\[
F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} F(\omega) d\omega \\
F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(t) dt
\]

Notice signs are different.

Pull out Fourier & Fourier Transformations.

Important: Be you can start w/ a difficult time-domain prob. That becomes easy in the frequency domain!

Time-Domain \hspace{1cm} \text{F.T.} \hspace{1cm} \text{Frequency-Domain}

- F(t) that solves some time-dependent diff eqn.
- Now it's just an algebraic sign (F(\omega))

Get your answer back in the time-domain. Inverse F.T.

\[ F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} F(t) dt \]

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(\omega) d\omega \]

\[ \text{Laplace Transform} \]

Normalization \rightarrow no hw or test problems.

\[ \delta(t) \]

\[ \text{dirac delta function introduced next time} \]
\[ F(t) = \frac{N}{\sqrt{2\pi}} e^{-t^2} \]

\[ \int F(t) \, dt \]

Root mean square

\[ \sigma = \sqrt{\sum (t-t_0)^2} \]

\[ \sigma = \sqrt{\int (t-t_0)^2 \, dt} \]

\[ \sigma \propto \Delta t \cdot \Delta \omega \]

\[ \Delta t \cdot \Delta \omega = \frac{\hbar}{2} \]

where \( \Delta t \cdot \Delta \omega = \hbar \) constant

For Gaussian: \( \Delta t \cdot \Delta \omega \geq \frac{\hbar}{2} \)

\( t \) = time \quad \omega = \text{angular frequency} \]

\( \hbar = \text{quantum mechanical energy} \)

\[ \Delta \omega \Delta \omega = \frac{\hbar}{2} \]

\( \Delta \omega \Delta \omega \geq \frac{\hbar}{2} \)

Heisenberg Uncertainty Relationship

\( \xi, \xi^2, \text{complementary observables} \)

you can only measure one or the other but not both at the same time

Some other complementary observables:

\( \xi P, \xi^2, \xi^3, P \xi^2, \xi^2 P, P \xi, \xi P \)

Another Example:

\[ \text{Angular momentum} \]

\[ \omega = \frac{2\pi}{\hbar} \]

\[ F(w) \]

\[ \text{finite wave train} \]

\[ \text{slow sine not finite} \]

\[ \text{Dirac delta function} \]
\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i \omega t} \, \text{Re} \{ F(i \omega) \} e^{i \omega u} \, du \, d\omega \]

-changing the minus sign -> taking complex conjugates —

\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \omega t} \, \text{Re} \{ F(i \omega) \} e^{-i \omega u} \, du \, d\omega \]

Gives You:

\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i \omega t} \, \text{Re} \{ F(i \omega) \} e^{i \omega u} \, du \, d\omega \]

rewritten as:

\[ F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(t') dt' \]

\[ \Theta = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i \omega t} \, e^{-i \omega u} \, du \]

acts as delta function for stuff.

\[ \delta(t-t') = \delta(x) \]

Dirac Delta "Function"

\( \delta(x) \) is not quite a function but almost a new category.

Functions exist. Distributions also exist:

\( \delta(x) \) & friends

\[ \delta(x) - \begin{cases} 
0 & x \neq 0 \\
\infty & x = 0 
\end{cases} \]

\[ \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \]

Think of \( \delta(x) \) as the limit of:

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \, \text{Peaks around zero} \]

Then take limit as \( \epsilon \to 0 \).
Dirac Delta Function is the Continuum Analog of Kronecker Delta

\[ f(x) = \sum_{n=1}^{\infty} f_n(x) = a f_1 + a f_2 + \ldots + 2 f_{\frac{1}{2}} + a f_{\frac{1}{n}} + \ldots \]

\[ f(x) = \int_{-\infty}^{\infty} g(y) f(y) \, dy \]

\[ \text{let } 4x = y, \quad 4dx = dy \]

\[ \int_{-\infty}^{\infty} s(4x) f(x) \, dx = \frac{1}{4} f(0) \]

\[ \text{generalize zero now} \]
Delta "function" was invented by Dirac to describe the charge density of a point particle at the origin.

\[ \text{THE PROTON: } R \approx 10^{-10} \text{m, } 2 \text{ fermi;} \]

Model of solid sphere - uniform charge (only depends on distance from charge)

\[ p(r) = \begin{cases} \frac{e}{4\pi \epsilon_0 R^3}, & r < R \\ 0, & r > R \end{cases} \]

(Inside the proton)

\[ p(r) \text{ dV} = \text{total charge} = e^2 \]

\[ \iiint p(r) \text{ dV} = \text{total charge} = e^2 \]

but, \[ \iiint p(r) \text{ dV} = \text{total charge} = e^2 \]

\[ p(r) = e \delta(r) = e \delta(x) \delta(y) \delta(z) \]

(Product of 3 \( \delta \)'s, 1 for each coordinate)

\[ \iiint p(r) \text{ dV} = \iiint [-e \delta(x) \delta(y) \delta(z)] \text{ dV} = e \]

\[ \iiint [-e \delta(x) \delta(y) \delta(z)] \text{ dV} = e \]
$f(x)$ has dimension, length

$$\int_{-\infty}^{\infty} f(x) \, dx = 1 \quad \text{(as long as integrals}
\quad \uparrow \quad \text{limits include 0)} \quad \text{make a limit of}
\quad \downarrow \quad \text{integration}
\quad \therefore \quad f(x) \text{ must have limits of length, though.}

\lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} f(x) \, dx = 0 \quad \text{for a real function}

\underline{Fourier Transform of a Delta Function:}

$$F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega t} \delta(t) \, dt \quad \text{where: } F(t) = \delta(t - T)

\therefore F(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(\omega t) dt

\text{Sifting Property: } F(w) = \frac{1}{\sqrt{2\pi}} e^{-i\omega T} \quad \text{or also:}

F(w) = \frac{1}{\sqrt{2\pi}} \left[ \cos(\omega T) + i \sin(\omega T) \right]

\uparrow \quad \text{Re}[F(w)]

\text{As } T \to 0, \text{ the cos flattens out and eventually forms a line } \omega T

\underline{Derivative of Dirac Delta Function:}

$$\delta'(t) = \frac{d}{dt} \delta(t)

\text{Remember:}

\begin{align*}
\delta(t) & \quad \frac{d}{dt} \delta(t) \\
\delta(t) & \quad \frac{d^2}{dt^2} \delta(t)
\end{align*}

\text{Then, dep.}

\begin{align*}
\delta(t) & \quad \frac{d^2}{dt^2} \delta(t) \\
\delta(t) & \quad \frac{d^3}{dt^3} \delta(t)
\end{align*}
\[ \int_{-\infty}^{\infty} S'(t) f(t) \, dt = \int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0) \]

**Special Property:**
- \( f(t) \) must have "bounded support," meaning \( f(t) \) is nonzero on a finite interval \([a, b]\)
- \( f(\pm \infty) \) (function is always zero at \( \pm \infty \))
  - must be bounded

\[ \alpha(uv) = v(duv) + u(duv) \]
\[ 2uv = \int_{a}^{b} d(uv) = \int_{a}^{b} vdu + \int_{a}^{b} udv \]

& integrate both sides

\[ \text{now, let } u = S(t) \text{ & } dv = f(t) \, dt \]
\[ v = \int_{a}^{b} S(t) f(t) \, dt = S(t) f(t) \Big|_{a}^{b} - \int_{a}^{b} S(t) f'(t) \, dt \]

\[ \text{surface term, } \int_{a}^{b} \text{ divergence theorem} \]

\[ \text{surface term, } \int_{a}^{b} \text{ divergence theorem} \]

\[ 0 = -f'(0) = -\frac{df}{dt} \mid_{t=0} \]

**EXAMPLE**
- another distribution

\[ \sin(at) \delta'(t) \text{ is a generalized distribution} \]
- simplify this
- distributions only make physical sense inside an integral, or multiplied by a "test" function

\[ \int_{-\infty}^{\infty} \sin(at) \delta'(t) f(t) \, dt \]

\[ \text{test function, } \]

\[ \begin{align*}
\int_{-\infty}^{\infty} \sin(at) & S'(t) f(t) \, dt \\
= \sin(at) & S'(t) f(t) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dt} [\sin(at) f(t)] S(t) \, dt \\
= -\int_{-\infty}^{\infty} & [\cos(at) f(t) + \sin(at) f'(t)] S(t) \, dt
\end{align*} \]
\[ \int_0^\infty \left[ \cos(x) f(t) + \sin(x) f(t) \right] dt \]

Undergraduate

Original line: \[ \int_0^\infty \sin(x t) f(t) dt \]

Can't see the distribution. If you go that far, you didn't start over.

\[ \int_0^\infty \sin(x t) f(t) dt = \text{some function} \]

Don't report ans. If it'll be you didn't start up \( f(t) \).

\[ \rightarrow \text{Similar to bonus problem} \]

Integration of Dirac Delta Function

\[ \Theta(t) = \int_{-\infty}^{t} \delta(x) dx = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \]

\( \Theta(t) \) at \( t = 0 \), you pick what \( \Theta(t) \) equals. Some popular choices are \( \pm, 0, \pm \).

\[ t - 3 < 0, \ \Theta = 0 \]
\[ t - 3 > 0, \ \Theta = 1 \]

The homework problem starts at \( x = 0 \) at height 4.
Algebraic Equations: \( 5x^2 = 2, \quad x^2 + 4 \)

Goal: Find the numbers \( c \) for \( x \) that make the equations true.

- Complex numbers: \( i^2 = -1 \), \( E = 2, \quad \bar{z} = \frac{1}{z} \)
- Good book on history of \( i \).

Differential Equations: first-order, ordinary, linear, diff.

\[ \frac{df(x)}{dx} = \cos x, \quad f(x) = \cos x + C \]

Goal: Find function(s) \( f(x) \) that make the equation true.

No guaranteed ways to do this; bit of guesswork.
Differential Equations

\[ \frac{dy(x)}{dx} - y(x) = 0 \]

Goal: find function \( y(x) \) that makes this true \( \forall x \) \( y(x) = c \) for all \( x \).

Guessing... \( y(x) = x^2 \)

\[ x^2 + 2x - x^2 = 0 \] it was a bad guess

A better guess... \( y(x) = e^x \)

\[ y(x) - y(x) = e^x - e^x = 0 \] \( \forall x \)

not most general equation \( Ae^x \) is the most general

where \( A \) is any constant

Highest derivative tells you # of arbitrary constants

\( y(t) = Ae^x \) has 1 arbitrary constant

so it's a first-order differential equation

Order: highest derivative \( y''(x) \)

\( y''(x) = 0 \) is first-order linear (in \( y \))

linear: derivatives of the function \( y(x) \), including the 0th derivative, occur to the first power

\[ \frac{d^2y(x)}{dx^2} + y(x) = 0 \]

\( y'' + y = 0 \) 2nd order, linear & homogeneous

\( y''(x) + y(x) = 0 \) non-linear

\( y''(x) + \sin(y(x)) = 0 \) also non-linear

\( [y''(x)]^2 - y(x) = 0 \rightarrow \left[ \frac{dy(x)}{dx} \right]^2 - y(x) = 0 \) also non-linear