$$\sum_{n=-\infty}^{\infty} \frac{1}{T}\left[\int_{-T/2}^{T/2} e^{-in\omega t'} F(t')\,dt'\right] e^{in\omega t}$$

$$F(t) = \int_{\omega=-\infty}^{+\infty} \frac{d\omega}{2\pi} \int_{t'=-\infty}^{t'=\infty} e^{-i\omega t'} F(t')\,dt' \; e^{i\omega t}$$

$d\omega$ always goes to end

forward $\boxed{\tilde{F}(\omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{t'=-\infty}^{\infty} e^{-i\omega t'} F(t')\,dt'}$ notice signs are diff

$$F(t) = \int_{\omega=-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{i\omega t}\,d\omega = \boxed{\frac{1}{\sqrt{2\pi}} \int_{\omega=-\infty}^{\infty} e^{+i\omega t} \tilde{F}(\omega)\,d\omega = F(t)}$$ reverse

pull out

- inverse -

Fourier & Reverse Fourier Transformations

Important b/c you can start w/ a difficult time-domain prob. that becomes easy in ~~freq.~~ domain!

Time-Domain (t) — F.T. convert → Frequency-Domain (ω)

- F(t) that solves some time-dependent diff eq → - now it's just an algebraic eqn ($\tilde{F}(\omega)$)

get your answer back in the time-domain ← convert back inverse F.T.

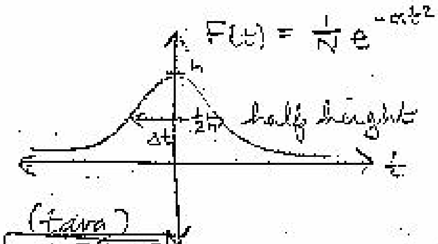
$$\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} e^{-i\omega t} F(t)\,dt$$ Fourier Transform

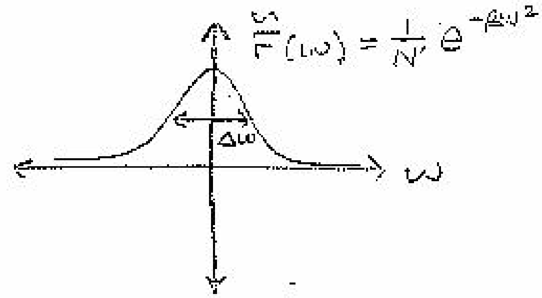
---

$f(s) = N \int e^{-st} F(t)\,dt \qquad i\omega \equiv s$ Laplace Transform

N ↓ normalization → no hw or test problems

* dirac delta function introduced next time





$\sqrt{\Sigma(\bar{t}^{(tavg)} - t)^2}$ root mean square

as $\Delta t \downarrow$, $\Delta\omega \uparrow$
& as $\Delta t \uparrow$, $\Delta\omega \downarrow$ } where $\Delta t \cdot \Delta\omega = C$ (some constant)

for Gaussians: $\Delta t \cdot \Delta\omega \geq \frac{1}{2}$

when t = time ω = angular frequency

$\hbar\omega$ = quantum mechanical energy

$\therefore \Delta t(\hbar\Delta\omega) \geq \frac{\hbar}{2}$ $\Delta t \Delta E \geq \frac{\hbar}{2}$

Heisenberg Uncertainty Relationship

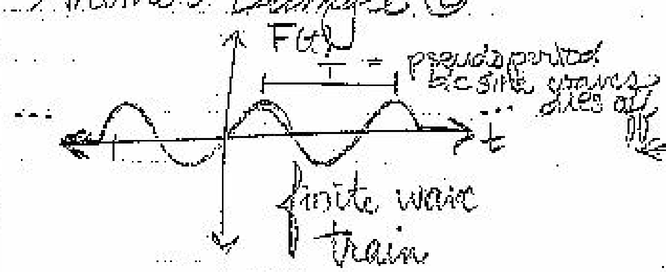
{t, E} complimentary observables

* you can only measure one or the other, but not both at the same time

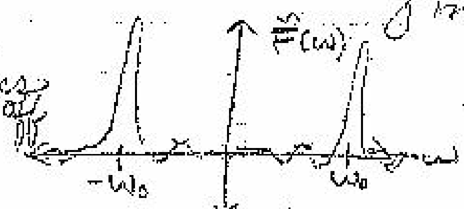
some other complimentary observables:

{X, P_x}, {y, P_y}, {z, P_z} {θ, L}
L = angular momentum

Another Example ②



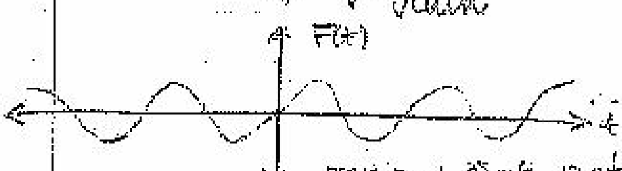
finite wave train



$\omega_0 = \frac{2\pi}{T}$ (pseudo frequency)

F(ω) = 0 everywhere but $\pm\omega_0$

dirac delta functions



∴ now one is not finite

$$\omega \to -z \qquad d\omega \to -dz$$

$$F(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}\int_{t'=-\infty}^{\infty} e^{-i\omega t'} F(t')\,dt'\, e^{i\omega t}\,d\omega$$

– changing the minus sign → taking complex conjugate –

$$F(t) = \frac{1}{2\pi}\int_{z=+\infty}^{-\infty}\int_{t'=-\infty}^{+\infty} e^{izt'} F(t')\,dt'\, e^{-izt}(-dz)$$

use this negative to flip integration limits

→ then let $z=\omega$ & $dz = d\omega$

Gives You:

$$\boxed{F(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty}\int_{t'=-\infty}^{+\infty} e^{+i\omega t'} F(t')\,dt'\, e^{-i\omega t}\,d\omega}$$

& rewritten as:

$$F(t) = \int_{t'=-\infty}^{+\infty} (?)\, F(t')\,dt'$$

acts as delta function for stuff ∞ diff

$$(?) = \frac{1}{2\pi}\int_{\omega=-\infty}^{+\infty} e^{i\omega(t'-t)}\,d\omega$$

$$= \delta(t'-t) = \delta(x)$$

Dirac Delta "Function"

$\delta(x)$ is not quite a function but almost ~ new category

generalized functions aka: distributions

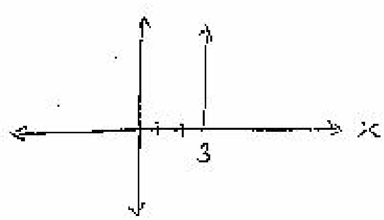
→ functions

→ $\delta(x)$ & friends

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

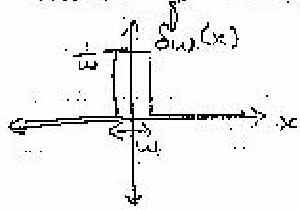
$$\int_{x=-\infty}^{+\infty} \delta(x)\,dx = 1$$

$\delta(x-3) =$



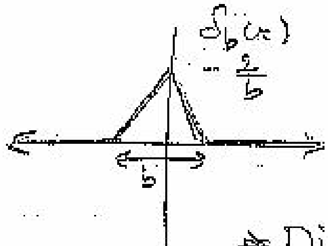
Think of $\delta(x)$ as the limit of:

* you can use any shape that peaks around zero



$\int \delta_\omega(x)\,dx = \text{Area} = \omega\left(\frac{1}{\omega}\right) = 1$

then take limit as $\omega \to 0$



$\lim_{b \to 0}$

* Dirac Delta Function is the Continuum Analog of Krönecker Delta

$$f_i = \sum_{n=1}^{300} \delta_{in} f_n = 0f_1 + 0f_2 + \dots + 1f_i + 0f_{i+1} + \dots$$
$$= \underline{\underline{f_i}}$$

krönecker delta

$$f(x) = \int_{y=-\infty}^{+\infty} \delta(x-y) f(y)\, dy$$

$-\rho$ or just something large

$$\int_{x=-\infty}^{+\infty} \delta(4x) f(x)\, dx \qquad \text{let } 4x = y \qquad 4dx = dy$$

scales

$$\frac{1}{4}\int_{y=-\infty}^{+\infty} \delta(y) f\left(\frac{y}{4}\right) dy = \frac{1}{4} f(0)$$

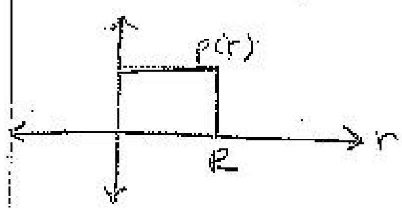
peaked @ zero now

Delta "Function" was invented by Dirac to describe the charge density of a pt. particle at the origin

eg (where you don't need the delta function)
THE PROTON: $R \approx 10^{-15}\,m = 1$ fermi
model of solid sphere - uniform charge
(only depends on distance from charge)

charge density $\rho(r) = \begin{cases} \frac{e}{4/3\pi R^3} & r<R \\ 0 & , r>R \end{cases}$ (outside the proton)

$1.60\times10^{-19}C$



eg electron (point-like) ~ no volume
(up to $10^{-18}m$) experimental error

$\rho(r) = \begin{cases} 0, & r>0 \\ \infty & r=0 \end{cases}$
b/c it's like $\frac{-e}{0\,(\text{volume})} = \infty$

but, $\iiint_V \rho(r)\,dV = $ total charge in V $= e^-$ ... V

$\rho(r) = -e\,\delta^{(3)}(\vec{r}) = -e\,\delta(x)\,\delta(y)\,\delta(z)$
product of 3 $\delta(r)$'s... 1 for each coordinate

works b/c they're w/ diff variables
you can't have $\delta^2(x)$

$\iiint_V \rho(r)\,dV = \int_x\int_y\int_z [-e\,\delta(x)\,\delta(y)\,\delta(z)]\,dx\,dy\,dz$

$= -e \underbrace{\int_x \delta(x)\,dx}_{1}\,\underbrace{\int_y \delta(y)\,dy}_{1}\,\underbrace{\int_z \delta(z)\,dz}_{1} = -e$

$\delta(x)$: has dimension $\frac{1}{\text{length}}$

$\int \delta(x)\,dx = 1$ (as long as integration limits include 0) ~ don't make 0 a limit of integration though

$\int$ ↑ length ↖ dimensionless

$\therefore$, $\delta(x)$ must have units of $\frac{1}{\text{length}}$

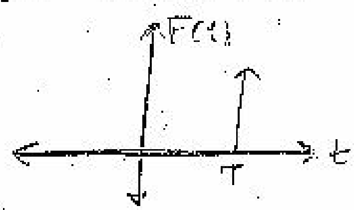
$\lim_{\epsilon \to 0} \int_{-\epsilon}^{\epsilon} f(x)\,dx = 0$ for a real function

Fourier Transform of a Delta Function:

$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} e^{i\omega t} F(t)\,dt \quad \text{where: } F(t) = \delta(t - T)$$

T: some constant

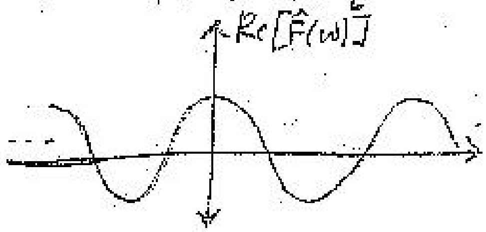
$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{t=-\infty}^{\infty} e^{i\omega t} \delta(t-T)\,dt$$



Sifting Property: $\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} e^{i\omega T}$

or also:

$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} [\cos(\omega T) + i \sin(\omega T)]$$

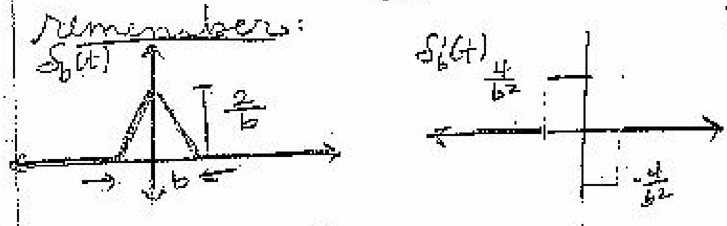


* as $T \to 0$, the cos flattens out and eventually forms a line @ 1

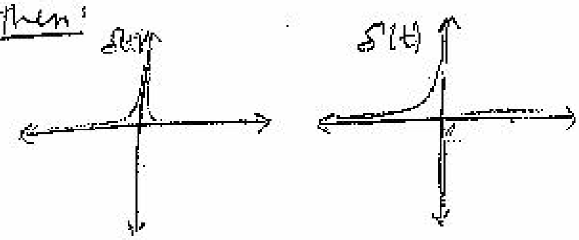
Derivative of Dirac Delta Function:

$$\delta'(t) = \frac{d}{dt} \delta(t)$$

remember:



Then:



$$\int_{t=-\infty}^{+\infty} \delta'(t)\, f(t)\,dt \qquad \int_{-\infty}^{\infty} \delta(t) f(t)\,dt = f(0)$$

Special Property:

f(t) must have "bounded support";
meaning: f(t) is nonzero on a finite interval [a,b]
(function is always zero at ±∞)
must be bounded ☺

$$d(uv) = v(du) + u(dv)$$

$$uv\Big|_a^b = \int_a^b d(uv) = \int_a^b v\,du + \int_a^b u\,dv \qquad \text{\& integrate both sides}$$

→ now, let $u = \delta'(t)$ & $dv = f(t)dt$ integration by parts

$$\int_{t=-\infty}^{\infty} \delta'(t) f(t)\,dt = \underbrace{\delta'(t) f(t)\Big|_a^b}_{\text{surface term}} - \int_a^b \delta(t) f'(t)\,dt$$

$du =$ $v =$

surface term → 0

divergence theorem = Gauss's Law

$$= -f'(0) \equiv -\frac{df}{dt}\Big|_{t=0}$$

EXAMPLE → another distribution

$\sin(\alpha t)\delta'(t)$ is a generalized distribution

SIMPLIFY THIS

→ distributions only make physical sense inside an integral & multiplied by a "test" function, f(t) w/ bounded support

$$\int_{-\infty}^{+\infty} \sin(\alpha t)\delta'(t) \underset{\text{test function}}{f(t)}\,dt$$

$$= \sin(\alpha t)\,\underline{\delta(t) f(t)}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dt}\left[\sin(\alpha t) f(t)\right] \delta(t)\,dt$$

$$= -\int_{-\infty}^{\infty} \left[\alpha \cos(\alpha t) f(t) + \sin(\alpha t) f'(t)\right] \delta(t)\,dt \implies$$

plug in 0 for all t's...

$$= -\int_{-\infty}^{\infty} \left[\alpha \cos(\alpha t) f(t) + \cancel{\sin(\alpha t)}^{\,0} f'(t)\right] \delta(t)\, dt$$

$$= -\int_{-\infty}^{\infty} \alpha \cdot 1\, f(t)\, \delta(t)\, dt = -\alpha f(0)$$ undergraduate

can't see the distribution if you go that far... b/c it didn't start w/ a function

original line: $\int_{-\infty}^{\infty} \sin(\alpha t)\, \delta'(t) f(t)\, dt$

$\sin(\alpha t)\, \delta'(t) = -\alpha \delta(t)$ graduate

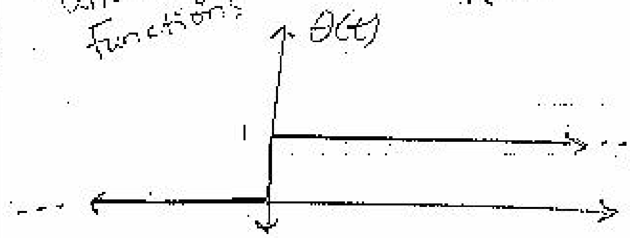
* don't report ans. w/ $f'(t)$ b/c you didn't start w/ $f(t)$...

→ SIMILAR TO BONUS PROBLEM ←

Integration of Dirac Delta Function

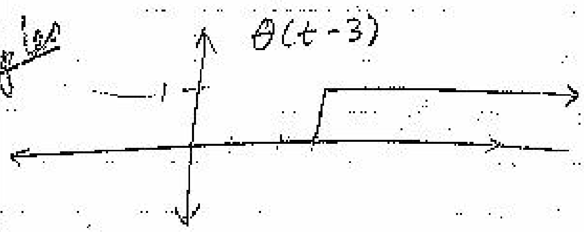
"Heavy-Side Functions" "Unit Step Functions"

$$\theta(t) \equiv \int_{x=-\infty}^{t} \delta(x)\, dx = \begin{cases} 0, & t<0 \\ 1, & t>0 \end{cases}$$

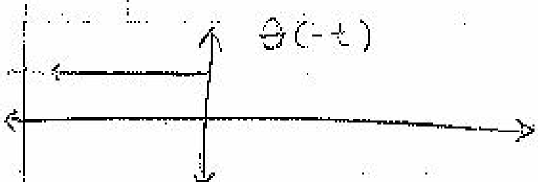


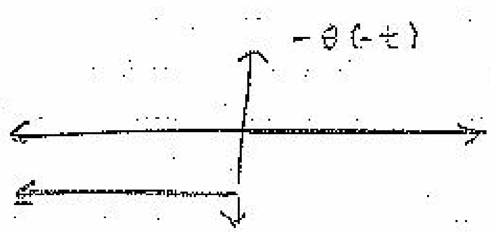
at $t=0$, you pick what $\theta(t)$ equals → some popular choices are $\frac{1}{2}$, 0 & 1

Some Examples

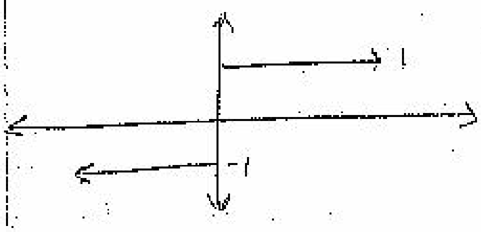


$t-3<0,\ \theta=0$
$t-3>0,\ \theta=1$





* homework problem starts at $3 \to 6$ & at height of 4



$2\theta(t)-1$
or
$\theta(t)-\theta(-t)$

electron: $\rho(\vec{r}) = -e\,\delta(x)\,\delta(y)\,\delta(z)$
later to be written in spherical coordinates

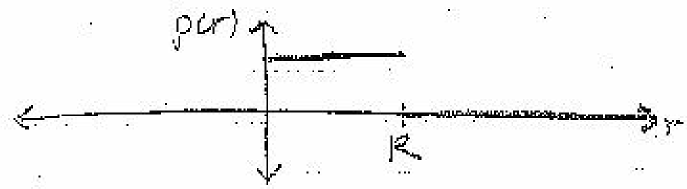
proton: model (solid ball of charge)

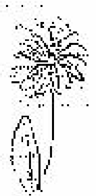
proton radius cubed

$$\rho(r)=\begin{cases}\frac{+e}{\frac{4}{3}\pi R^3}, & r<R\\ 0, & r>R\end{cases}$$

like $\theta(t)$ function → on and then it turns off

$\rho(r)=\frac{+e}{\frac{4}{3}\pi R^3}\,\theta(R-r)$ rewritten w/ $\theta(t)$ rather than piecewise function





Algebraic Equations: $5-x=2$, $x^2=4$
Goal: Find the number(s) for x that make the equations true
w/ complex numbers $i^2=-1$ $\xi - i,\ i\xi = \sqrt{-1}$
good book for history of i

n & homogeneous ~ (vs. partial) types of diffeq's...
differential equations: first-order, ordinary, linear
$\frac{df(x)}{dx}=\cos x$ $f'(x)=\cos x$ $f(x)=\sin x + C$

Goal → find function(s), $f(x)$ that make the equation true
*no guaranteed ways to do this - bit of guesswork ☺

- Differential Equations -

$$\frac{dy(x)}{dx} - y(x) = 0$$

Goal: find function, $y(x)$ that makes this true $\forall x$ (for all x)

$y'(x) - y(x) = 0$

Guessing... $y(x) = x^2$ $\frac{dy(x)}{dx} = 2x$

plug in → $2x - x^2 \neq 0$ ∴ it was a bad guess

A better guess... $y(x) = e^x$ $y'(x) = e^x$

$y'(x) - y(x) = e^x - e^x = 0 \quad \forall x$

not most general equation → Ae^x is the most general where A is any constant

Highest derivative tells you # of arbitrary constants
∴ $y(t) = Ae^x$ has 1 arbitrary constant b/c it's a first-order differential equation

Order - highest derivative in function, not variable

$y'(x) - y(x) = 0$ is first-order, linear (in y) & homogeneous

Linear - derivatives of the function $y(x)$, including the 0^{th} derivative) occur to the first power.

$$\frac{d^2y(x)}{dx^2} + y(x) = 0 \qquad y'' + y = 0$$

2^{nd} order - linear & homogeneous

$y''(x) + y^2(x) = 0$ non-linear

$y'(x) + \sin(y(x)) = 0$ also non-linear

$[y''(x)]^2 - y(x) = 0 \rightarrow \left[\frac{dy(x)}{dx}\right]^2 - y(x) = 0$ also non-linear