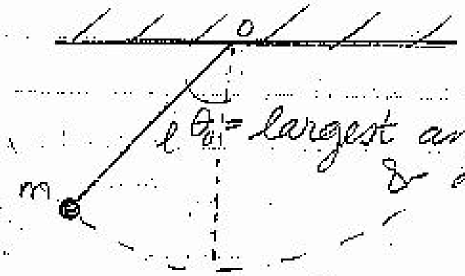
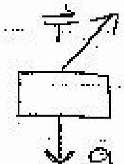


Free Body Diagram



This system accelerates,
So the forces don't all cancel out

Analog of $\vec{F} = m\vec{a}$
$\Sigma \tau = I\alpha$
using angular components

axis of torques and moment of inertia (I)

*Must pick an origin: Two choices:
① Center of Mass (even if it accelerates)
OR ② any non-accelerating pt.

#2 is the better choice, in this case origin = O

$$\Sigma T_0 =$$
$$-mgl\sin[\theta(t)] = ml^2\ddot{\theta}(t)$$

b/c the torque & angle are always oppositely directed

ml^2: I for a pt. particle

$\ddot{\theta}(t)$: α angular acceleration

$$\ddot{\theta}(t) + \frac{g}{l}\sin[\theta(t)] = 0$$

2nd order, nonlinear, homogeneous, ordinary

nonlinear: b/c of $\sin[\theta(t)]$

*usually you let $\sin\theta \sim \theta$ b/c of Taylor Expansion

$$\theta - \frac{\theta^3}{3} + \frac{\theta^5}{5} - \ldots \sim \theta$$

Then look @ your tolerance, (1%)

$$\frac{\theta^3}{3!} \ll \frac{1}{100} \Rightarrow \theta \ll \sqrt[3]{\frac{6}{100}} \sim .39\,\text{rad} \sim \underline{22^\circ}$$

largest θ you can have before approximation breaks down!

$\ddot{\theta}(t) + \frac{g}{\ell}\theta(t) = 0$

2nd-order, linear, homogeneous

$\theta(t) = A\sin(\omega t + B)$

where $\omega \equiv \sqrt{\frac{g}{\ell}}$

pg. 162 in Marion book solves this another way.

Now Solving the Non-linear Equation

$$\ddot{\theta}(t) + \omega_0^2 \sin[\theta(t)] = 0$$

find $\theta(t)$ ⇝ it's gonna be an infinite Taylor Series

*Trick: multiply both sides by $2\dot{\theta}(t)$:

$$2\dot{\theta}(t)\ddot{\theta}(t) + 2\omega_0^2\dot{\theta}(t)\sin[\theta(t)] = 0$$

$$2\dot{\theta}(t)\ddot{\theta}(t) = -\frac{2g}{\ell}\dot{\theta}(t)\sin[\theta(t)]$$

$$\frac{d}{dt}\left[\dot{\theta}(t)^2\right] = \frac{2g}{\ell}\frac{d}{dt}\left[\cos(\theta(t)) + C\right]$$

first constant of integration

but we will represent As $-\cos(\theta_0)$

& integrate both sides (remove $\frac{d}{dt}$'s)

$$[\dot{\theta}(t)]^2 = \frac{2g}{\ell}[\cos\theta(t) - \cos\theta_0]$$

Boundary condition: when $\dot{\theta}(t) = 0$ when $\theta = \theta_0$ (built into above equation)

Could get this far with Energy Conservation!

"Energy" is called a first integral of motion

$$\tfrac{1}{2}m\ell^2(\dot{\theta}(t))^2 = mg\ell\,[\cos\theta(t) - \cos\theta_0]$$

kinetic potential

– Energy Conservation –

& Now we have a 1st order differential eqn:

$$\dot{\theta}(t) = \sqrt{\tfrac{2g}{\ell}[\cos\theta(t) - \cos\theta_0]}$$

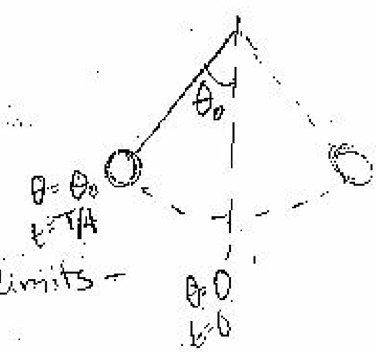
To solve, get time dependence on one side & θ dependence on the other side

$$\frac{d\theta(t)}{dt} = \dot{\theta}(t) = \sqrt{\tfrac{2g}{\ell}[\cos\theta(t) - \cos\theta_0]}$$

Separate & Integrate

take absolute value

$$\left|\int_{\theta=\theta_0}^{0} \frac{d\theta}{\sqrt{\tfrac{2g}{\ell}[\cos\theta - \cos\theta_0]}}\right| = \int_{t=0,\ \text{when } \theta=\theta_0}^{T/4\ \leftarrow\ \text{when } \theta=0} dt = T/4$$



$$\therefore T = 4\int_0^{\theta_0} \frac{d\theta}{\sqrt{\tfrac{2g}{\ell}[\cos\theta - \cos\theta_0]}}$$

rewrite w/ half angle formula

$$\cos\theta = 1 - 2\sin^2\left(\tfrac{\theta}{2}\right)$$

& pull out constants

$$T = 2\sqrt{\tfrac{\ell}{g}}\int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\left(\tfrac{\theta_0}{2}\right) - \sin^2\left(\tfrac{\theta}{2}\right)}}$$

* carefully pick a new variable

Change Variable: so now we have definite limits

$\sin\left(\frac{\theta}{2}\right) \rightarrow \sin\frac{\theta_0}{2}\sin\varphi$

$\underset{(min)}{\theta=0} \Rightarrow \varphi=0 \qquad \underset{(max)}{\theta=\theta_0} \Rightarrow \varphi=\frac{\pi}{2}$

You also need the derivative of both sides:

$\frac{1}{2}\cos\left(\frac{\theta}{2}\right)d\theta \rightarrow \sin\left(\frac{\theta_0}{2}\right)\cos\varphi\, d\varphi$

solve for $d\theta$ ↑ constant — product rule (first term dragged out)

(1) $T = 2\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\frac{\pi}{2}} \frac{2\sin(\theta_0/2)\cos\varphi\, d\varphi}{\cos\left(\frac{\theta}{2}\right)\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\left(\frac{\theta}{2}\right)}}$ — from half-angle formula earlier

(2) Answer: $4\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\frac{d\varphi}{\cos\left(\frac{\theta}{2}\right)}$ ⇓ from there to here for next time (1)

How to get from (1) to (2):

$$T = 2\sqrt{l/g}\int_{\varphi=0}^{\pi/2}\frac{2\sin(\theta_0/2)\cos\varphi\, d\varphi}{\cos(\theta/2)\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}$$

& use change of variable here too

$$T = 2\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\frac{2\sin(\theta_0/2)\cos\varphi\, d\varphi}{\cos\left(\frac{\theta}{2}\right)\sqrt{\cancel{\sin^2\frac{\theta_0}{2}}-\cancel{\sin^2\left(\frac{\theta_0}{2}\right)}\sin^2\varphi}}$$

$$T = 2\sqrt{\frac{l}{g}}\int_{\varphi}^{\pi/2}\frac{2\cancel{\cos\varphi}\, d\varphi}{\cos\left(\frac{\theta}{2}\right)\sqrt{1-\sin^2\varphi}} \qquad \sqrt{1-\sin^2\varphi} = \cancel{\cos\varphi}$$

$$T = 4\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\frac{d\varphi}{\cos(\theta/2)} = 4\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\frac{d\varphi}{\sqrt{1-\sin^2\theta/2}}$$

$$T = 4\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\frac{d\varphi}{\sqrt{1-\sin^2\left(\frac{\theta_0}{2}\right)\sin^2\varphi}}$$

$x < 1$ — plays the role of x below... → use binomial expansion & watch it converge b/c $\sin\theta$ is less than 1...

$$\frac{1}{\sqrt{1-x}} = (1-x)^{1/2} = 1 + \frac{x}{2} + \frac{1}{2}\left(\frac{3}{4}\right)x^2 + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)x^3 + \ldots$$

includes small angle limiting case

$$T = 4\sqrt{\frac{l}{g}}\int_{\varphi=0}^{\pi/2}\left[1 + \frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2\varphi + \frac{3}{8}\sin^4\frac{\theta_0}{2}\sin^4\varphi + \dots\right]d\varphi$$

integrated & evaluated — integrated & evaluated

$$T = 4\sqrt{\frac{l}{g}}\left[\frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2}\left(\frac{\pi}{4}\right) + \frac{3}{8}\sin^4\frac{\theta_0}{2}\left(\frac{3\pi}{16}\right) + \dots\right]$$

$T(\theta_0)$ b/c period really is a function of the initial angle

$$T(\theta_0) = 2\pi\sqrt{\frac{l}{g}}\left[1 + \frac{1}{2}\cdot\frac{1}{2}\sin^2\left(\frac{\theta_0}{2}\right) + \left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\sin^4\frac{\theta_0}{2} + \dots\right]$$

expand until tolerance is met

— THE END —

*error is approx. equal to the size of the last term you leave off

→ Must use this version for angles greater than 22° rather than using the $\sin\theta \cong \theta$ approximation because you will be wrong... ☺

& it's frictionless

① What is the $v(\theta)$?

② where does it come off (at what angle?)?

With Energy Conservation first:

Assume energy conservation b/c gravity is the only force doing work & it's conservative

$K_i = 0$ — $U_i = 0$ b/c you can only measure difference in potential

$K_f = \frac{1}{2}mv_f^2$ — $U_f = -mgR[1-\cos\theta]$

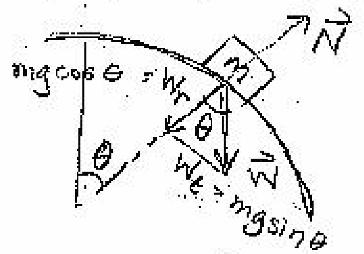
$K_i + U_i = K_f + U_f$ → not dependent on mass of object

$0 = \frac{1}{2}\cancel{m}v^2 - \cancel{m}gR[1-\cos\theta]$

$\underline{v = \sqrt{2gR[1-\cos\theta]}}$ valid for $0 < \theta < \theta_{critical}$

where it comes off surface

& now find $\theta_{critical}$ (where object leaves the surface)
This occurs when $\vec{N} = 0$



decompose $\vec{W}$ in radial & tangential components

Newton's 2nd Law:

$$\Sigma \vec{F} = m\vec{a}$$

looking @ radial direction

$$\Sigma F_r = m a_r \quad \text{where } \hat{r} \text{ is radially in}$$

$$W_r - N = m\left(\frac{v^2}{R}\right) \sim \text{radial acceleration}$$

$$mg\cos\theta - N = m\left(\frac{v^2}{R}\right)$$

Solve for N & set equal to 0

$$\cancel{m}g\cos\theta_{crit} = \frac{\cancel{m}v^2}{R}$$

not dependent on mass ☺

& plug in velocity from above

& you get: $\underline{\theta_{crit} = \arccos\left(\frac{2}{3}\right)}$

A little slower...

$$\cancel{g}\cos\theta_{crit} = \frac{v^2}{R} = \frac{2\cancel{g}\cancel{R}[1-\cos\theta_{crit}]}{\cancel{R}}$$

$$\cos\theta_{crit} = 2[1-\cos\theta_{crit}] = 2 - 2\cos\theta_{crit}$$

$$\frac{3\cos\theta_{crit}}{3} = \frac{2}{3} \qquad \underline{\theta_{crit} = \arccos\left(\frac{2}{3}\right)}$$

$$\Sigma F_t = m a_t$$

$$mg\sin\theta = m\frac{dv}{dt} \quad \text{use chain rule}$$

ω = angular speed (not a vector quantity)

$$\cancel{m}g\sin\theta = \cancel{m}\frac{dv}{d\theta}\frac{d\theta}{dt}$$

& thus removed the t

$$\underline{\omega = \frac{v}{R}}$$

$$g\sin\theta = \frac{dv}{d\theta}\frac{v}{R} \rightarrow Rg\sin\theta = v'(\theta)v(\theta)$$

nonlinear first order, nonhomogeneous still integrable

$$\ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0$$

→ first integral of this is energy

$$V'(\theta)V(\theta) = Rg\sin\theta$$

$$Rg\sin\theta d\theta = v dv$$

$$Rg\int_{\theta'=0}^{\theta} \sin\theta' d\theta' = \int_{v'=0}^{v} v' dv'$$

→ the tick marks are just to differentiate them from the final pt.

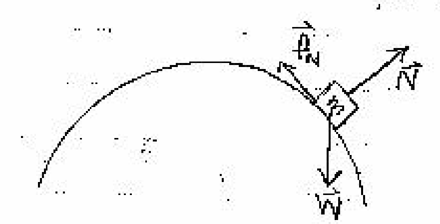
$$Rg(-\cos\theta')\Big|_{\theta'=0}^{\theta} = \frac{1}{2}(v')^2\Big|_{v'=0}^{v(\theta)}$$

$$Rg[1-\cos\theta] = \frac{1}{2}[v(\theta)]^2$$

$$v(\theta) = \sqrt{2Rg[1-\cos\theta]}$$

same as previous result w/ Energy Conservation

SAME PROBLEM, NOW WITH FRICTION



$$\vec{f}_k = \mu_k \vec{N}$$

radial: $\Sigma F_r = ma_r$ same as previously for radial direction

$$mg\cos\theta - N = m\frac{v^2}{R} \Rightarrow N = mg\cos\theta - \frac{mv^2}{R}$$

tangential: $\Sigma F_t = ma_t$

$$mg\sin\theta - f_k = m\frac{dv}{dt} = m\frac{dv}{d\theta}\cdot\frac{d\theta}{dt} = m\frac{dv}{d\theta}\frac{v}{R}$$

$= \omega$ (t)

$$mg\sin\theta - \mu_k N = m\frac{dv}{d\theta}\frac{v}{R}$$

$$\cancel{m}g\sin\theta - \mu_k\left[\cancel{m}g\cos\theta - \frac{\cancel{m}v^2}{R}\right] = \cancel{m}\frac{dv}{d\theta}\frac{v}{R}$$ doesn't depend on mass

put nonhomogeneous terms together

& move R to other side

$$2Rg[\sin\theta - \mu_k\cos\theta] + 2\mu_k v^2 = \frac{dv}{d\theta}v = \frac{d}{d\theta}\left(\frac{1}{2}v^2\right)$$

*put in standard form & multiply out by 2

$$\frac{d(v^2(\theta))}{d\theta} - 2\mu_k v^2(\theta) = 2Rg[\sin\theta - \mu_k\cos\theta]$$

sin of θ, not v(θ), so it's still linear!

let $f(\theta) = v^2(\theta)$

$$\frac{df(\theta)}{d\theta} - 2\mu_k f(\theta) = 2Rg[\sin\theta - \mu_k\cos\theta]$$

$$f'(\theta) - 2\mu_k f(\theta) = 2Rg[\sin\theta - \mu_k\cos\theta]$$

first-order, linear in $f(\theta)$, nonhomogeneous & ordinary diff eq.

Find the $f(\theta)$ that makes this true!

$$f(\theta) = f_c(\theta) + f_p(\theta)$$

general; complimentary (one arbitrary constant b/c it's a first-order diff eq); particular

Complementary:
(Solution to Homogeneous Eqn)

$$f_c'(\theta) - 2\mu_k f_c(\theta) = 0$$

guess: $Ae^{r\theta} = f_c(\theta)$
$f_c'(\theta) = rAe^{r\theta}$

$$rAe^{r\theta} - 2\mu_k Ae^{r\theta} = 0$$

$r - 2\mu_k = 0$ characteristic equation
$r = 2\mu_k$ ☺ $\quad f_c(\theta) = Ae^{2\mu_k\theta}$

Particular:

$$f_p'(\theta) - 2\mu_k f_p(\theta) = 2Rg[\sin\theta - \mu_k\cos\theta]$$

guess $f_p(\theta) = b\cos\theta + c\sin\theta$

$$f_p(\theta) = b\cos\theta + c\sin\theta$$
$$f'_p(\theta) = -b\sin\theta + c\cos\theta \quad \text{\& plug in!}$$
$$(-b\sin\theta + c\cos\theta) - 2\mu_k(b\cos + c\sin\theta) = 2Rg[\sin\theta - \mu_k\cos\theta]$$

coefficients must match on either side ☺

$$\sin\theta[-b - 2\mu_k c - 2Rg] + \cos\theta[c - 2\mu_k b + 2Rg\mu_k] = 0 \quad \forall\theta$$

must equal 0 independently... for all s.h. driven SHO

$$\left.\begin{array}{l} -b - 2\mu_k - 2Rg = 0 \\ c - 2\mu_k b + 2Rg\mu_k = 0 \end{array}\right\} \text{\& solve}$$

Mathematica gives us:

$$b = \frac{2(-gR + 2g\mu_k^2 R)}{1+4\mu_k^2}$$

$$c = \frac{-6g\mu_k R}{1+4\mu_k^2}$$

$$f(\theta) = f_c(\theta) + f_p(\theta)$$

$$v^2(\theta) = \underset{\text{(general)}}{f(\theta)} = \underset{\text{complementary)}}{Ae^{2\mu_k\theta}} + \underset{\text{(particular)}}{\left(\frac{-2gR + 4\mu_k^2 gR}{1+4\mu_k^2}\right)\cos\theta - \left(\frac{6\mu_k gR}{1+4\mu_k^2}\right)\sin\theta}$$

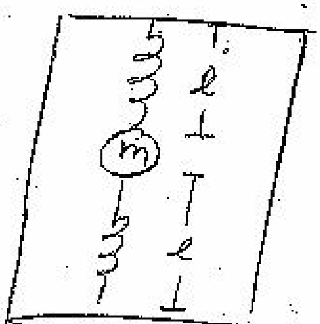
$$v(\theta) = \sqrt{\text{all the above}} \text{ ☺}$$

A is fixed by initial (boundary) conditions
& for this problem w/ friction, the condition $v(\theta = 0) = 0$ is not allowed

b/c otherwise, it wouldn't go anywhere

Coordinates - 2D polar

Graduate Homework Explanation:



a nonlinear diffeq

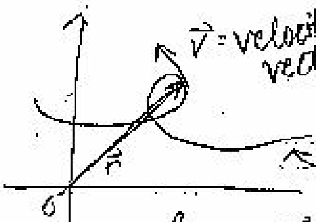
$\vec{V}(t)$ ← scalar expand in Cartesian Coordinates

$$\vec{V}(t) = V_x(t)\,\hat{e}_x + V_y(t)\,\hat{e}_y + V_z(t)\,\hat{e}_z$$

↑ vector function

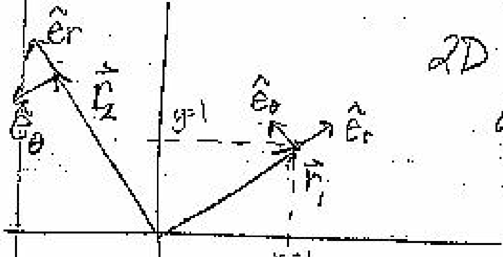
$\hat{e}_i$ time independent aka: $\frac{d\hat{e}_i}{dt} = 0$

$$\vec{A}(t) = \frac{d\vec{V}(t)}{dt} = \frac{dV_x(t)}{dt}\hat{e}_x + \frac{dV_y(t)}{dt}\hat{e}_y + \frac{dV_z(t)}{dt}\hat{e}_z$$



"coordinate-free representation"

where $\vec{r}$ = position vector, displacement



2D polar (r, θ)

unit vectors $(\hat{e}_r, \hat{e}_\theta)$

"^" hat symbols imply that they are length one

$\hat{e}_r \cdot \hat{e}_r = 1 \quad \hat{e}_\theta \cdot \hat{e}_\theta = 1 \quad \hat{e}_r \cdot \hat{e}_\theta = 0$

$r = \sqrt{x^2 + y^2}$ (positive square root)

$\theta = \arctan\left(\frac{y}{x}\right)$ (maybe $\pm \pi$ radians) * must draw picture to know which quadrant it's in

$$\hat{e}_r = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{e}_x + y\hat{e}_y}{\sqrt{x^2+y^2}}$$

$$\hat{e}_\theta = \frac{-y\hat{e}_x + x\hat{e}_y}{\sqrt{x^2+y^2}}$$

$$\dot{r} = \frac{dr}{dt} = \frac{d(\vec{r})}{dt} \neq \frac{d\vec{r}}{dt}$$

$$\dot{r} = \frac{d}{dt}(x^2+y^2)^{1/2} = \frac{1}{2}(x^2+y^2)^{-1/2}\left(2x\overset{\dot{x}}{\frac{dx}{dt}} + 2y\overset{\dot{y}}{\frac{dy}{dt}}\right)$$

$$\dot{r} = \frac{x\dot{x}+y\dot{y}}{\sqrt{x^2+y^2}} \neq |\dot{\vec{r}}| = |\vec{v}|$$

take magnitudes then derivative

take time derivative first, then magnitude

\# Number Three

quotient rule

$$\dot{\theta} = \frac{1}{1+(\frac{y}{x})^2}\frac{d}{dt}\left(\frac{y}{x}\right) = \frac{1}{1+(\frac{y}{x})^2} \cdot \frac{x\dot{y}-y\dot{x}}{x^2}$$

$$\dot{\theta} = \frac{x\dot{y}-y\dot{x}}{x^2+y^2}$$

$$\dot{\hat{e}}_r = \frac{d\hat{e}_r}{dt} = \frac{d}{dt}\left[\frac{x\hat{e}_x + y\hat{e}_y}{\sqrt{x^2+y^2}}\right] \qquad \underbrace{\dot{\hat{e}}_x = 0 \quad \dot{\hat{e}}_y = 0}_{\text{in cartesian}}$$

$$\dot{\hat{e}}_r = \frac{\dot{x}\hat{e}_x + \dot{y}\hat{e}_y}{\sqrt{x^2+y^2}} - (x\hat{e}_x + y\hat{e}_y)\frac{d}{dt}\left[(x^2+y^2)^{-1/2}\right]$$

$$\dot{\hat{e}}_r = \frac{\dot{x}\hat{e}_x + \dot{y}\hat{e}_y}{\sqrt{x^2+y^2}} - (x\hat{e}_x + y\hat{e}_y)\left(-\frac{1}{2}\right)(x^2+y^2)^{-3/2}(2x\dot{x} + 2y\dot{y})$$

$$\dot{\hat{e}}_r = \frac{(\dot{x}\hat{e}_x + \dot{y}\hat{e}_y)(x^2+y^2) - (x\hat{e}_x + y\hat{e}_y)(x\dot{x}+y\dot{y})}{(x^2+y^2)^{3/2}}$$

next page →