10-2 THE INTRINSIC MAGNETIC MOMENT OF SPIN 1/2 PARTICLES

The existence of spin has immediate consequences for physical systems. We shall see later that the spin of the electron alters the Hamiltonian for the hydrogen atom (and by extension, for other atoms). The reason is that the electron has an intrinsic magnetic dipole moment by virtue of its spin. Under certain circumstances it is possible to treat the electron spin as the only degree of freedom that an electron possesses: This happens when an electron is localized inside a crystal lattice. When this is the case, the coupling of the magnetic moment to an externally imposed magnetic field has consequences that will be explored in this and the next section.

The magnetic moment is[1]

$$\mathbf{M} = -\frac{eg}{2m_e}\mathbf{S} \tag{10-21}$$

Where g, the gyromagnetic ratio, is very close to 2.[2] For such a localized electron, the Hamiltonian in the presence of a magnetic field **B** is just the potential energy

$$H = -\mathbf{M}\cdot\mathbf{B} = \frac{eg\hbar}{4m_e}\boldsymbol{\sigma}\cdot\mathbf{B} \tag{10-22}$$

The Schrödinger equation for the state $\psi(t) = \begin{pmatrix}\alpha_+(t)\\ \alpha_-(t)\end{pmatrix}$ is

$$i\hbar\frac{d}{dt}\psi(t) = \frac{eg\hbar}{4m_e}\boldsymbol{\sigma}\cdot\mathbf{B}\psi(t) \tag{10-23}$$

If **B** is taken to define the z-axis, and if we write $\psi(t) = e^{-i\omega t}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}$, then the equation becomes

$$\hbar\omega\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix} = \frac{eg\hbar B}{4m_e}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha_+\\ \alpha_-\end{pmatrix}$$

One eigenvalue is $\omega_0 = egB/4m_e$ and the corresponding solution is $\begin{pmatrix}1\\ 0\end{pmatrix}$. The other eigenvalue is $\omega_0 = -egB/4m_e$ and the eigenvector is $\begin{pmatrix}0\\ 1\end{pmatrix}$. If the initial state is

$$\psi(0) = \begin{pmatrix}a\\ b\end{pmatrix} \tag{10-24}$$

then the state at a later time will be

$$\psi(t) = \begin{pmatrix}ae^{-i\omega_0 t}\\ be^{i\omega_0 t}\end{pmatrix} \tag{10-25}$$

---

[1] A classical charge $-e$ moving in a circle with angular momentum **L** will form a current loop whose magnetic moment is $\mathbf{M} = -e\mathbf{L}/2m$. Since spin has no "classical limit," the justification for (10-21) has to be found elsewhere. The relativistic Dirac equation yields this result, as well as the value $g = 2$.

[2] The value of $g = 2(1.0011596\ ...)$ is one of the most accurately measured (and calculated) numbers in all of physics. The theoretical and experimental values agree to the last decimal place, with the only theoretical uncertainties coming from the still poorly understood short-distance properties of the fundamental particles.

Suppose that at time $t = 0$ the spin is in an eigenstate of S_x with eigenvalue $\hbar/2$, so that it "points in the x-direction." This means that

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

that is,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{10-26}$$

At a later time,

$$\psi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} \tag{10-27}$$

At the time t,

$$\langle S_x \rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(e^{i\omega_0 t} \; e^{-i\omega_0 t})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{2}\cos 2\omega_0 t \tag{10-28}$$

Similarly,

$$\langle S_y \rangle = \frac{\hbar}{2}\frac{1}{\sqrt{2}}(e^{i\omega_0 t} \; e^{-i\omega_0 t})\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega_0 t} \\ e^{i\omega_0 t} \end{pmatrix} = \frac{\hbar}{2}\sin 2\omega_0 t \tag{10-29}$$

Thus the spin precesses about the direction of B with a frequency

$$2\omega_0 = \frac{egB}{2m_e} \equiv g\omega_c \tag{10-30}$$

$\omega_c = eB/2m_e$ is called the *cyclotron frequency*. The same frequency of precession occurs if the spin makes an angle θ with the z-axis, rather than lying in the x-plane. For a magnetic field of 1T, the magnitude of $\omega_c = 0.9 \times 10^{11}$ radians/s.

It is worth commenting that the "motion" of the spin can easily be interpreted when written in the Heisenberg picture. There we have

$$\frac{d\mathbf{S}}{dt} = \frac{i}{\hbar}[H, \mathbf{S}(t)] \tag{10-31}$$

With

$$H = \frac{eg}{2m_e}\mathbf{S} \cdot \mathbf{B} \equiv \gamma \mathbf{S} \cdot \mathbf{B} \tag{10-32}$$

we can calculate

$$\begin{aligned} \frac{dS_1(t)}{dt} &= \frac{i\gamma}{\hbar}\{B_1[S_1, S_1] + B_2[S_2, S_1] + B_3[S_3, S_1]\} \\ &= \frac{i\gamma}{\hbar}\{-i\hbar B_2 S_3 + i\hbar B_3 S_2\} = \gamma(\mathbf{B} \times \mathbf{S})_1 \end{aligned}$$

Similar calculation for the other components yields

$$\frac{d\mathbf{S}(t)}{dt} = \gamma(\mathbf{S} \times \mathbf{B}) \tag{10-33}$$

This, however, is exactly the equation for the precession of a classical angular momentum vector (associated with a magnetic dipole) subject to a torque due to a magnetic field. In this form we easily see that the precession is not restricted to the case of spin 1/2. Given that the magnetic moment of a system with angular momentum $\mathbf{J}$ is $\gamma\mathbf{J}$, the same equation holds.

10-3 PARAMAGNETIC RESONANCE

In a solid the gyromagnetic factor g of an electron is affected by the nature of the forces acting in the solid. A knowledge of g provides very useful constraints on what these forces could be, and it is therefore important to be able to measure g. This can be done by the *paramagnetic resonance method.* The principle of the method is the following: We have a magnetic field pointing in the z-direction, and the electron spin precesses about that direction. How fast does it do so? If we could introduce a magnetic field that is perpendicular to the z-axis and rotates with the spin, then the field would "see" an electron spin at rest. The component of the electron spin that points in the x-y plane would preferentially align in a direction opposite to the magnetic field to reach the minimum energy state. For those electrons not already aligned in the minimum energy direction, a transition to the lowest energy will take place, and in the process, energy—in the form of radiation—is given up. This can be detected and measured.

It is not practical to have a magnetic field rotating with a frequency of the order of 10^{11} radians per second. If, however, we have a magnetic field that points in the x-direction, say, and oscillates with a frequency ω, it may be viewed as a superposition of a field rotating in the x-y plane clockwise with frequency ω, and a field rotating counterclockwise with the same frequency, with the phase arranged so that the net effect is in the x-direction. (This is analogous to obtaining a linear polarization out of the sum of two circular polarizations.) Only one of the components will travel in the same direction as the precessing spin. The other component will move in a direction opposite to the spin precession, and its effect on the electron spin averages out to zero.

Consider an electron whose only degrees of freedom are the spin states, under the influence of a large magnetic field B_0 pointing in the z-direction and constant in time, and a small oscillating field $B_1 \cos \omega t$ pointing in the x-direction. The Schrödinger equation now reads

$$i\hbar \frac{d}{dt}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{eg\hbar}{4m_e}\begin{pmatrix} B_0 & B_1 \cos \omega t \\ B_1 \cos \omega t & -B_0 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \tag{10-34}$$

If $B_1 = 0$,

$$i\frac{da(t)}{dt} = \frac{egB_0}{4m_e}a(t) \equiv \omega_0 a(t)$$
$$i\frac{db(t)}{dt} = -\omega_0 b(t)$$

so that $a(t) = a(0)e^{-i\omega_0 t}$ and $b(t) = b(0)e^{i\omega_0 t}$. This suggests that if $B_1 \neq 0$ then we introduce $A(t)$ and $B(t)$ as follows:

$$\begin{aligned} A(t) &= a(t)e^{i\omega_0 t} \\ B(t) &= b(t)e^{-i\omega_0 t} \end{aligned} \tag{10-35}$$

where $a(t)$ and $b(t)$ obey (10-34).
If we also introduce the notation

$$\omega_1 = \frac{egB_1}{4m_e} \tag{10-36}$$

then the equations for $A(t)$ and $B(t)$ read as follows

$$\begin{aligned} i\frac{dA(t)}{dt} &= -\omega_0 A(t) + i\frac{da(t)}{dt}e^{i\omega_0 t} \\ &= -\omega_0 A(t) + (\omega_0 a(t) + \omega_1 b(t) \cos \omega t)e^{i\omega_0 t} \\ &= \omega_1 B(t) \cos \omega t e^{2i\omega_0 t} \\ &\approx \tfrac{1}{2}\omega_1 B(t) e^{(2i\omega_0 - \omega)t} \end{aligned} \tag{10-37}$$

In exactly the same way we obtain

$$i\frac{dB(t)}{dt} \approx \frac{1}{2}\,\omega_1 A(t)e^{-i(2\omega_0-\omega)t} \tag{10-38}$$

To obtain these equations in their final form we made use of the *rotating wave* approximation

$$\cos\,\omega t e^{2i\omega_0 t} = \frac{1}{2}\,(e^{2i\omega_0 t+i\omega t} + e^{2i\omega_0 t-i\omega t}) \approx \frac{1}{2}\,e^{i(2\omega_0-\omega)t} \tag{10-39}$$

The resonance condition $\omega = 2\omega_0$ implies that the first term above will oscillate very rapidly, and it contributes nothing on the average. We only used the second term in the rotating wave approximation.

The two first-order differential equations can be combined into a single second-order equation. We differentiate (10-37) and get

$$\begin{aligned}\frac{d^2A(t)}{dt^2} &= -\frac{i\omega_1}{2}\frac{d}{dt}\,(e^{i(2\omega_0-\omega)t}B(t))\\ &= \frac{\omega_1}{2}\,(2\omega_0 - \omega)e^{i(2\omega_0-\omega)t}B(t) - \frac{i\omega_1}{2}\,e^{i(2\omega_0-\omega)t}\,\frac{dB(t)}{dt}\\ &= (2\omega_0 - \omega)i\,\frac{dA(t)}{dt} - \left(\frac{\omega_1}{2}\right)^2 A(t)\end{aligned} \tag{10-40}$$

To solve this differential equation, let us try the solution $A(t) = A(0)e^{i\Omega t}$. We get

$$-\Omega^2 = -(2\omega_0 - \omega)\Omega - \left(\frac{\omega_1}{2}\right)^2$$

This quadratic equation has two roots:

$$\Omega_\pm = \left(\omega_0 - \frac{\omega}{2}\right) \pm \sqrt{\left(\omega_0 - \frac{\omega}{2}\right)^2 + \frac{\omega_1^2}{4}} \tag{10-41}$$

We now write

$$A(t) = A_+e^{i\Omega_+ t} + A_-e^{i\Omega_- t} \tag{10-42}$$

After a couple of lines of algebra this leads to

$$B(t) = -\frac{2}{\omega_1}\,(\Omega_+A_+e^{-i\Omega_- t} + \Omega_-A_-e^{-i\Omega_+ t}) \tag{10-43}$$

We can now finally write the solutions for $a(t)$ and $b(t)$. They are

$$\begin{aligned}a(t) &= A(t)e^{-i\omega_0 t}\\ b(t) &= B(t)e^{i\omega_0 t}\end{aligned} \tag{10-44}$$

Suppose that at time $t = 0$ the spin is in the "up" spin state χ_+. This means that $a(0) = 1$, $b(0) = 0$, which translates into

$$\begin{aligned}A_+ + A_- &= 1\\ \Omega_+A_+ + \Omega_-A_- &= 0\end{aligned} \tag{10-45}$$

The solution is

$$\begin{aligned}A_+ &= \frac{\Omega_-}{\Omega_- - \Omega_+}\\ A_- &= -\frac{\Omega_+}{\Omega_- - \Omega_+}\end{aligned} \tag{10-46}$$

We may now calculate the probability that at a time t, the system is in the "down" spin state χ_-. That probability is

$$\begin{aligned} P_-(t) = |b(t)|^2 &= \frac{4}{\omega_1^2}|\Omega_+A_+e^{i\Omega_+t} + \Omega_-A_-e^{i\Omega_-t}|^2 \\ &= \frac{4}{\omega_1^2}\left(\frac{\Omega_+\Omega_-}{\Omega_- - \Omega_+}\right)^2|e^{i\Omega_+t} - e^{i\Omega_-t}|^2 \\ &= \frac{8}{\omega_1^2}\left(\frac{\Omega_+\Omega_-}{\Omega_- - \Omega_+}\right)^2(1 - \cos(\Omega_- - \Omega_+)t) \end{aligned} \qquad (10\text{-}47)$$

At resonance, when

$$\omega = 2\omega_0 \qquad (10\text{-}48)$$

(10-41) yields

$$\Omega_\pm = \pm\frac{\omega_1}{2}$$

This means that

$$P_{\text{res}}(t) = \tfrac{1}{2}(1 - \cos\omega_1 t) \qquad (10\text{-}49)$$

Off-resonance we have

$$P_-(t) = \frac{1}{2}\frac{\omega_1^2}{(2\omega_0 - \omega)^2 + \omega_1^2}(1 - \cos\sqrt{(2\omega_0 - \omega)^2 + \omega_1^2}t) \qquad (10\text{-}50)$$

which is small, since $\omega \gg \omega_1$. At resonance the probability becomes of the order of unity. Since the energy of the "up" state is different from that of the "down" state, this energy difference, absorbed from the external field, is enhanced at resonance, when it matches $2\omega_0$. This allows us to determine g.

10-4 ADDITION OF TWO SPINS

In classical mechanics, angular momenta add vectorially. For example, if the angular momentum of the moon about the earth's axis is **S** and the orbital angular momentum of the earth about the sun is **L**, then there is meaning to the statement that the total angular momentum of the moon about the sun is

$$\mathbf{J} = \mathbf{L} + \mathbf{S} \qquad (10\text{-}51)$$

Does this carry over to quantum mechanics? We will show that we can "add" two angular momenta, but that the addition is not vectorial in the usual sense. We begin by considering the *addition of two spins*. An example of a two-spin system might be the ground state of helium, which has two electrons, both with zero orbital angular momentum in the lowest state. Thus the only contribution to the total angular momentum comes from the spins. We ignore all other aspects of the problem and deal only with the spins.

Suppose we have two electrons, whose spins are described by the operators $\mathbf{S}_1$ and $\mathbf{S}_2$. Each of these sets of operators satisfies the standard angular momentum commutation relations

$$[S_{1x}, S_{1y}] = i\hbar S_{1z} \qquad \text{(and cycl.)} \qquad (10\text{-}52)$$

and

$$[S_{2x}, S_{2y}] = i\hbar S_{2z} \qquad \text{(and cycl.)} \qquad (10\text{-}53)$$