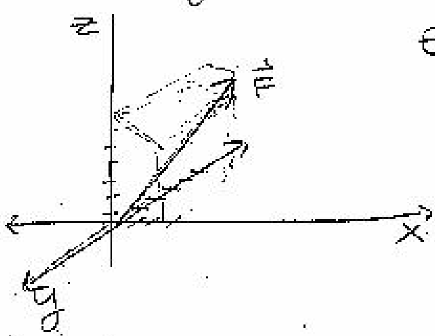
http://www.physics.smu.edu/~scalise/p4321
~ weekly updated ~
Midterm & Final
labs 11 & 25 for Mathematica

Fourier Series

Analogy w/ vector decomposition



expand $\vec{F}$ in terms of components

Std. Basis:

$\{e_1, e_2, e_3\}$ or $\{e_x, e_y, e_z\}$

$$\vec{F} = \sum_{i=1}^{3} a_i \hat{e}_i = a_1\hat{e}_1 + a_2\hat{e}_2 + a_3\hat{e}_3$$

coefficients — basis vectors

$\therefore \vec{F} = 3\hat{e}_1 + 5\hat{e}_2 + 7\hat{e}_3$

Linear Algebra Review

Another Basis: orthonormal basis:

$\hat{u}_1 = (1, 0, -1)$ $\hat{u}_2 = (1, 1, 1)$ $\hat{u}_3 = (1, -2, 1)$

* orthogonal to one another & normalized (below)

$\hat{u}_1 = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ $\hat{u}_2 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ $\hat{u}_3 = (\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}})$

all $\perp$ to one another, resulting in:

9 relations:

$\hat{u}_1 \cdot \hat{u}_1 = 1$	$\hat{u}_2 \cdot \hat{u}_1 = 0$	$\hat{u}_3 \cdot \hat{u}_1 = 0$
$\hat{u}_1 \cdot \hat{u}_2 = 0$	$\hat{u}_2 \cdot \hat{u}_2 = 1$	$\hat{u}_3 \cdot \hat{u}_2 = 0$
$\hat{u}_1 \cdot \hat{u}_3 = 0$	$\hat{u}_2 \cdot \hat{u}_3 = 0$	$\hat{u}_3 \cdot \hat{u}_3 = 1$

when indices are the same $\delta_{ij} = 1$, otherwise $\delta_{ij} = 0$

$\hat{u}_i \cdot \hat{u}_j = \delta_{ij}$ 'Kronecker Delta' b's are just some more constant coefficients like the a's before... what we wanna solve for...

$$\vec{F} = \sum_{i=1}^{3} b_i \hat{u}_i = b_1\hat{u}_1 + b_2\hat{u}_2 + b_3\hat{u}_3$$

now find the b's... $\delta_{ji} = \hat{u}_j \cdot \hat{u}_i$

$$* \; \hat{u}_j \cdot \vec{F} = \sum_{i=1}^{3} b_i \underbrace{[\hat{u}_j \cdot \hat{u}_i]}_{\delta_{ji}} = \sum_{i=1}^{3} b_i \delta_{ji} = b_j \; *$$

b/c $\delta_{ji} = 0$ unless $j=i$, in which case you get: $\hat{u}_j \cdot \vec{F} = 0 + 0 + b_j$

substitutes i in for j

multiply in

coefficients from e_i basis

$$b_1 = \hat{u}_1 \cdot \vec{F} = \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right) \cdot (3, 5, 7) = \frac{-4}{\sqrt{2}} = \frac{3}{\sqrt{2}} + 0 - \frac{7}{\sqrt{2}}$$

$$b_2 = \hat{u}_2 \cdot \vec{F} = \frac{15}{\sqrt{3}}$$

$$\boxed{b_j = \hat{u}_j \cdot \vec{F}}$$

$$b_3 = \hat{u}_3 \cdot \vec{F} = 0$$

$$\therefore \; \vec{F} = -\frac{4}{\sqrt{2}}\hat{u}_1 + \frac{15}{\sqrt{3}}\hat{u}_2 + \cancel{0}\hat{u}_3$$

Basis vectors must be $\perp$, but not necessarily normal, (although advised)

$|\vec{F}|$ in $\hat{e}_i$: $\sqrt{3^2\hat{e}_1\hat{e}_1 + 5^2\hat{e}_2\hat{e}_2 + 7^2\hat{e}_3 \cdot \hat{e}_3}$ *other terms are dropped b/c they're $\perp$

must equal:

$|\vec{F}|$ in $\hat{u}_i$: $\sqrt{\frac{16}{2}\hat{u}_1\hat{u}_1 + \frac{225}{3}\hat{u}_2 \cdot \hat{u}_2 + 0^2\hat{u}_3\hat{u}_3}$

Now with functions:

Idea: Expand a function $F(t)$ in an "orthonormal" set of basis ~~vecto~~ functions $\{u_i(t)\}$ infinite # of these functions

Analog of Dot Product: Dot Product: $\Sigma a_i b_i$

*normalize it.

$$u_i \cdot u_j \overset{?}{=} \langle u_i(t), u_j(t) \rangle = \frac{1}{N}\int_{t_1}^{t_2} u_i(t)\, u_j(t)\, dt = \delta_{ij}$$

handle separately

Fourier Basis Functions:

$\rightarrow$ even functions

$\{ 1 = \cos(\overset{\text{zero}}{0}\omega t), \cos(1\omega t), \cos(2\omega t)$, etc...

... $\sin(1\omega t), \sin(2\omega t)$, etc... $\}$ $\rightarrow$ odd functions

together, this set is orthonormal & complete $\rightarrow$ can expand any function $f(t)$

* Homework on Fridays * ☺

$\langle 1 | 1 \rangle \Rightarrow$ dot product of 1 and 1

$= \frac{1}{N}\int_{t_1}^{t_1+nT} 1 \cdot 1\, dt = \frac{2}{nT}\int_0^T 1 \cdot 1\, dt = 2$ → integrating over 1 period is plenty

where T = period $T = \frac{2\pi}{\omega}$

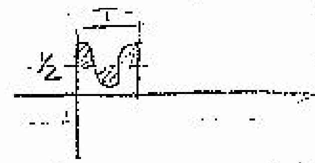
ω = angular freq

just make these 1 for simplicity

(*) case that gives us trouble, should've equaled 1

(*) $\langle 1 | \cos(\omega t)\rangle = \frac{2}{T}\int_0^T 1 \cdot \cos(\omega t)\, dt = 0$

$\langle \cos(\omega t) | \cos(\omega t)\rangle = \frac{2}{T}\int_0^T \cos^2(\omega t)\, dt$



$= \frac{2}{T}\cdot\frac{1}{2}T = 1$ ✓

$\langle \cos(n\omega t) | \cos(p\omega t)\rangle = \delta_{np}$ $n, p \geq 1$

when same you get 1, 0 elsewhere

$\langle \sin(n\omega t) | \sin(p\omega t)\rangle = \delta_{np}$

$\langle \cos(n\omega t) | \sin(p\omega t)\rangle = 0$, always

$$F(t) = \frac{a_0}{2}\cdot 1 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

* Now find a_n & b_n's ... like we did earlier to find b_n

multiply both sides by 1 and integrate $\frac{2}{T}\int_0^T \ldots dt$

lhs

$\langle 1 | F(t)\rangle = \frac{2}{T}\int_0^T 1 \cdot F(t)\, dt$ refer back to (*)

rhs: $\frac{2}{T}\int_0^T 1 \cdot \frac{a_0}{2}\cdot 1\, dt + \sum_{n=1}^{\infty}\frac{2}{T}\int_0^T 1\left[a_n \underbrace{\cos(n\omega t)}_{0} + b_n \underbrace{\sin(n\omega t)}_{0}\right]$

$\frac{2}{T}\int_0^T 1 \cdot F(t)\, dt = a_0$ $\frac{a_0}{2}$ is the time average of $F(t)$

multiply both sides by cos(pwt) and integrate $\frac{2}{T}\int_0^T \ldots dt$ where $p \geq 1$

$$\langle \cos(p\omega t) | F(t) \rangle = \frac{a_0}{2} \langle \cos(p\omega t) | 1 \rangle + \sum_{n=1}^{\infty} a_n \langle \cos(p\omega t) | \cos(n\omega t) \rangle + b_n \langle \cos(p\omega t) | \sin(n\omega t) \rangle]$$

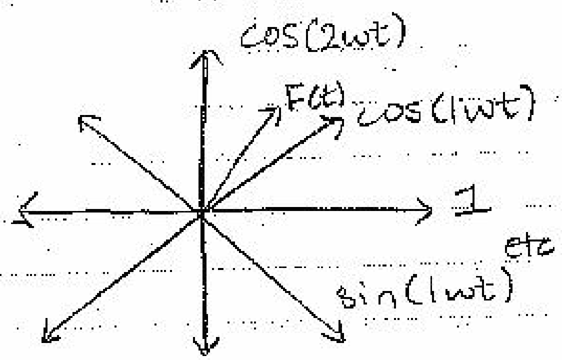
$\langle \cos(p\omega t) | 1 \rangle = 0$, $\langle \cos(p\omega t) | \cos(n\omega t) \rangle = \delta_{np}$, $\langle \cos(p\omega t) | \sin(n\omega t) \rangle = 0$

$$\sum_{n=1}^{\infty} a_n \delta_{np} = a_p$$

turns all n's to p's & removes Σ & δ ☺

$$\boxed{a_p = \frac{2}{T}\int_0^T \cos(p\omega t) F(t) dt}$$

$$\boxed{b_p = \frac{2}{T}\int_0^T \sin(p\omega t) F(t) dt}$$



infinite dimensions to establish F(t)

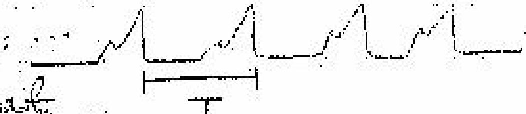
* If F(t) is even: F(t) = F(−t), then $b_p = 0$ $\forall p$ ~~for~~ all p's

* If F(t) is odd: F(t) = −F(−t), then $a_p = 0$ $\forall p$

If F(t) is neither even, nor odd, then you need all ap's & bp's...

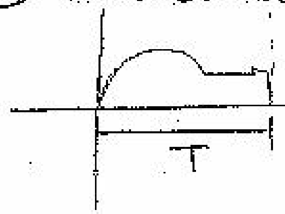
⟹ sines & cosines are periodic, ∴ the function F(t) to be expanded in <u>denumerably</u> many basis functions must therefore be periodic as well

① truly periodic -- heartbeat:



you can use more than 1 heartbeat to do a period, but be easy on yourself & keep it to 1

② F(t) could be defined on a finite time interval

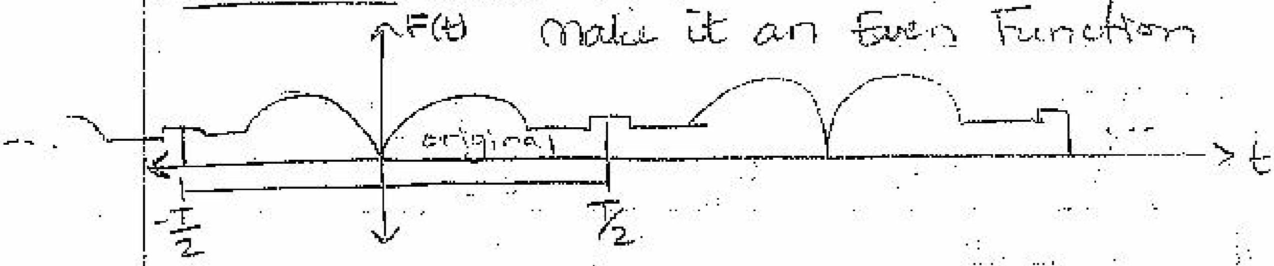


& we don't care what it is outside of the range, so just pretend that it's periodic & expand to make life easier

not smartest way to do it b/c the function is neither even nor odd- so you need all a's & b's

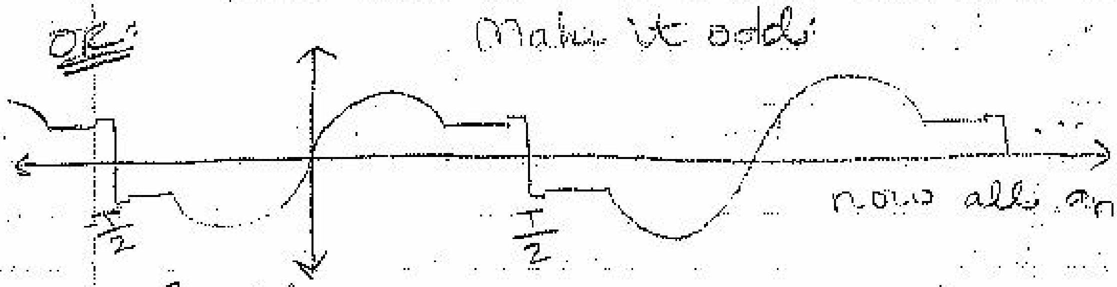
SMARTER WAY:

make it an Even Function



Now F(t) is even & all b_n's are 0...

OR: Make it odd:



now all a_n's are 0

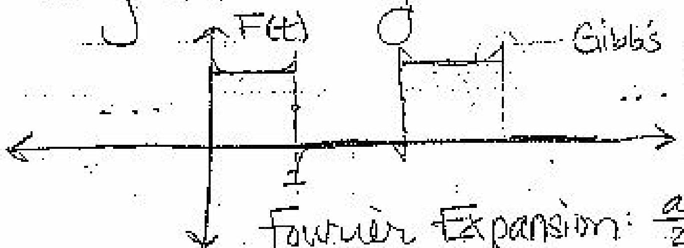
Consider:

~What if the interval is not finite & F(t) is not periodic, although defined from $-\infty < t < \infty$

→ Then you need all frequencies (remove "denumerabl[e]" from previous sentence) not just the harmonics→

SPS Thurs. 6:30pm
Lab 11
w/ pizza ☺

discontinuous functions can be fudged into working this way:



Gibb's phenomenon → no matter how precise you get, these discrepancies will still appear

Fourier Expansion: $\frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega t) + b_n \sin(n\omega t)]$

☀ Show the completeness of a function:

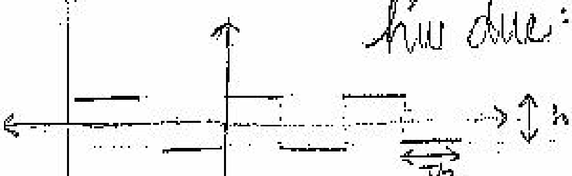
$\vec{F} = \sum_{n=1}^{3} b_n \hat{u}_n \qquad \underline{b_n = \hat{u}_n \cdot \vec{F}}$ substitute

$\vec{F} = \sum_{n=1}^{3} (\hat{u}_n \cdot \vec{F}) \hat{u}_n \qquad \vec{F} = \sum_{n=1}^{3} (\vec{F} \cdot \hat{u}_n) \hat{u}_n$

$\vec{F} = \vec{F} \cdot \underbrace{\sum_{n=1}^{3} \hat{u}_n \cdot \hat{u}_n}_{\text{must equal identity matrix}}$

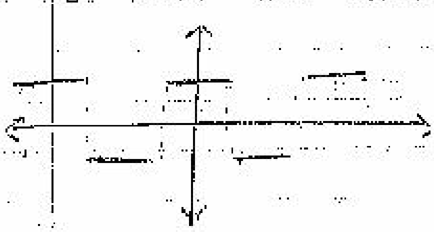
orthonormality
$\hat{u}_n \cdot \hat{u}_p = \delta_{np}$
→ orthonormality & completeness are rather the same just different indices

hw due: week from Friday ☺



50% duty cycle – on half the time

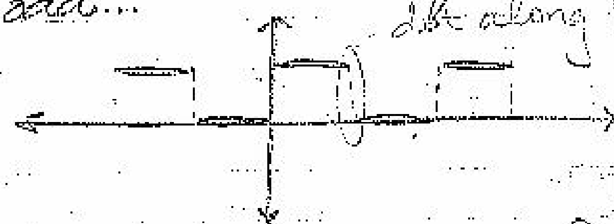
* putting x-axis there makes the function odd (like sin), so that all a_n's (including a_0) → 0



* putting y-axis here creates an even function → all b_n's are 0 + a_0 is 0 b/c that is the average

Choice where function is neither even nor odd...

dt along here is invisible to fourier transform



$a_1 \ldots a_\infty = 0$

$a_0 = \frac{h}{2}$

b/c it's basically an odd function (just moved down by a constant)

on $0 \le t \le T$ $\quad F(t) = \begin{cases} h & 0 \le t \le \frac{T}{2} \\ 0 & \frac{T}{2} \le t \le T \end{cases}$

"convergence in the mean"

dot product

$$a_0 = \langle 1 | F(t) \rangle = \frac{2}{T}\int_0^T 1 \cdot F(t)\, dt$$

function is defined piecewise → do 2 integrals

$$a_0 = \left[\int_0^{T/2} 1 \cdot h\, dt + \int_{T/2}^{T} 1 \cdot \cancel{0\, dt}\right] = \frac{2}{T} h \frac{T}{2} = h \checkmark$$

0

$\frac{a_0}{2}$ = average value of function $\therefore \frac{a_0}{2} = \frac{h}{2}$ used in Fourier Transform

$$a_n = \langle \cos(n\omega t) | F(t) \rangle = \frac{2}{T}\int_0^T \cos(n\omega t)\, F(t)\, dt$$

from cos

$$a_n = \frac{2}{T}\left[\int_0^{T/2} \cos\left(\frac{2\pi n t}{T}\right) h\, dt + \int_{T/2}^{T} 0\, dt\right]$$

$\omega = \frac{2\pi}{T}$ $\qquad$ F(t) from 0 → T/2

$$a_n = \frac{2}{T} h \left(\frac{T}{2\pi n}\right) \sin\left(\frac{2\pi n t}{T}\right)\Big|_0^{T/2} = \frac{h}{\pi n}\left[\cancel{\sin(n\pi)}^{\,0} - \cancel{\sin(0)}^{\,0}\right]$$

chain rule

$\therefore$ all a_n's $= 0$

$$b_n = \langle \sin(n\omega t) | F(t) \rangle = \frac{2}{T}\int_0^T \sin(n\omega t)\, F(t)\, dt$$

$$b_n = \frac{2}{T}\left[\int_0^{T/2} \sin(n\omega t)\, h\, dt + 0\right]$$

F(t) from 0 → T/2

$$b_n = -\frac{2}{T} h \frac{1}{n\omega} \cos(n\omega t)\Big|_0^{T/2} = -\frac{2}{T} h \frac{T}{2\pi n} \cos\left(\frac{2\pi n t}{T}\right)\Big|_0^{T/2}$$

chain rule

$$b_n = \frac{h}{n\pi}\left[\cos(0) - \cos(n\pi)\right] = \frac{h}{n\pi}\left[1-(-1)^n\right]$$

$b_1 = \frac{h}{\pi}[2] = \frac{2h}{\pi}$ $\quad$ $b_2 = \frac{h}{2\pi}[0] = 0$ & for all even values of n, $b_n = 0$

always give you factor of 2 when n is odd

$\boxed{b_{n\,(n=\text{odd})} = \frac{2h}{n\pi}}$ & graph.

* moving origin vertically only changes a_0
* moving origin horizontally "messes everything up"

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$$F(t) = \frac{a_0}{2} + b_1 \sin(1\cdot\omega t) + b_3 \sin(3\omega t) + b_5 \sin(5\omega t) + \dots$$

$$F(t) = \frac{h}{2} + \frac{2h}{\pi}\sin\left(\frac{2\pi t}{T}\right) + \frac{2h}{3\pi}\sin\left(\frac{6\pi t}{T}\right) + \frac{2h}{5\pi}\sin\left(\frac{10\pi t}{T}\right) + \dots$$

Condensed Versions:

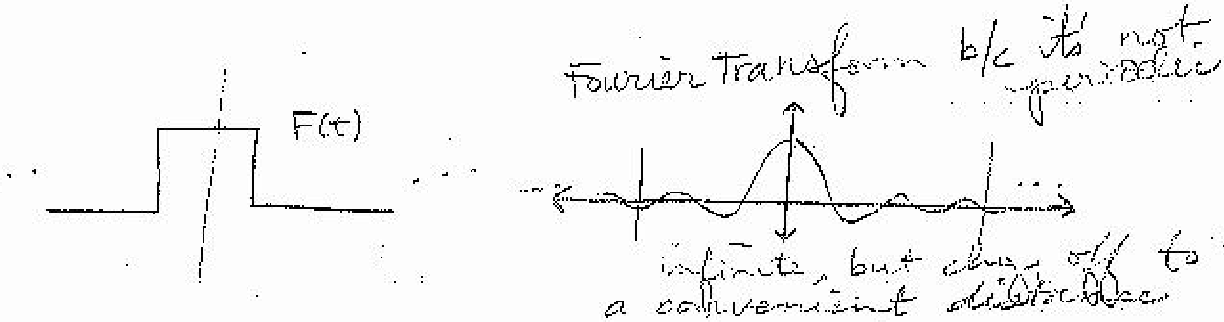
Sum over odds: $F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{\text{odd } n=1}^{\infty} \frac{1}{n}\sin\left(\frac{2\pi n t}{T}\right)$ $\quad$ $n = 2k+1$

Sum over all: $F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)}\sin\left[\frac{2\pi(2k+1)t}{T}\right]$

so they're always odd

shifted from 0 all values: $F(t) = \frac{h}{2} + \frac{2h}{\pi}\sum_{j=1}^{\infty} \frac{1}{(2j-1)}\sin\left[\frac{2\pi(2j-1)t}{T}\right]$

* Take it out to enough terms to look like original function → you be the judge



$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]$$

$\omega = \frac{2\pi}{T}$ $\quad a_n = \frac{2}{T}\int_0^T \cos(n\omega t)F(t)\,dt$ $\quad$ Here if $F(t)$ is real, a_n & b_n are real...

$$b_n = \frac{2}{T}\int_0^T \sin(n\omega t)F(t)\,dt$$

Euler: $e^{i\theta} = \cos\theta + i\sin\theta$

→ look @ Taylor expansions for $\cos\theta$ & $i\sin\theta$ → add them & you'll get the Taylor expansion for $e^{i\theta}$

COMPLEX $\quad F(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$ $\quad$ c_n is complex

find c_n's... $\quad e^{-im\omega t}F(t) = e^{-im\omega t}\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$

$$\underbrace{\left(\frac{1}{T}\right)}_{\text{normalize}}\int_0^T e^{-im\omega t}F(t)\,dt = \sum_{n=-\infty}^{\infty} c_n \underbrace{\left(\frac{1}{T}\right)\int_0^T e^{-im\omega t}e^{in\omega t}\,dt}_{\delta_{mn}}$$

normalize

– change sign – take the complex conjugate of original exp.

$$= \sum_{n=-\infty}^{\infty} c_n \delta_{mn} \quad \rightarrow \text{changes all } n\text{'s to } m\text{'s}$$

$$\frac{1}{T}\int_0^T e^{-im\omega t}F(t)\,dt = c_m$$

– multiply out below –

PROOF: $\frac{1}{T}\int_0^T e^{-im\omega t}e^{+in\omega t}\,dt = \frac{1}{T}\int_0^T \left[\cos(m\omega t) - i\sin(m\omega t)\right]\cdot\left[\cos(n\omega t) + i\sin(n\omega t)\right]dt$

– first term –

$= \frac{1}{T}\int_0^T \cos(m\omega t)\cos(n\omega t)\,dt = \frac{1}{2}\delta_{mn}$ → defined earlier in notes → normalization is only $\frac{1}{2}$

– last term –

$+ \frac{1}{T}\int_0^T \sin(m\omega t)\sin(n\omega t)\,dt = \frac{1}{2}\delta_{mn}$

next page for cross-terms

$$-i \frac{1}{T}\int_0^T \underbrace{\sin(m\omega t)\cos(n\omega t)}_{0 \text{ b/c they're } \perp} dt = 0$$

* imaginary cross-terms disappear & real ones are zero

* when F(t) is real, there's a symmetry between the c_n's...

$$c_n = \frac{1}{T}\int_0^T e^{-in\omega t} F(t)\,dt$$

Euler's Eqn: $c_n = \frac{1}{T}\int_0^T [\cos(n\omega t) - i\sin(n\omega t)] F(t)\,dt$

$$c_n = \frac{1}{T}\int_0^T \cos(n\omega t) F(t)\,dt - i\frac{1}{T}\int_0^T \sin(n\omega t) F(t)\,dt$$

$$= \frac{1}{2}a_n - i\frac{1}{2}b_n = \frac{1}{2}(a_n - ib_n)$$

* only valid when $n>0$

c_{-n} (just change n's to -n's!)

$$c_{-n} = \frac{1}{T}\int_0^T e^{+in\omega t} F(t)\,dt = \frac{1}{2}(a_n + ib_n) = c_n^*$$

c_{-n} is complex conjugate of c_n

-only works under the assumption that F(t) is real-

$$c_0 = \frac{1}{T}\int_0^T e^0 F(t)\,dt = \frac{1}{T}\int_0^T F(t)\,dt = F_{avg} = \frac{a_0}{2}$$

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t} \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} e^{-in\omega t'} F(t)\,dt$$

plug in

t' an integration variable → don't confuse w/ t's in F(t)!

$$F(t) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_0^T e^{-in\omega t'} F(t')\,dt'\right] e^{in\omega t}$$

as long as it's a full period

$T \to \infty$, $\omega \to 0$

"spacing" ω

let $T \to \infty$ $\quad n\omega$ = discrete variable

$\frac{1}{T} = \frac{\omega}{2\pi} \to \frac{d\omega}{2\pi}$ in limit $\quad \omega$ = continuous, real variable in lim form

$\sum_n = \int_\omega$ in lim