0)\,]_{N=0},$$

$$[\,(1,0,0), (0,1,0), (0,0,1)\,]_{N=1},$$

$$[\,(2,0,0), (0,2,0), (0,0,2), (1,1,0), (1,0,1), (0,1,1)]_{N=2},$$

$$[(3,0,0), (0,3,0), (0,0,3), (2,1,0), (2,0,1), (1,2,0), (1,0,2), (0,1,2), (0,2,1), (1,1,1)]_{N=3}, \ldots\}$$

continuing thus as $N \to \infty$; for instance, the $N = 4$ group includes (4,0,0), (1,0,3), (2,1,1), (2,2,0) and so on. The partition function for this system of identical bosons with variable N becomes

$$Z_{\text{bosons}} = [1]_{N=0} + [x_1 + x_2 + x_3]_{N=1}$$

$$+ [x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_1x_3]_{N=2}$$

$$+ [x_1^3 + x_2^3 + x_3^3 + x_1^2x_2 + x_1^2x_3 +$$

$$x_1x_2^2 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 + x_1x_2x_3\,]_{N=3} + \ldots$$

$$= (1 + x_1 + x_1^2 + \ldots)(1 + x_2 + x_2^2 + \cdots)(1 + x_3 + x_3^2 + \ldots). \tag{21}$$

Each factor is a geometric series,

$$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}, \tag{22}$$

which converges for $|x| < 1$, and thus

$$Z_{\text{bosons}} = \prod_{i=1}^{3}(1 - x_i)^{-1}. \tag{23}$$

For identical fermions, in this instance where each one has three possible microstates, the list of N-particle states is much shorter:

$$\{\tilde{n}\} = \{[\,(0,0,0)\,]_{N=0},$$

$$[\,(1,0,0), (0,1,0), (0,0,1)\,]_{N=1},$$

$$[\,(1,1,0), (1,0,1), (0,1,1)]_{N=2},$$

$$[(1,1,1)]_{N=3}\},$$

which yields

$$Z_{\text{fermions}} = 1 + x_1 + x_2 + x_3$$

$$+ x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3$$

$$= \prod_{i=1}^{3}(1 + x_i). \tag{24}$$

More generally, with M one-particle microstates available to each member of an ensemble of identical particles, the partition function reads

$$Z = \prod_{i=1}^{M}(1 \pm x_i)^{\pm 1}, \tag{25}$$

where the plus sign in Z holds for fermions and the minus sign for bosons.

In allowing the possibility of an open system of variable N, the E_i are shifted. The first law of thermodynamics says that the internal energy of a closed system increases with the addition of heat and/or the performance of work; in the sign conventions typically used by physicists, we write $dU = \text{đ}Q - \text{đ}W$. But allowing particles to enter or leave the system can also change the internal energy. For instance, the new particles might undergo exothermic chemical reactions with the original ones. In such circumstances dU picks up an addtional term μdN,

$$= TdS - PdV + \mu dN, \tag{26}$$

where dN is the particle number increment (modeled as continuous since N is typically huge) and μ denotes the chemical potential,[14] the increase in internal energy per particle when particles are added to the system. The effect of μ can be starkly seen in processes of constant entropy and volume. In a closed system $(\Delta U)_{S,V} = 0$, so that by Eq. (2),

$$0 = \Delta(N\langle E\rangle) = N\Delta(\textstyle\sum_i E_i P_i). \tag{27}$$

In contrast, in an open system, $(\Delta U)_{S,V} = \mu\Delta N$, which with the help of Eqs. (1), (2), and (26) says

$$0 = \Delta(N\langle E\rangle - \mu N)$$

$$= N\,\Delta[\,\textstyle\sum_i (E_i - \mu)P_i\,]\,. \tag{28}$$

Thus the x_i in Eq. (11) and thereafter get replaced with

$$x_i = e^{-\beta(E_i - \mu)}. \tag{29}$$

To calculate the average number of molecules $\langle n_i\rangle$ found in state i, sum over the multiparticle states $\{\tilde{n}\}$, each with probability $P_{\tilde{n}} = \frac{1}{Z}e^{-\beta E_{\tilde{n}}}$:

$$\langle n_i\rangle = \sum_{\tilde{n}} n_i\, P_{\tilde{n}}$$

$$= \frac{1}{Z}\textstyle\sum_{\tilde{n}} n_i\, e^{-\beta\sum_j n_j(E_j - \mu)}$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial E_i}\ln Z, \tag{30}$$

where $\partial E_j/\partial E_i = 1$ for $i = j$ and 0 for $i \neq j$. For indistinguishable fermions, Eq. (29) becomes

$$\langle n_i\rangle_F = \frac{g_i}{e^{\beta(E_i - \mu)} + 1}, \tag{31}$$

and for indistinguishable bosons,

$$\langle n_i\rangle_B = \frac{g_i}{e^{\beta(E_i - \mu)} - 1}\,. \tag{32}$$

Since the partition function is a sum over *states* and not merely a sum over *energies,* these distribution functions on the right-hand sides of Eqs. (31) and (32) also pick up a factor g_i for any polarization or spin multiplicities.

Although N was allowed to vary in order to derive these distributions, in applications they hold whether or not N is varied. Let us illustrate their use in Bose-Einstein condensation (BEC).

Its story begins in 1924 when Satyendra Bose sent to Albert Einstein a paper about counting the quantum states of photons. Bose had trouble getting his paper accepted, but Einstein was impressed, translated the paper from English into German, and submitted it on Bose's behalf to the *Zeitschrift für Physik*, where it was published under Bose's name.[15] Then Einstein extended Bose's arguments to massive particles such as atoms. He showed that if atoms with integer spin were cooled to very low temperatures, a critical temperature exists below which the atoms coalesce into their state of lowest energy, forming a macroscopic object.

Picking up this idea, in 1938 Fritz London suggested that such condensates explained the superfluidity of helium-4.[16] Helium was first liquefied by Kamerlingh Onnes in 1908, when he allowed it to reach its boiling point at 4.2 K. Below 2.17 K, a discontinuity in the heat capacity occurs, the density drops, and some of the liquid becomes a "superfluid" with zero viscocity. It will creep through microscopic holes in vessels thought to be leak proof, and climb up a vertical tube or wall (Fig. 1).

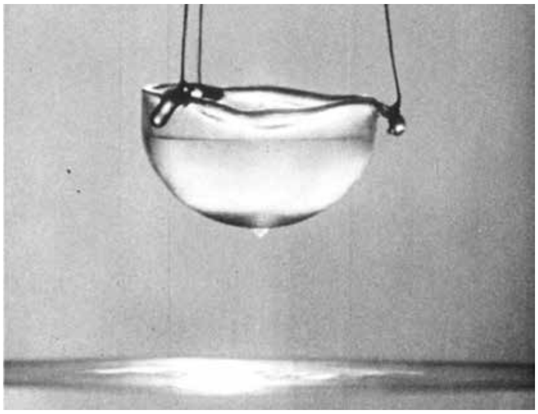

Fig. 1. Liguid helium creeping up the sides of the cup, going over the rim, and dropping outside. Photo by Alfred Leitner (1963), public domain.

The number density of atoms needs to be high, and the temperature low, so the de Broglie wavelengths are long and overlap. To make Bose-Einstein statistics relevant, the atoms must be identical, a purity difficult to achieve when starting from macroscopic samples. Under these stringent conditions, the overlapping wave functions of the individual helium-4 atoms coalesce into a coherent state with macroscopic effects.

Superconductivity, first demonstrated by Onnes in 1911 with mercury, occurs below a critical temperature (4.1 K for Hg), when the metal loses all electrical resistance. The mechanism for such type I superconductivity in pure metals (distinct from type II high-temperature superconductivity in alloys) was not understood until the mid-1950s, with the so-called BCS theory of John Bardeen, Leon Cooper, and Robert Schreiffer. The crucial insight was that electrons, which are fermions, can pair up through interactions with the metal's lattice. The negatively charged electrons repel each other but are attracted to the positive charges in the lattice. Two electrons can thereby interact acoustically through lattice vibrations, even though they are far apart compared to the lattice spacing, quenching their Coulomb repulsion. These "Cooper pairs" carry integer spin, are subject to boson statistics, and can condense into a composite state whose energy leaves a gap of some 10^{-3} eV below the excited states. When kT is less than the gap, collisions are minimized, resulting in no resistance.[17]

Because photons are bosons, we might wonder if macroscopic light waves are some sort of collective state. Coherent radiation produced by a laser, where so many photons have the same phase, polarization, and frequency, suggests a candidate occurrence, at room temperature.

We can recreate the BEC prediction by starting with Eq. (32), assuming a gas of nonrelativistic, noninteracting bosons, each of mass m and carrying kinetic energy $E = p^2/2m$. The microstate i means $(\mathbf{p},\mathbf{r})$, a state of specific momentum and location. The number of bosons within volume dV and having momentum between p and $p+dp$ is therefore

$$dn = \langle n_{(\mathbf{p},\mathbf{r})}\rangle_B \, 4\pi p^2 dp \, dV/h^3, \tag{33}$$

where h is Planck's constant, here taking the role of a phase-space pixel to make the integration measure a dimensionless number for counting particles. I am supposing the momentum vectors point randomly in all directions, and assume spin 0 so that $g_i = 1$ (otherwise a g gets carried along). Equation (32) becomes

$$\langle n_{(\mathbf{p},\mathbf{r})}\rangle_B = \frac{w\, e^{-\beta p^2/2m}}{1-we^{-\beta p^2/2m}}, \tag{34}$$

where $w = e^{\beta\mu}$. Our objective is to calculate, as a function of temperature, the number of bosons n_0 residing in the lowest-energy microstate. That ground state has $p = 0$, which from Eq. (34) gives the constraint $n_0 = w/(1-w)$, implying that $0 \le w \le 1$ since n_0 can range from 0 to ∞. Thus we complete Eq. (33) by counting bosons in the zero-momentum ground state:

$$N = n_o + \frac{4\pi V}{h^3}\int_0^\infty \frac{we^{-\beta p^2/2m}}{1-we^{-\beta p^2/2m}}\, p^2 dp. \tag{35}$$

With the change of variable $y^2 = p^2/2m$, and using the geometric series of Eq. (22) to integrate term by term, we find that

$$N = n_o + \frac{V\Gamma(w)}{\lambda^3}, \tag{36}$$

where a temperature-dependent correlation length λ emerges,

$$\lambda = \left(\frac{h^2}{2\pi mkT}\right)^{\frac{1}{2}} \tag{37}$$

and

$$\Gamma(w) = \sum_{j=1}^{\infty}\frac{w^j}{j^{3/2}}. \tag{38}$$

These results turn Eq. (36) into

$$\frac{n_o}{V} = \frac{N}{V} - \frac{\Gamma(w)}{\lambda^3}, \tag{39}$$

which will be >0 provided $N/V > \Gamma(w)/\lambda^3$. The maximum value of $\Gamma(w)$ occurs at $w = 1$, where $\Gamma(1) = \zeta(3/2) \approx 2.612$ offers an instance of the Riemann zeta function. If $N/V > \zeta(3/2)/\lambda^3$ then $n_0 > 0$. The critical number density occurs when $N/V = \zeta(3/2)/\lambda^3$, giving a critical temperature

$$T_c = \frac{h^2}{2\pi mk}\left(\frac{N/V}{2.612}\right)^{2/3}, \qquad (40)$$

so that, for $T < T_c$, Eq. (39) may be written

$$\frac{n_o}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}, \qquad (41)$$

with $n_o/N = 0$ for $T > T_c$ (an example of a so-called second-order phase transition).

The first demonstration of BEC with non-helium atoms occurred in 1995 when a group at the University of Colorado and NIST-JILA led by Eric Cornell and Carl Wieman made a condensate with rubidium-87 atoms cooled to 170 nK. A few weeks later, Wolfgang Ketterle's group at MIT produced a condensate with sodium-23 atoms. For this achievement, Cornell, Wieman, and Ketterle shared the 2001 Nobel Prize in Physics.[18]

What about laser light as a collective effect of massless spin-1 bosons? In November 2012 the first BEC in photons was demonstrated.[19]

In the logic of physics, what are we to make of the spin-statistics theorem itself? The *statement* of the spin-statistics theorem is analogous, in its relation to the rest of physics, to the Planck-Einstein postulate $E = h\nu$; easy to state but (so far) not derived from any deeper principle. The "proof of the spin-statistics theorem" is not analogous to the demonstration that the work-energy theorem follows from $\mathbf{F} = m\mathbf{a}$; rather, the spin-statistics proof shows consistency between the rest of known physics and the hypothesis of connecting spin to the plus/minus sign of Eq. (20). Sudarshan and Duck observed:

> *"The spin-statistics theorem could conceivably be an essential ingredient of a more fundamental view of the world…*
>
> *With such a point of view forced upon us, we should modify the meaning of 'understand,' and at the same time reduce our expectations of any proof the spin-statistics theorem. What is proved…is that the existing theory is consistent with the spin-statistics relation. What is not demonstrated is a reason for the spin-statistics relation…"*
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> *…It is difficult to imagine a fundamental mechanism for the Pauli exclusion principle—upon which all depends—which predicates it…Must we reduce our demands on physics to require only consistency[?] Does an understanding of the 'Why?' of the spin-statistics relation have no direct answer in physics? Or must physics be formulated to include it[?] The Pauli result does not explain the spin-statistics relation and cannot. [Those who seek an elementary explanation] must remain unsatisfied because the consistency of relativistic quantum mechanics and quantum field theory with the Pauli exclusion principle has every reason to be as complicated as these subjects are, not as simple and direct as the Pauli exclusion principle itself."*[20]

For the foreseeable future we will have to leave it there!

Acknowledgments

Deep thanks to Professor George Sudarshan for his inspiration and kindness. I am grateful to Toni Sauncy for reading a draft of this manuscript and offering insightful suggestions.

References

[1] E.C.G. Sudarshan and I. Duck, *Pauli and the Spin-Statistics Theorem* (World Scientific, Singapore, 1997), undoubtedly offers the most comprehensive survey anywhere of spin, the exclusion principle, and the spin-statistics theorem. It includes all the important original papers, with discussion. See also R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That* (W.A. Benjamin, New York, NY, 1964).

[2] E.C. Stoner, "The distribution of electrons among atomic levels," *Philos. Mag.* **48**, 719 (1924); Sudarshan and Duck, Ref. 1, pp. 21-22.

[3] Wolfgang Pauli, "About the Relationship of the Closing of Electron Shells in Atoms with the Complex Structure of the Spectrum," *Zeit. f. Physik* **31**, 765 (1925); Sudarshan and Duck, Ref. 1, Ch. 1. Electron spin was postulated by G.E. Uhlenbeck and S.A. Goudsmit in *Naturwissenschaften* **47**, 953 (1925), a few months after Pauli's "two-valued quantum degree of freedom."

[4] Wolfgang Pauli, "The Connection between Spin and Statistics," *Phys. Rev.* **58**, 717 (1940); Sudarshan and Duck, Ref. 1, p. 345.

[5] An important example of quantum mechanics applied to distinguishable particles may be found in the Oppenheimer-Phillips effect concerning the deuteron: J. Robert Oppenheimer and Melba N. Phillips, "Note on the Transmutation Function for Deuterons," *Phys. Rev.* **48**, 500 (1935).

[6] Erwin Schrödinger, *Statistical Mechanics* (Cambridge Univ. Press, 1946, 1967), Ch. II.

[7] Kerson Huang, *Statistical Mechanics* (Wiley, New York, NY, 1963), Chs. 7-9.

[8] R. P. Feynman, R.B. Leighton, and M. Sands, *The Feynman Lectures on Physics,* Vol. III (Addison-Wesley, Reading, MA, 1963), Ch. 3.

[9] David Griffiths, *Introduction to Quantum Mechanics,* 2nd ed. (Pearson-Prentice Hall, Upper Saddle River, NJ, 2005), pp. 207-208. One might wonder how atoms bind into molecules through covalent bonds. It is the entire spin-space state that must be antisymmetric for the shared electrons.

[10] Feynman *et al.*, Ref. 8, Ch. 4, Sec. 1.

[11] D.E. Neuenschwander, in the "Question and Answers" column, "Question # 7: The spin-statistics theorem," *Am. J. Phys.* **62**(11), 972 (1994).

[12] "Answer to Question # 7, 'The spin-statistics theorem:'" Roy Gould, *Am. J. Phys.* **63** (2), 109 (1995); Henri Bacry, *Am. J. Phys.* **63**(4), 297-298 (1995); Robert Hilborn, *Am. J. Phys.* **63**(4), 298-299 (1995); Thomas von Foerster, *Am. J. Phys.* **64**(5), 526 (1996); E.C.G. Sudarshan and I. Duck, "Toward an understanding of the spin-statistics theorem," *Am. J. Phys.* **66**(94), 284-303 (1998). See also R.T. Deck and J.D. Walker, "The Connection between Spin and Statistics," *Physica Scripta* **63**, 7-14 (2001); J.A. Morgan, "Spin and statistics in classical mechanics," *Am. J. Phys.* **72**(11), 1408-1417 (2004); Luis Boya and E.C.G. Sudarshan, "The Spin-Statistics Theorem in Arbitrary Dimensions," arXiv:0711.1111v1[hep-th] 7 Nov 2007.

[13] Sudarshan and Duck, Ref. 1, pp. 3-4.

[14] Ashley Carter, *Classical and Statistical Thermodynamics* (Prentice Hall, Upper Saddle River, NJ, 2001), Ch. 9.

[15] Ronald Clark, *Einstein: The Life and Times* (Avon Books, New York, NY, 1972), pp. 408–409.

[16] Fritz London, "The λ-Phenomenon of Liquid Helium and the Bose–Einstein Degeneracy" *Nature* **141** (3571): 643–644 (1938); *Superfluids* Vols. I and II (reprinted by Dover, New York, NY, 1964).

[17] http://hyperphsics.phy-ast.gsu.edu/hbase/solids/coop.html#c; I.J.R. Aitchison and A.G.G. Hey, *Gauge Theories in Particle Physics* (Adam Hilger, Bristol, UK, 1982), Ch. 9.

[18] Barbara G. Levi, "Cornell, Ketterly, and Wieman Share Nobel Prize for Bose-Einstein Condensates," *Physics Today* **54**, 12, 14 (2001); "New State of Matter Seen Near Absolute Zero," NIST, http://physics.nist.gov/News/Update/950742.html.

[19] Jan Klaers, Julian Schmitt, Frank Vewinger, and Martin Weitz, "Bose–Einstein condensation of photons in an optical microcavity," *Nature* **468**(7323): 545–548(2010).

[20] Sudarshan and Duck, Ref. 1, pp. 4-5.