4321 and 7305

1. Consider the Fibonacci sequence: 1,1,2,3,5,8,13,21,34, ... where $F(1) = 1$, $F(2) = 1$, $F(3) = 2$, ... with recursion relation $F(n+2) = F(n+1) + F(n)$. Define the vectors $\vec{v}(n) = \begin{pmatrix} F(n) \\ F(n+1) \end{pmatrix}$. Then the real symmetric matrix $S = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ generates $\vec{v}(n+1)$ from $\vec{v}(n)$ as follows S $\vec{v}(n) = \vec{v}(n+1)$

   (a) Find the eigenvalues and eigenvectors of S.

   (b) Let Q be the matrix whose columns are the normalized eigenvectors of S. Show that $Q^T S Q$ is the diagonal matrix D.

   (c) Show that Q is an orthogonal matrix, that is $Q^T Q = I$.

   (d) Show that $S^n = Q^T D^n Q$.

   (e) Find a formula for the nth Fibonacci number $F(n)$.

Bonus:

1. Spectrally decompose the 2×2 hermitian matrix $M = \begin{pmatrix} 4\pi & i\pi \\ -i\pi & 4\pi \end{pmatrix}$ as

$$M = \sum_{i=1}^{2} \rho_i \; \vec{u}_i \vec{u}_i^\dagger$$

   where ρ_i is the ith eigenvalue and $\vec{u}_i$ is the ith orthonormalized column eigenvector. That is,

$$M\vec{u}_i = \rho_i \vec{u}_i \qquad \text{and} \qquad \vec{u}_i^\dagger \cdot \vec{u}_j = \delta_{ij} \; .$$

   (a) Show that $M^2 = \sum_{i=1}^{2} (\rho_i)^2 \; \vec{u}_i \vec{u}_i^\dagger$ without using the numbers specific to this problem.

   (b) Find ρ_1, $\vec{u}_1$, ρ_2, and $\vec{u}_2$.

   (c) Show completeness: $\sum_{i=1}^{2} \vec{u}_i \vec{u}_i^\dagger = I_2$ (the 2×2 identity matrix).

   (d) Find $\sin(M)$ two ways:

      i. By taking advantage of the spectral decomposition:
      $\sin(M) = \sum_{i=1}^{2} \sin(\rho_i) \; \vec{u}_i \vec{u}_i^\dagger$

      ii. By using the first few (?!) terms of a Taylor expansion:
      $\sin(M) = M - M^3/3! + M^5/5! + \ldots$

      iii. Comment on the convergence of the series.